Is this matrix singular?

András Recski

Budapest University of Technology and Economics

Paris, 2009

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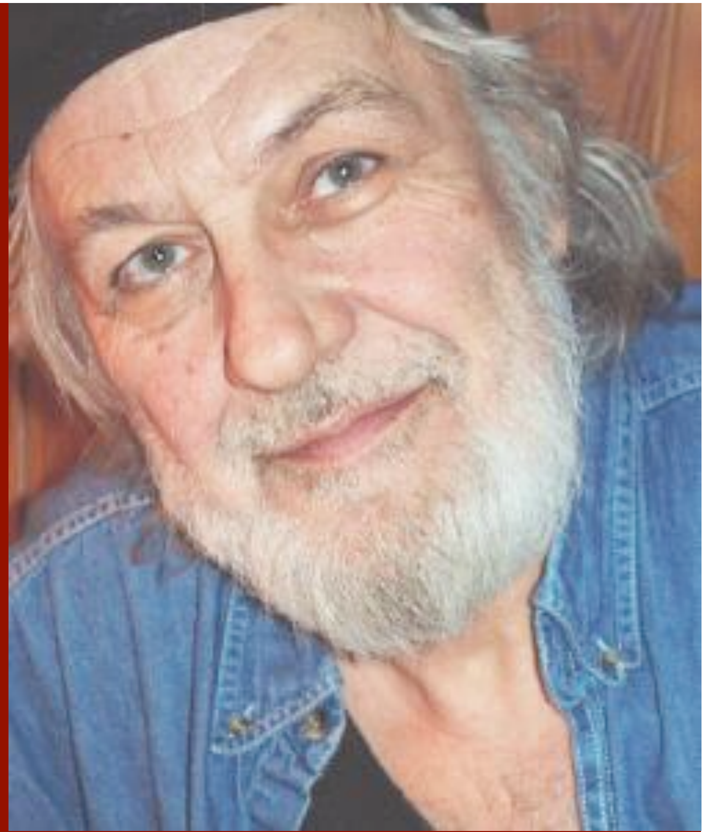

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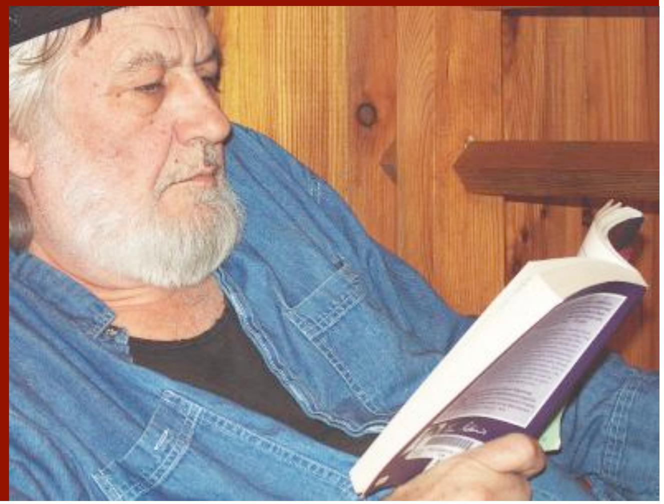

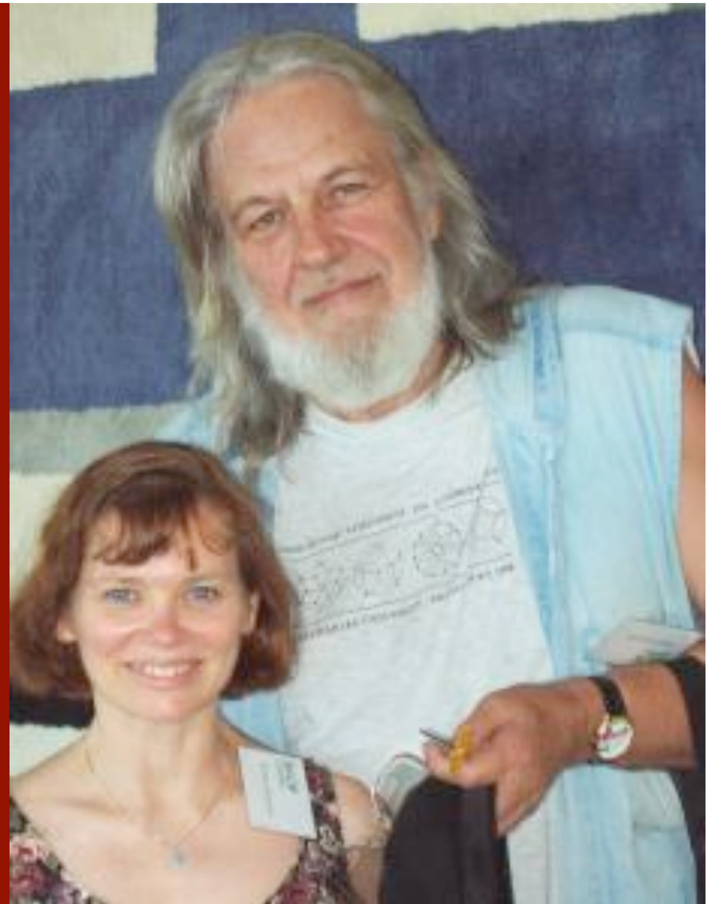

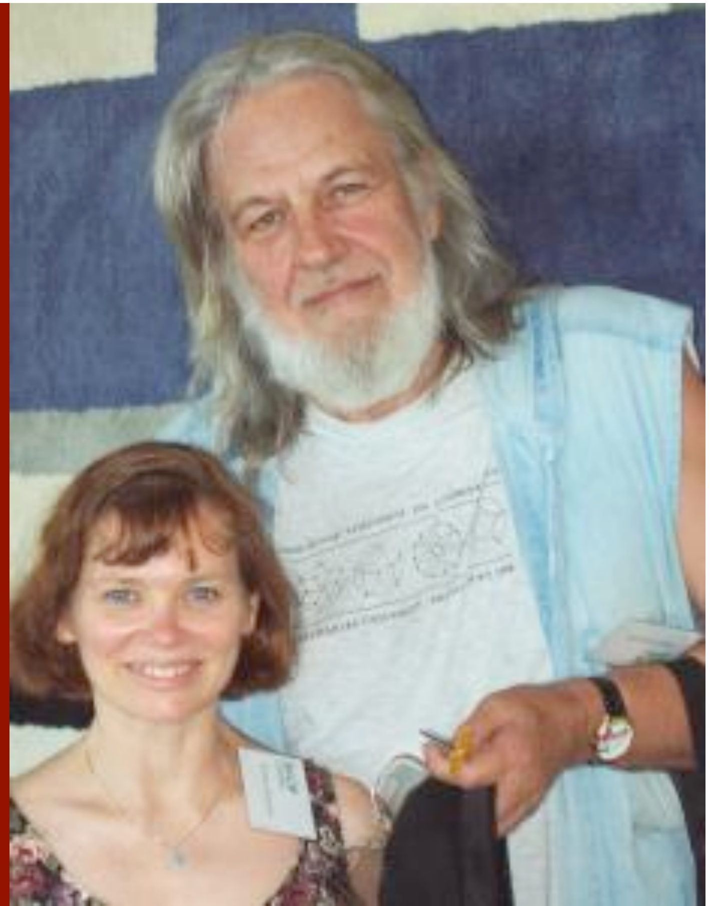

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Jack in Budapest in 1994

The Anonymus group – the students of Dénes Kőnig, including Pál Erdős, Pál Turán, György Szekeres, Tibor Gallai and many others

When is a matrix singular?

Let *A* be an *n*X*n* matrix.

det *A* can be determined effectively if the entries are from a field.

When is a matrix singular?

Let *A* be an *n* x *n* matrix.

det *A* can be determined effectively if the entries are from a field.

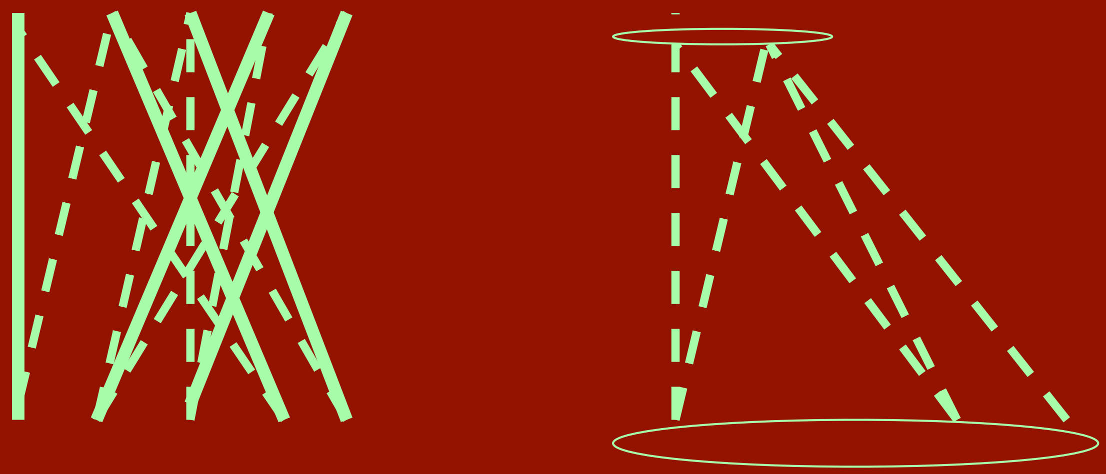
But what if they are from a commutative ring?

A classical case

D. Kőnig, 1915

If the nonzero entries are distinct variables (or real numbers, algebraically independent over the field of the rationals) then we can describe the zero-nonzero pattern of the matrix by a bipartite graph and check whether the graph has a perfect matching.

$$A_1 = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & 0 & 0 & 0 \\ 0 & x_{11} & x_{12} & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 \end{pmatrix}$$



If the nonzero entries are different variables (or real numbers, algebraically independent over the field of the rationals)

then rank equals term rank.

Systems of Distinct Representatives and Linear Algebra*

Jack Edmonds

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(November 16, 1966)

Some purposes of this paper are: (1) To take seriously the term, "term rank." (2) To make an issue of not "rearranging rows and columns" by not "arranging" them in the first place. (3) To promote the numerical use of Cramer's rule. (4) To illustrate that the relevance of "number of steps" to "amount of work" depends on the amount of work in a step. (5) To call attention to the computational aspect of SDR's, an aspect where the subject differs from being an instance of familiar linear algebra. (6) To describe an SDR instance of a theory on extremal combinatorics that uses linear algebra in very different ways than does totally unimodular theory. (The preceding paper, Optimum Branchings, describes another instance of that theory.)

Key Words: Algorithms, combinatorics, indeterminates, linear algebra, matroids, systems of distinct representatives, term rank.

1. Introduction

'he well-known concept of term rank [5, 6],[1] is wn here to be a special case of linear-algebra rank. s observation is used to provide a simple linear-ebra proof of the well-known SDR theorem. Except familiar linear algebra, the paper is self-contained. ncidentally to SDR's, an algorithm is presented for nputing the determinant or the rank of any matrix r any integral domain. It is a variation of Gaussian ., linear) elimination which has certain advantages. s observed to be an interestingly bad algorithm for nputing term rank.

'he final part of the paper discusses some simple troidal aspects of SDR's.

However, here the word "transversal" will be used differently.)

3. Matrices of Zeros and Ones

The subject of SDR's is frequently treated in the context of matrices of 0's and 1's. The *incidence matrix* of the family Q of subsets of E is the matrix $A=[a_{ij}]$, $i \epsilon E$, $j \epsilon Q$, such that $a_{ij}=1$ if $i \epsilon j$, and $a_{ij}=0$ otherwise.

A *matching* in a matrix is a subset of its positions (i, j) such that first indices (rows) of members are all different and second indices (columns) of members are all different. A *transversal* (*column transversal*) of a matrix is a matching in the matrix which has a member

Edmonds, 1967

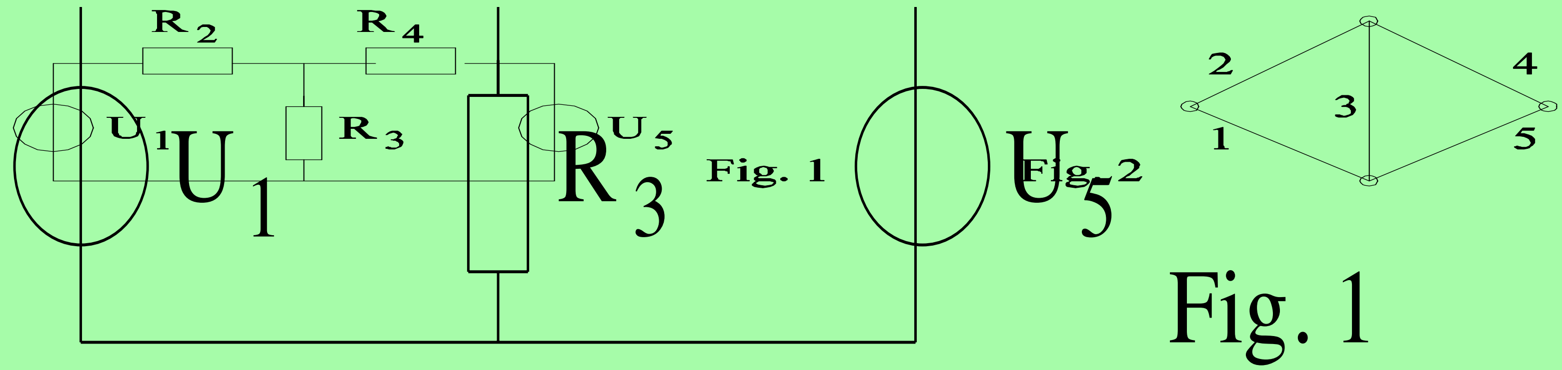
Theorem 1. The term rank of a 0,1 matrix **A** is the same as the linear algebra rank of the matrix obtained by replacing the 1's in **A** by distinct indeterminates over any integral domain.

Another classical case

If the matrix was obtained during the analysis of an electric network consisting of resistors, voltage and current sources, then…

Kirchhoff, 1847


DDR
5
GUSTAV ROBERT KIRCHHOFF
1824
1887
30
DEUTSCHE BUNDESPOST BERLIN



2-terminal devices (like resistors, voltage sources) are represented as edges of a graph

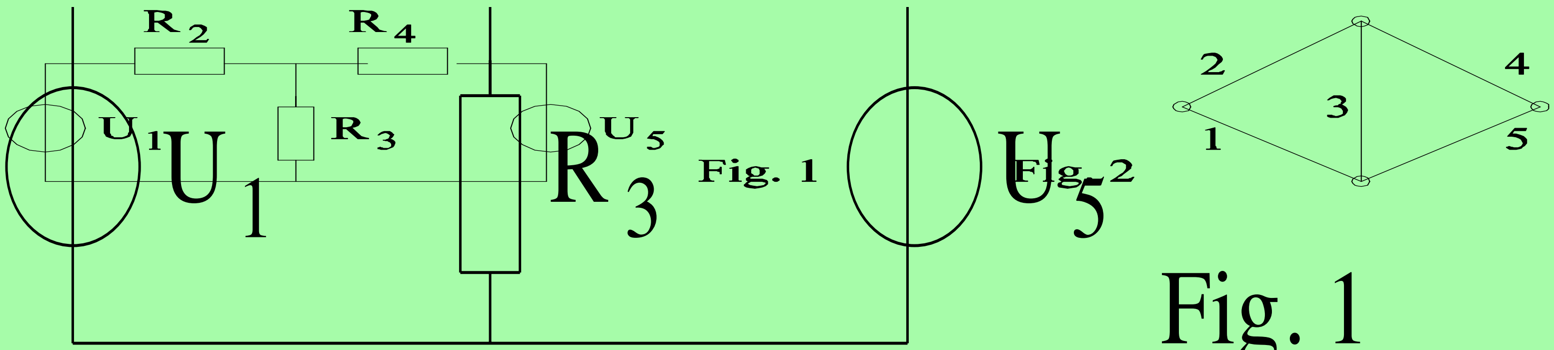
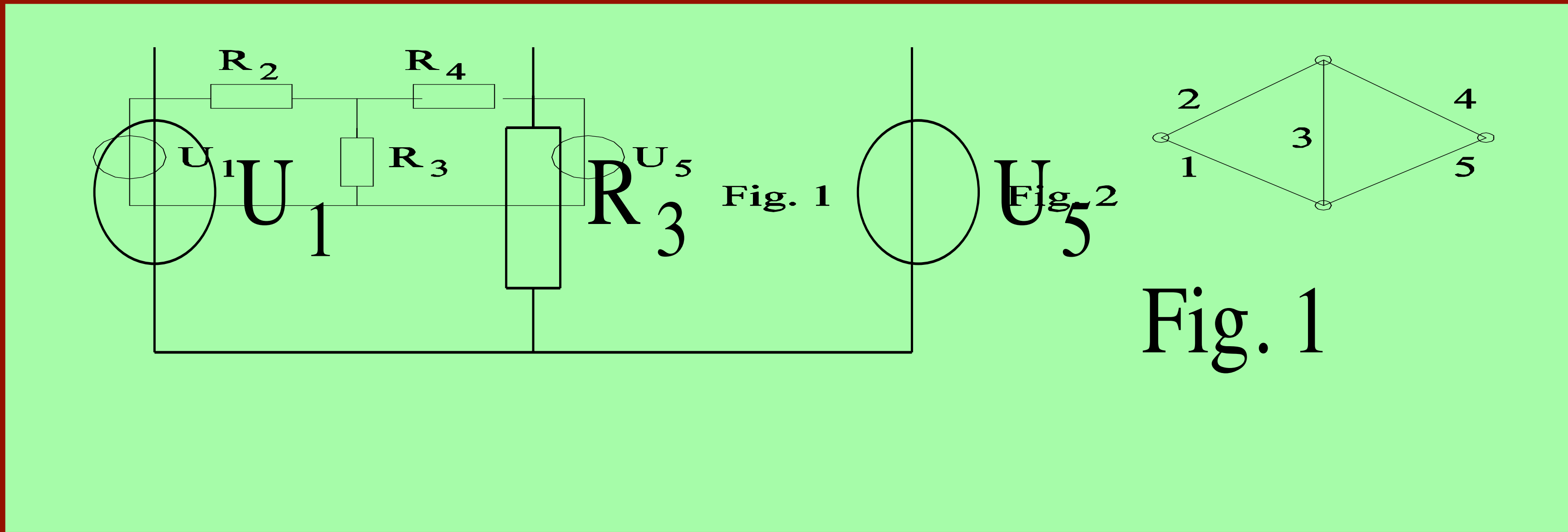


Fig. 1

2-terminal devices (like resistors, voltage sources) are represented as edges of a graph, relations among voltages (or among currents) are described with the help of the circuits (cut sets, respectively) of the graph.



$$i_3 = (R_4 u_1 + R_2 u_5) / (R_2 R_3 + R_2 R_4 + R_3 R_4)$$

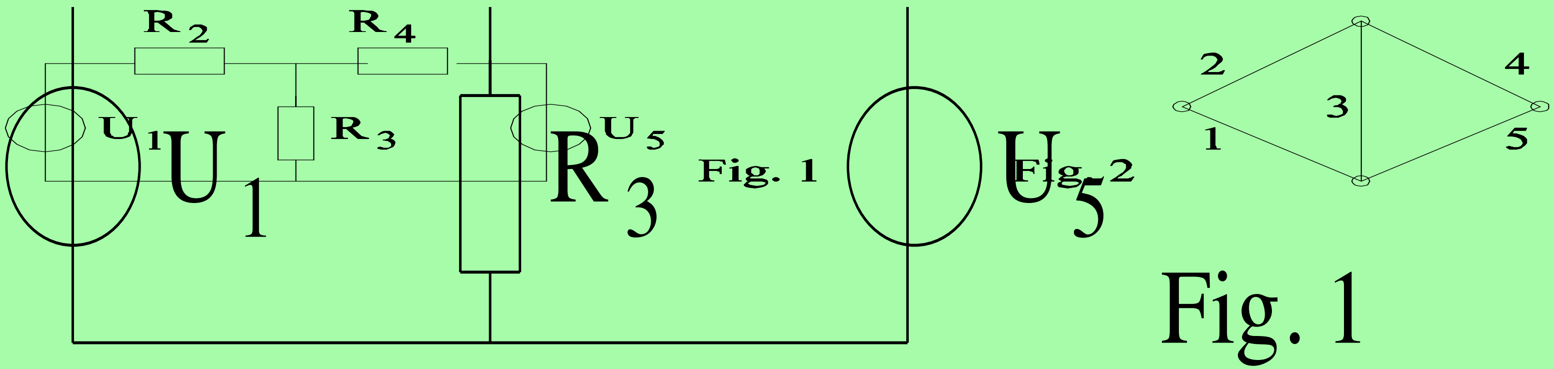


Fig. 1

$$i_3 = (R_4 u_1 + R_2 u_5) / (R_2 R_3 + R_2 R_4 + R_3 R_4)$$

$$= (Y_2 Y_3 u_1 + Y_3 Y_4 u_5) / (Y_2 + Y_3 + Y_4)$$

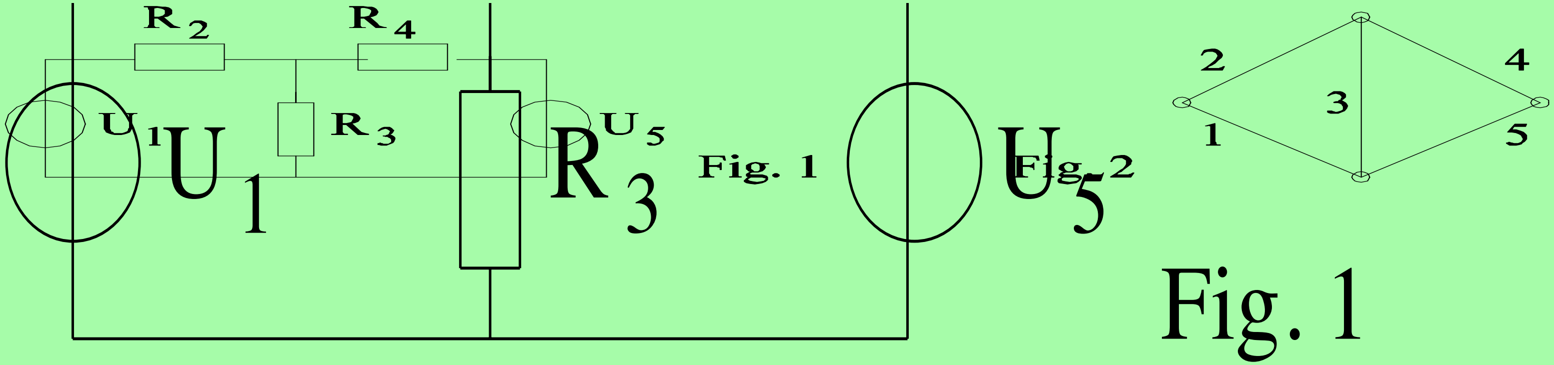


Fig. 1

$$i_3 = (R_4 u_1 + R_2 u_5) / (R_2 R_3 + R_2 R_4 + R_3 R_4)$$

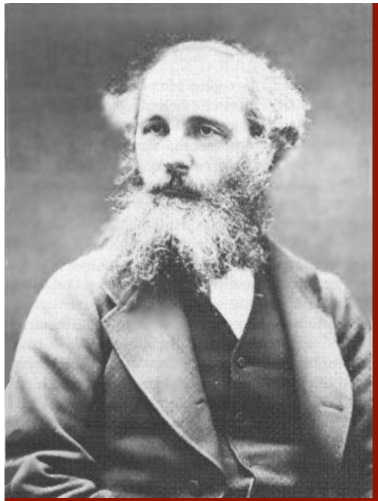
$$= (Y_2 Y_3 u_1 + Y_3 Y_4 u_5) / (Y_2 + Y_3 + Y_4)$$

$$W_Y(G) = \sum_T \prod_{j \varepsilon T} Y_j$$

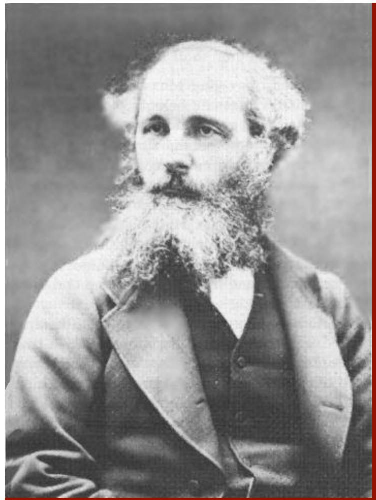
(Kirchhoff, 1847; Maxwell, 1892)

If the matrix **A** was obtained during the analysis of an electric network consisting of resistors, voltage and current sources, then the nonzero expansion members of det **A** are in one-one correspondence with those trees of the network graph which contain every voltage source and none of the current sources.

…the nonzero expansion members of det **A** are in one-one correspondence with those trees of the network graph which contain every voltage source and none of the current sources. Hence if the physical parameters are distinct indeterminants then nonsingularity $\longleftrightarrow$ the existence of such a tree.



Maxwell



Maxwell

Generalization of this classical case

If the matrix was obtained during the analysis of an electric network consisting of resistors, voltage and current sources ***and more complex devices like ideal transformers, gyrators, operational amplifiers etc.*** then what?

Example 1 – Ideal transformers

$$u_2 = k \cdot u_1, \quad i_1 = -k \cdot i_2$$

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Both the tree and the tree complement must contain exactly one of the two port edges.

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Both the tree and the tree complement must contain exactly one of the two port edges.

If the number of the ideal transformers is part of the input, one needs the ***matroid partition algorithm*** (Edmonds, 1968).

Example 2 – Gyrators

$$u_2 = -R \cdot i_1, \quad u_1 = R \cdot i_2$$

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If the number of the ideal transformers is part of the input, one needs the ***matroid matching algorithm*** (Lovász, 1980).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r\begin{pmatrix} a & b & e & f \\ c & d & g & h \end{pmatrix} = 2.$$

How can we generalize the above observations to arbitrary 2-ports?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r\begin{pmatrix} a & b & e & f \\ c & d & g & h \end{pmatrix} = 2.$$

- $|\{p_1, p_2\} \cap T| \leq 1$ if $ad = bc$ holds;

[redacted]

- $|\{p_1, p_2\} \cap T| \geq 1$ if $eh = fg$ holds;

[redacted]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r \begin{pmatrix} a & b & e & f \\ c & d & g & h \end{pmatrix} = 2.$$

- $|\{p_1, p_2\} \cap T| \leq 1$ if $ad = bc$ holds;
- $\{p_1, p_2\} \cap T \neq \{p_1\}$ if $ah = fc$ holds;
- $\{p_1, p_2\} \cap T \neq \{p_2\}$ if $bg = de$ holds;
- $|\{p_1, p_2\} \cap T| \geq 1$ if $eh = fg$ holds;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r\begin{pmatrix} a & b & e & f \\ c & d & g & h \end{pmatrix} = 2.$$

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- $|\{p_1, p_2\} \cap T| \geq 1$ if $eh = fg$ holds;
- $|\{p_1, p_2\} \cap T| \neq 1$ if $(a + b)(g - h) = (c + d)(e - f)$ holds.

Theoretically there are infinitely many possible algebraic relations among these 8 numbers but only these five can lead to singularities (R., 1980).

- $|\{p_1, p_2\} \cap T| \leq 1$ if $ad = bc$ holds;
- $\{p_1, p_2\} \cap T \neq \{p_1\}$ if $ah = fc$ holds;
- $\{p_1, p_2\} \cap T \neq \{p_2\}$ if $bg = de$ holds;
- $|\{p_1, p_2\} \cap T| \geq 1$ if $eh = fg$ holds;
- $|\{p_1, p_2\} \cap T| \neq 1$ if $(a + b)(g - h) = (c + d)(e - f)$ holds.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r\begin{pmatrix} a & b & e & f \\ c & d & g & h \end{pmatrix} = 2.$$

Does the column space matroid of this 2 X 4 matrix contain every important qualitative information about the 2-ports?

Obviously not. Compare

$$u_1 = Ri_2 \,, \quad u_2 = -Ri_1$$

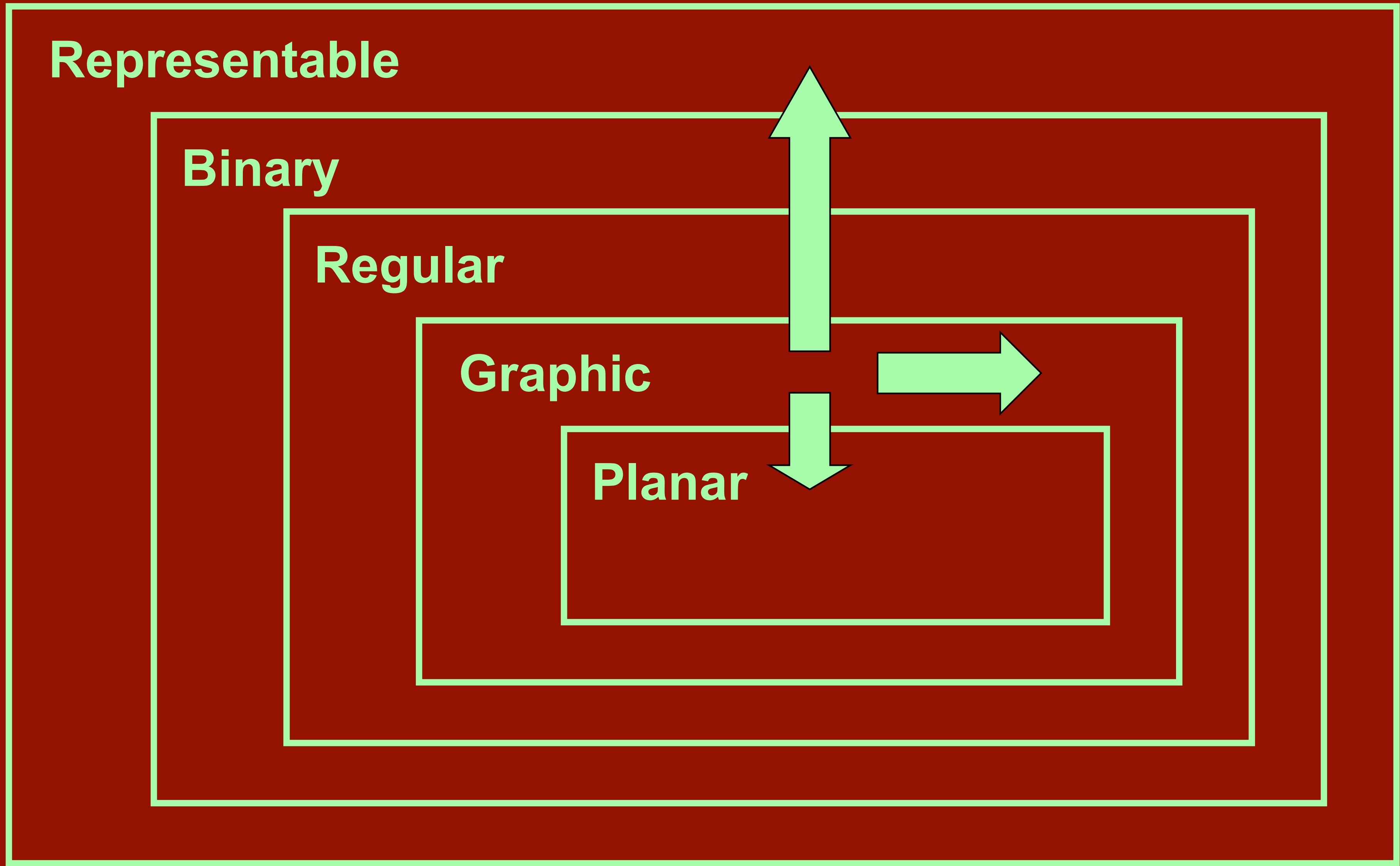
and

$$u_1 = Ri_2 \,, \quad u_2 = -2Ri_1$$

Finally, a conjecture:

The sum of two graphic matroids is either graphic or nonbinary

Algebraic representation

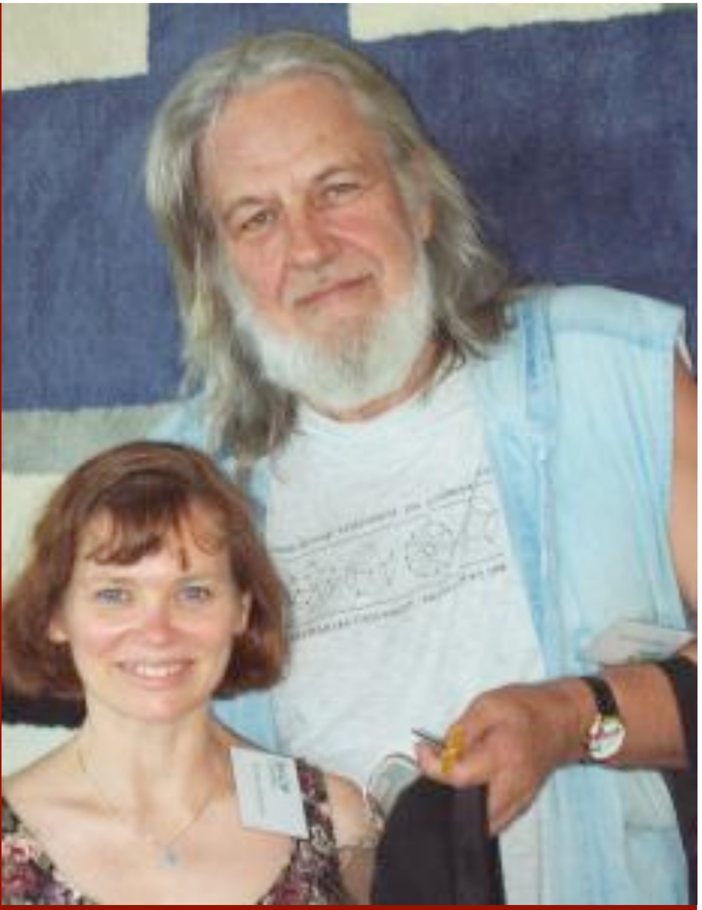


Finally, a conjecture:

The sum of two graphic matroids is either graphic or nonbinary

Known to be true if the two matroids are equal

(Lovász-R., 1973)



Happy birthday, Jack!