A Decomposition for Chordal graphs and Applications

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Joint work with Vincent Limouzy and Juraj Stacho

Pretty Structure, Existencial Polytime
Jack Edmonds' Birthday, 7-9 april 2009

Schedule

Chordal graphs

Schedule

Schedule

Schedule

Chordal graphs

Reduced Clique Graph

First properties of reduced clique graphs

Decomposition and split minors

Schedule

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Fundamental objects to play with

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under inclusion

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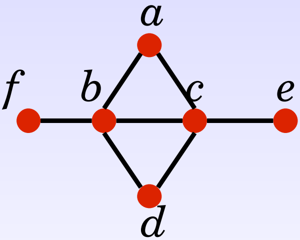
Minimal Separators

A subset of vertices S is a minimal separator if G if there exist $a, b \in G$ such that a and b are not connected in $G - S$.
and S is minimal for inclusion with this property .

An example



3 minimal separators $\{b\}$ for f and a, $\{c\}$ for a and e and $\{b, c\}$ for a and d.

Maximal Clique trees

A maximal clique tree (sometimes clique tree) is a tree T that satisfies the following three conditions :

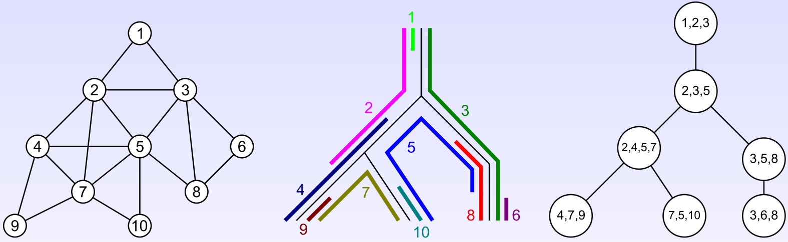
- Vertices of T are associated with the maximal cliques of G
- Edges of T correspond to minimal separators.
- For any vertex $x \in G$, the cliques containing x yield a subtree of T.

Using results of Dirac 1961, Fulkerson, Gross 1965, Buneman 1974, Gavril 1974 and Rose, Tarjan and Lueker 1976 :

The following statements are equivalent and characterize chordal graphs :

(i) G has a simplicial elimination scheme
(ii) Every minimal separator is a clique
(iii) G admits a maximal clique tree.
(iv) G is the intersection graph of subtrees in a tree.
(v) Any MNS (LexBFS, MCS) provides a simplicial elimination scheme.

An example



Size of a maximal clique tree in a chordal graph

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- Therefore any maximal clique tree is bounded by $|V| + |E|$. Similarly the size of the minimal separators is linear.

Reduced Clique Graph

Definition

For a chordal graph G we define its Maximal cliques graph denoted by $\mathcal{C}_r(G)$ whose vertices are the maximal cliques of G and we put an edge between two maximal cliques C, C' if $C \cap C'$ is a minimal separator.

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- Note that this is a subgraph of the intersection graph of the maximal cliques of G.
- Is $\mathcal{C}_r(G)$ chordal ? What is the size of $\mathcal{C}_r(G)$ in terms of n and m ?

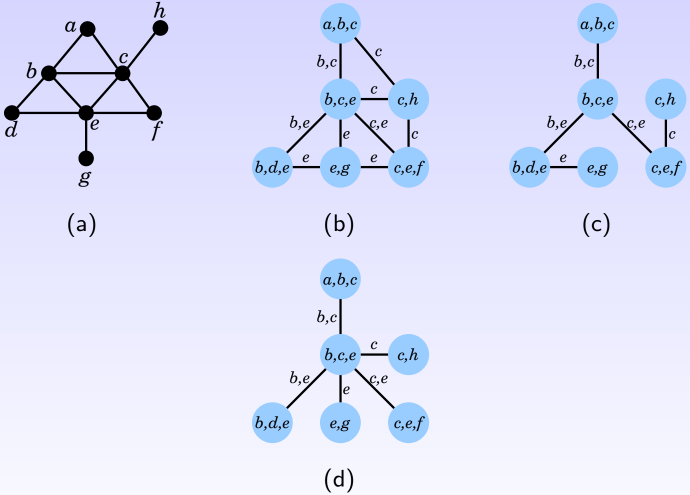
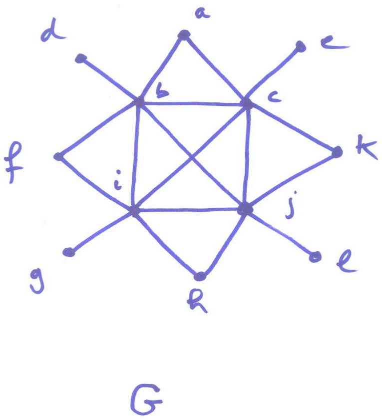


FIG.: A chordal graph (a), its reduced clique-graph (b), $\{b, d, e\} \cap \{c, e, f\} = \{e\}$ the edge is missing.

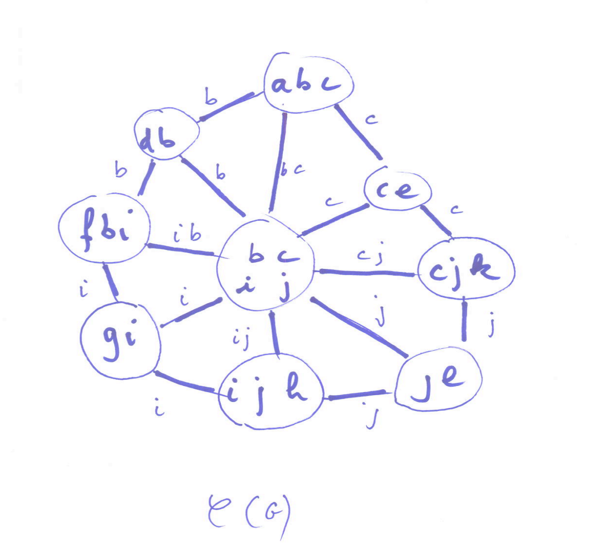
Size of $\mathcal{C}_r(G)$

Considering a star on n vertices,
shows $|\mathcal{CS}(G)| \in O(n^2)$
Not linear in the size of G

Is $\mathcal{C}_r(G)$ chordal ?



$\mathcal{C}_r(G)$ is not chordal!



Combinatorial structure of $\mathcal{C}_r(G)$

Lemma 1 : M.H and C. Paul 95

If C_1, C_2, C_3 is a cycle in $\mathcal{C}_r(G)$, with S_{12}, S_{23} and S_{23} be the associated minimal separators then two of these three separators are equal and included in the third.

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Lemma 2 : M.H. and C. Paul 95

Let C_1, C_2, C_3 be 3 maximal cliques, if $C_1 \cap C_2 = S_{12} \subset S_{23} = C_2 \cap C_3$ then it yields a triangle in $\mathcal{C}_r(G)$

Theorem : Blayr and Payton 93 but also Gavril 87 and Shibata 88

Maximal clique trees are exactly the maximum spanning trees of $\mathcal{C}_r(G)$.

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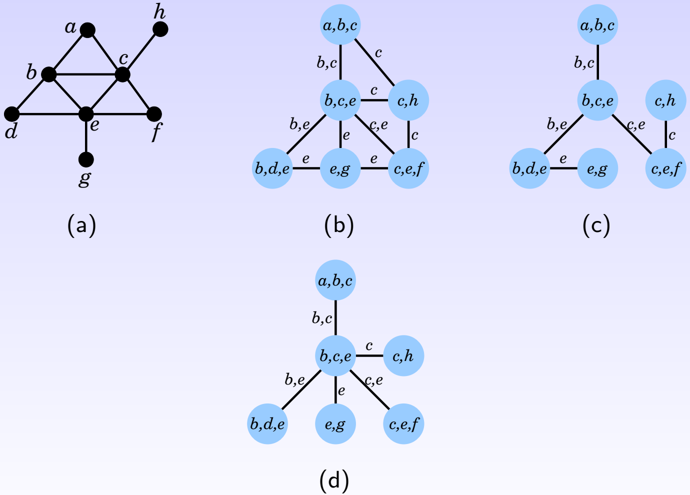
The weight of an edge being the size of the minimal separator it represents.

- $\mathcal{C}_r(G)$ is the union of all maximal clique trees of G.
- From one maximal clique tree to another there always exists a path of exchanges on triangles.

Lemma 3 : Equality case

Let C_1, C_2, C_3 be 3 maximal cliques, if $S_{12} = S_{23}$ then :

- either the $C_1 \cap C_3 = S_{13}$ is a minimal separator
- or the edges C_1C_2 and C_2C_3 cannot belong together to a maximal clique tree of G.


a
b
c
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e
f
g
h
(a)
a,b,c
b,c
c
b,c,e
c,h
b,e
e
c,e
c
b,d,e
e,g
c,e,f
(b)
(c)
(d)

Cannonical representation

- For an interval graph, its PQ-tree represents all its possible models and can be taken as a cannonical representation of the graph (for example for graph isomorphism)

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- For an interval graph, its PQ-tree represents all its possible models and can be taken as a cannonical representation of the graph (for example for graph isomorphism)
- But even path graphs are isomorphism complete. Therefore a canonical tree representation is not obvious for chordal graphs.
- $\mathcal{C}_r(G)$ is a Pretty Structure to study chordal graphs.

  To prove structural properties of all maximal clique trees of a given chordal graph.

First example

Theorem : Lévêque, Maffray, Preissmann 2008

There always exists a maximal clique tree with a leaf labelled by a maximal minimal separator.

Proof

1. Compute a maximal clique tree T

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Proof

1. Compute a maximal clique tree T
2. Sort the minimal separators according to their size
3. Consider an edge $ab \in T$ labelled by S of maximum size
4. Either in T_b all edges adjacent to b are labelled with separators included or equal to S
5. Or recurse on S' a minimal separator T_b incomparable with S and of maximal size.

At step 4, consider an edge bc labelled with a minimal separator U

- if $U \subset S$.
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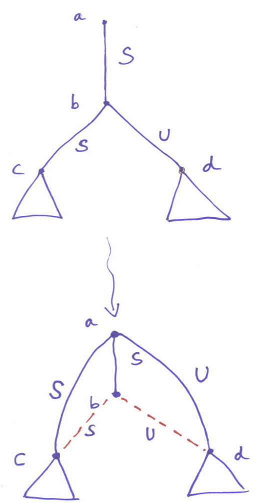
- Then ab become an pending edge.


a
S
b
S
U
c
d
a
S
S
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S
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c
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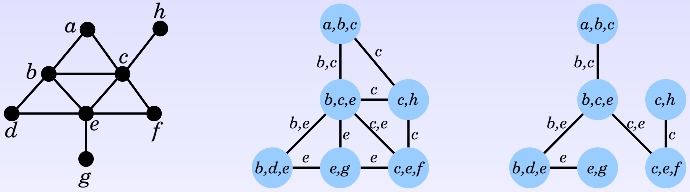
Algorithm



(e) (f) (g)

Complexity

This tree can be linearly computed.

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It always exists a simplicial elimination scheme following a linear extension of the containment ordering of the minimal separators.

- Notice that not all linear extensions are available.
- These schemes are interesting for the structure of path graphs
- Can these linear extensions be computed linearly, using for example some search on G ?

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- Each clique tree uses exactly $k-1$ edges of the multipartite split

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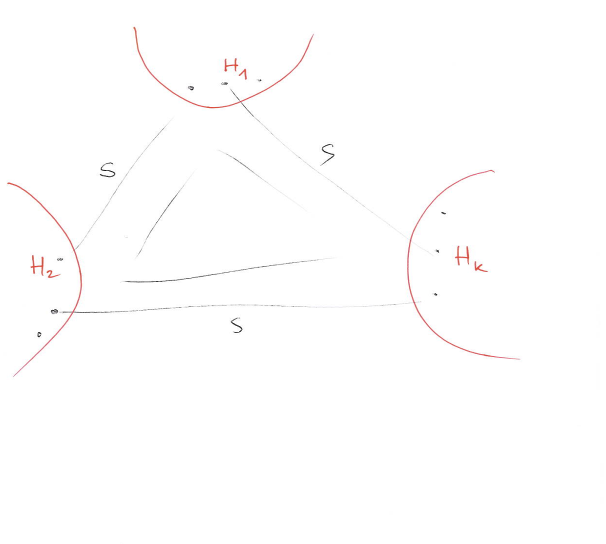
- Each clique tree uses exactly $k-1$ edges of the multipartite split
- A clique tree of G is connected in each component C_i


H1
S
S
H2
Hk
S

Split minors

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Definition

H is a split-minor of $\mathcal{C}_r(G)$, if H can be obtained from $\mathcal{C}_r(G)$ using L1, L2 and L3.

Theorem

Every $\mathcal{C}_r(G)$ is totally decomposable with the operations L1, L2 and L3.

Asteroidal number

Definition

For a graph G, a set A of vertices is asteroidal, if for each $v \in A$, $A - v$ belongs to one connected component of $G - N(v)$.
The asteroidal number $a(G)$ is the size of the maximum asteroidal set in G.

Computing $a(G)$ is NP-hard for planar graphs but polynomial for HDD-free graphs Kloks, Krastch, Muller 1997.

Theorem M.H., J. Stacho 2009

For a chordal graph $a(G) < k$ iff no labeled k-star is a split-minor of $\mathcal{C}_r(G)$

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G is interval iff no labeled claw is a split-minor of $\mathcal{C}_r(G)$

Leafage

Definition (Lin, McKee, West 1998)

For a chordal graph G, the leafage $l(G)$ is the minimum number of leaves in a maximal clique tree of G.

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Applications

If $l(G) = k$, an optimal model provides a good implicit representation.
Max clique, coloration, ... in $O(k.n)$.

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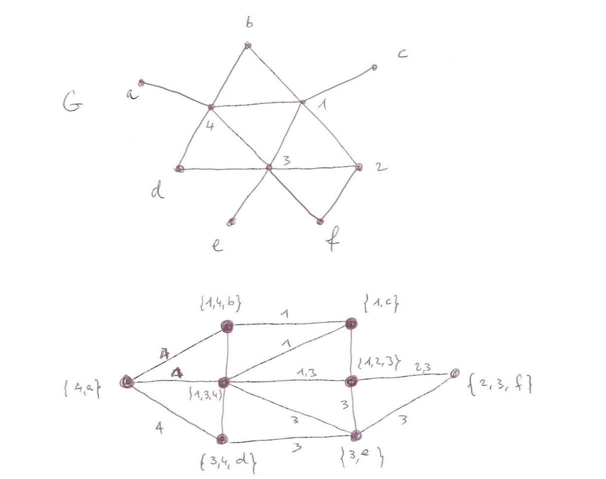
1. Use tokens in the multipartite splits (corresponding to half edges) and propagate them
2. Construct augmenting paths in an associated directed graph preserving the degrees of the tree.


G
a
b
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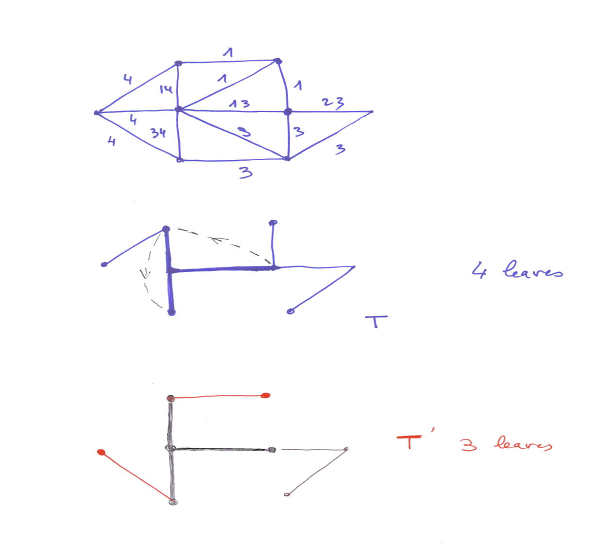
d
e
f
1
2
3
4
{1,4,b}
{1,c}
{4,a}
{1,3,4}
{1,2,3}
{2,3,f}
{3,4,d}
{3,e}
1
4
1,3
2,3
3


4 leaves
T
T'
3 leaves

Results so far on $\mathcal{C}_r(G)$ as a labelled graph

	Maximum w. Hamilton Path	Leafage
G arbitrary	NP-complete	NP-complete
Labelled $\mathcal{C}_r(G)$	linear Interval graph recognition	polynomial $O(n^3)$

Two algorithmic problems :

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1. Can we compute in linear time the leafage of a chordal graph ?
2. Since it is linear to check if the vertex leafage of a chordal graph is 2 (i.e. the recognition of path graphs).
   Is it polynomial to compute the vertex leafage of a chordal graph ?

Jack's possible question :

There must be a min-max theorem for the leafage ?

Thank you for your attention !

Happy Birthday Jack !