To make long stories short … from the last 24 years

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Kathie: We would like the talks to be ***easy to understand.***
Jack prefers talks on stuff he should have known years ago but has forgotten or never got to.

2008

Integer Decomposition (ID)

Polyhedron P has the *ID property*,

if $v \in kP$, and v, k integer $\Rightarrow$

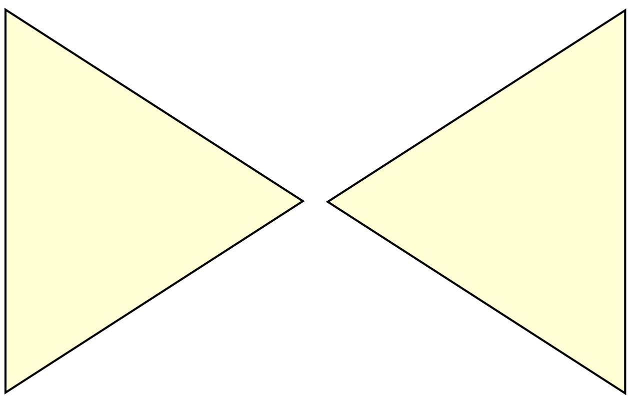
$v = v_1 + \dots + v_k,\ \ v_1, \dots, v_k \in P \cap \mathbf{Z}^n$

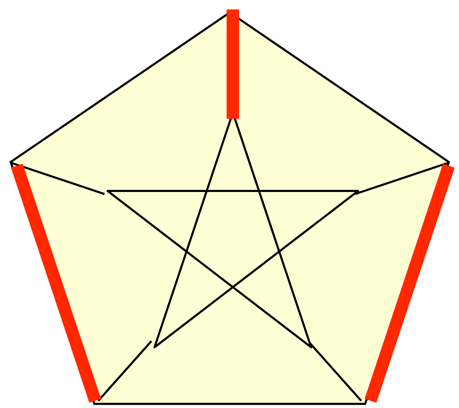
Examples. **ID**: matchings in bipartite graphs

independent sets of matroids

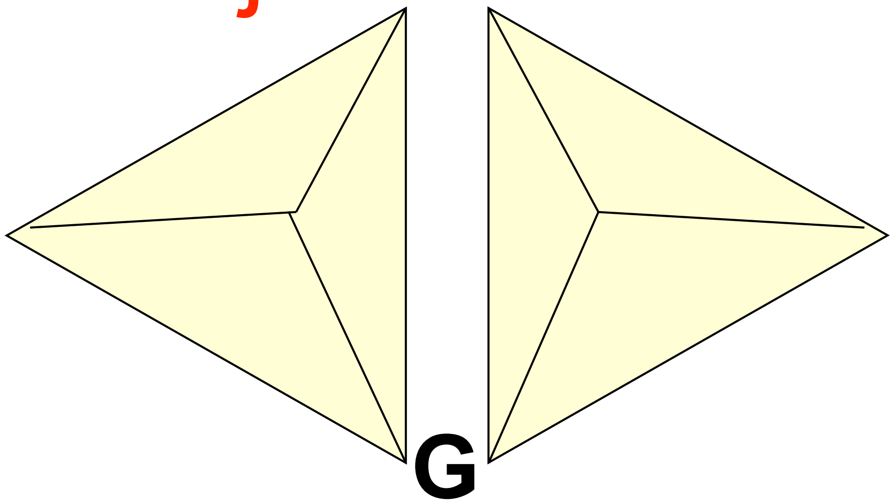
by Jack's matroid partition thm.

NOT ID:



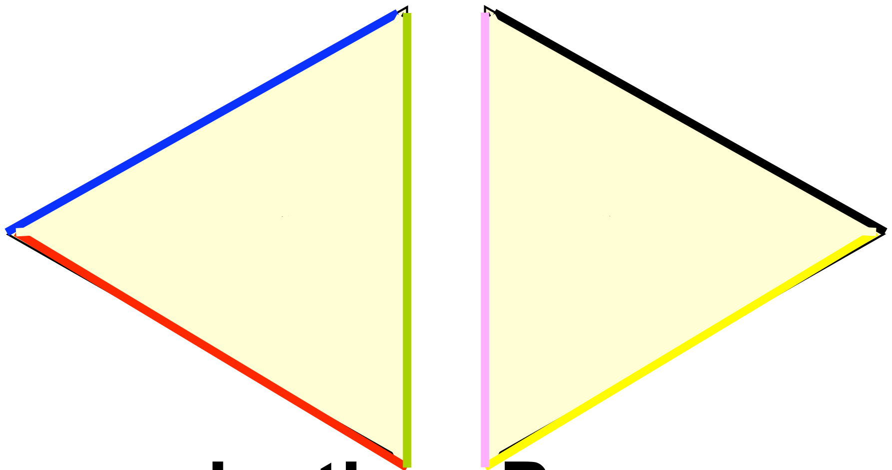


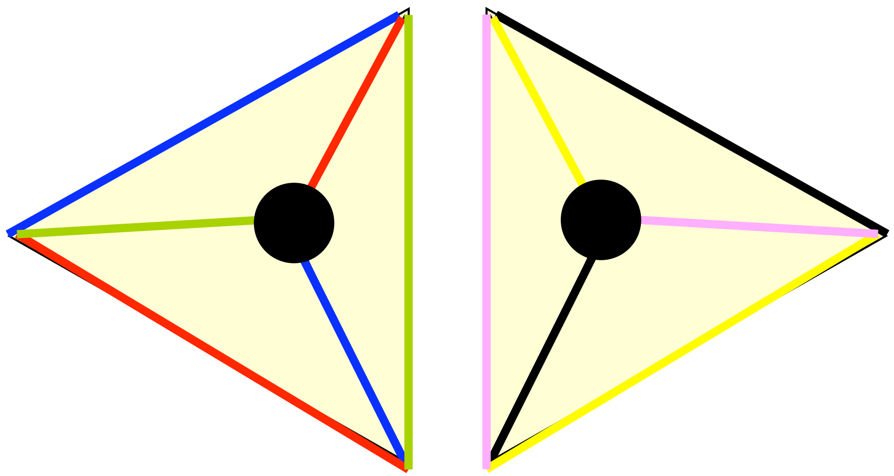
Projection of TU is not necessarily TU
Projection of ID is not necessarily ID



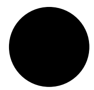
Q = conv(PM(G))

PM(G):= set of perfect matchings of G





projection P :
½ sum of colors ∈ 3 P

The lift up is 3/2 in 

Baum Trotter '77: A is TU $\Rightarrow$ $Q := \{y : yA \leq c\}$ is ID

Projections of Q are also ID (S. august 2008)

Proof : $Q \subseteq \mathbf{Z}^{n'}$, let $n < n'$ and

$$P := \{(y_1, \ldots, y_n) : y \in Q\}$$

Let $z \in kP$, z integer .

$z/k \in P$, so $\exists\, z'$: $(z/k, z') \in Q$, so $(z, kz') \in kQ$

kz' is a solution of $y'A' \leq b'$ $(= kc - zA_{(all\ ;\ 1,\ldots,n)})$

So **kz' can be chosen to be integer**. DONE

Kathie's and Jack's coflows are box TDI and Projection of TU . Cameron, Edmonds (1992)

Other applications ?

Generalizing Greene, Kleitman, Bessy-Thomasse I-II

(Gallai's conjecture, variant of Berge and Linials'conjectures)

Theorem (S. 2007)

max union of k stable sets

$\geq \min\{|X| + k|\mathcal{C}|$: $\mathcal{C}$ set of circuits

X = vertices uncovered by $\mathcal{C}$ }

Corollaries: k=1, and k = ∝ integer rounding

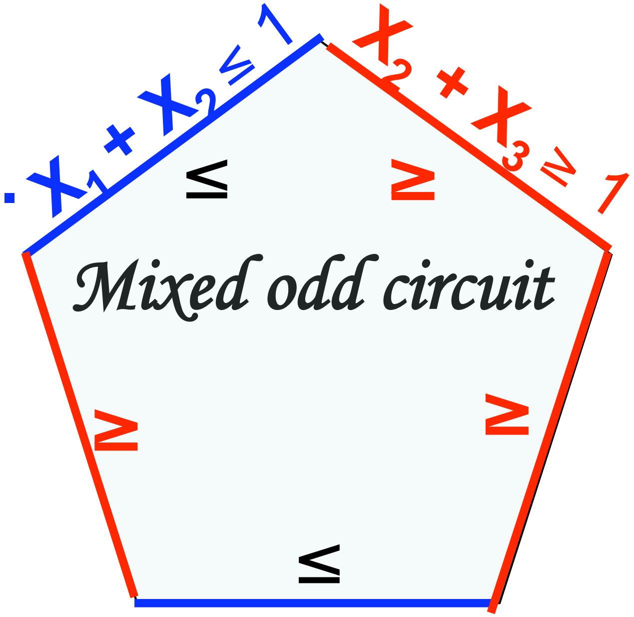
chromatic number = round up of …

1998

MINIMAL NONINTEGER 0-1

Minors : generalizing min-imp, min non-ideal

$A^{\leq}$, $A^{\geq}$ 0-1 matrices,n columns.



$$A^{\leq} x \leq 1$$
$$A^{\geq} x \geq 1$$
$$x \geq 0$$

Theorem: Minimal noninteger ⇔ 1 frac point :

- Minimal imperfect (vertex: constant 1/r vector)
- Minimal nonideal (constant 1/r or degen proj …)
- Mixed odd circuit

Divisibility Lemma

Let A, B: 0-1, A r-regular

$$AB = \begin{pmatrix} \mu_1 & 1 & 1 & \dots & 1 \\ 1 & \mu_2 & 1 & \dots & 1 \\ \dots & & \dots & & \dots \\ 1 & \dots & 1 & & \mu_n \end{pmatrix}$$

Then : either $\mu_1 = \dots = \mu_n =: \mu$

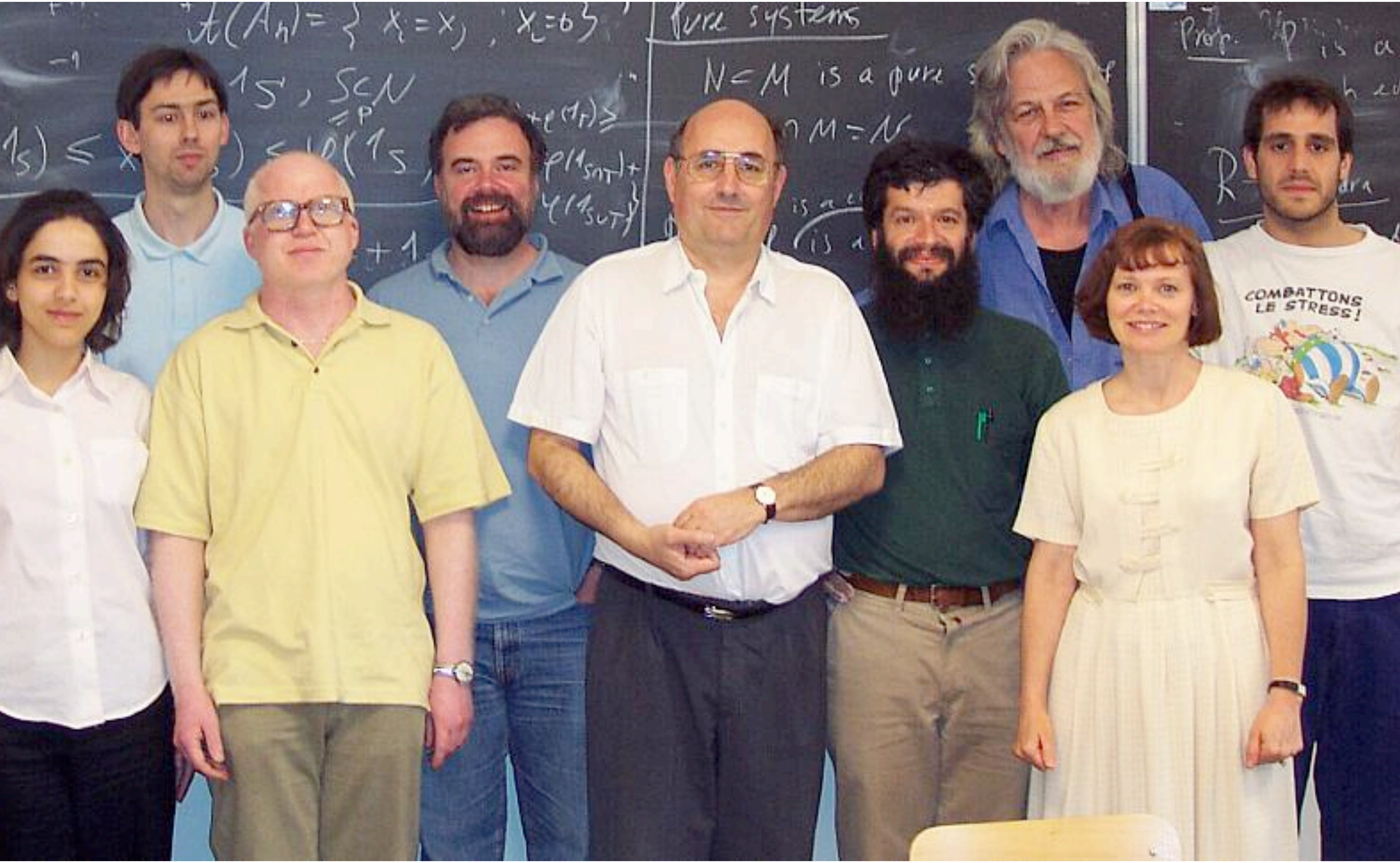
or $\{\mu_1, \mu_2, \dots, \mu_n\} = \{0, r\}$

Proof: $0 \le \mu_i \le r$: $r \mid \mu_i + n - 1 \in$ r+1 consec,

Proof of the Theorem: $\mu=0$; $\mu>1$; $r=2$

2008


FINE HALL


Pure systems
COMBATTONS LE STRESS!

Waterloo Folklore $\stackrel{?}{=}$ Jack's …

clothes

beard

oral teaching

$T \subseteq V(G)$. T-*join* : $F \subseteq E(G)$, where T=vertices of odd degree.

Jack's Chinese Postman Problem

G connected, $T \subseteq V(G)$ even $\Rightarrow \exists$ T-join.

Every T-join can be edge-partitioned into paths whose end-points partition T.

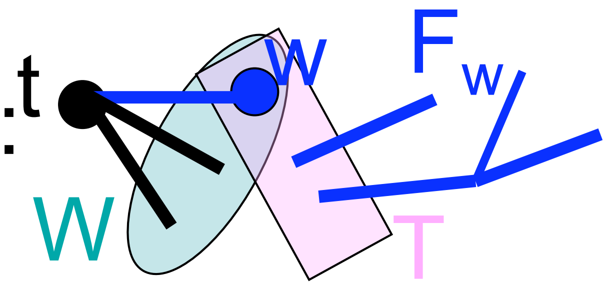
T-cut : $\{xy: x \in X, y \in Y\}$: $\{X,Y\}$ T-odd partition of V.

$\tau(G,T)=\min\{|F|: F \text{ T-join}\} \geq \nu(G,T)=\max$ T-cut-packing

Virtual Private Networks (VPN)

Goyal, Olver and Shepherd (2008) proved the 'VPN tree conjecture' through the following :

Theorem: $G = (V,E)$; $T, W \subseteq V$, T odd; $\forall w$:
F_w $T\Delta\{w\}$-join and edge-disjoint for different w.
Then $\exists$ $r \in V$ s.t. $\exists$ edge-disj r,w paths ($w \in W$).

Proof:


$V' = V \cup \{t\}$, $E' = E \cup \{tw: w \in W\}$,
$T' = T \cup \{t\}$. Apply to $\{t, V\}$:

Exercise: $\{X,Y\}$ min T-cut $\Rightarrow$ $\exists$ $x \in X \cap T$, $y \in X \cap T$ s.t. it is a min (x,y)-cut. (Hint:GH-tree ~Padberg-Rao, Rizzi's T-pairing.)

2008

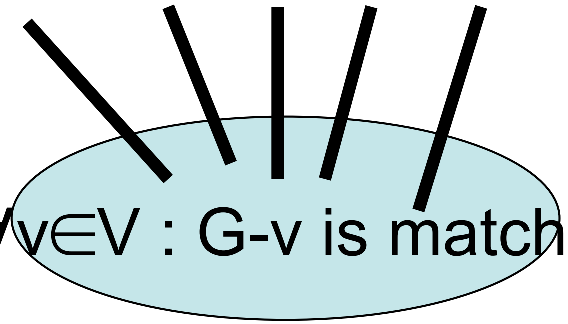
Seymour Graphs

G=(V,E) s.t. for all T : $\tau(G,T) = \nu(G,T)$ Eg bip; sp.

Conjecture S'89, Proof: Ageev, Kostochka, Szigeti '97.

Particular case: G planar for all F⊆E

Cut Condition ⇔ multiflow

Factorcritical-contraction  keeps it !

∀v∈V : G-v is matchable

∀*bicritical* : v∈V : G-v is factorcrit.

Theorem (Ageev, S., Szigeti) G Seymour ⇔

It can be fact.-contracted to a bicritical graph ≠ |

Open : **Recognition of Seymour graphs.**

1985

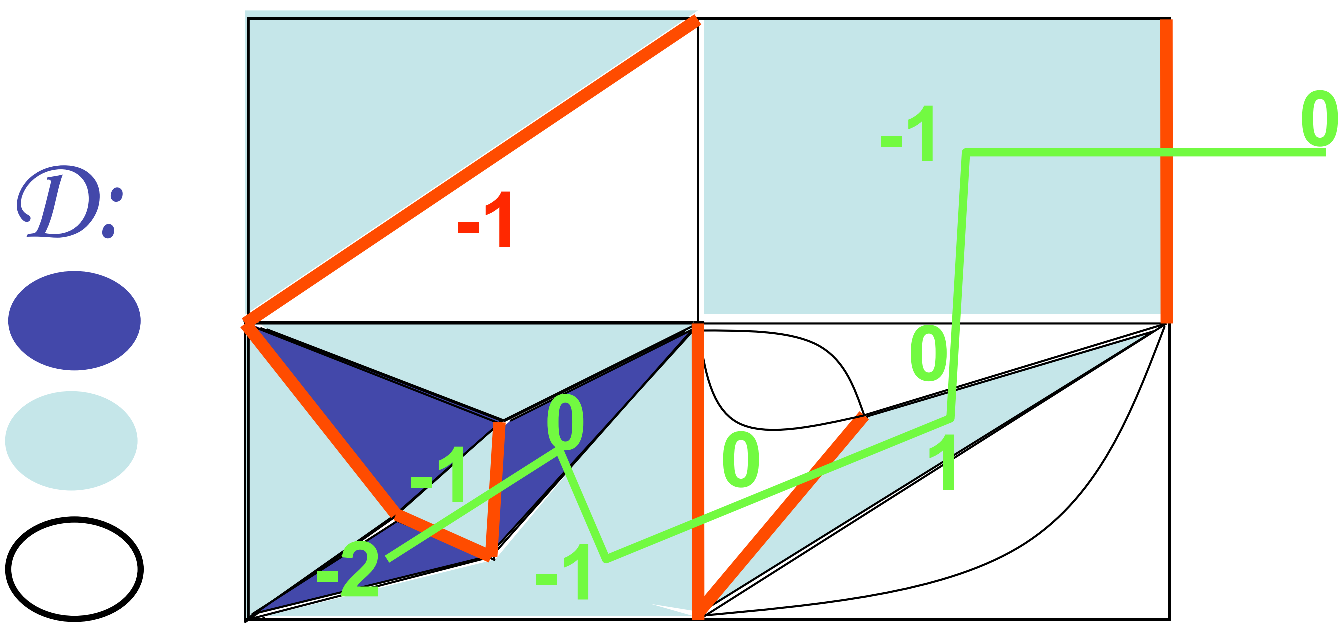
Edmonds-Johnson Theorem on T-joins

Quick proof of Seymour's theorem on T-joins.

Edmonds and Johnson's theorem :

G=(V,E) plane, ±1-weighted, no 'dual <0 circuit'

(cut condition) ⇒ ∃ fractional multiflow » .



1990 ?

- Allitas : Adott egy G ir▧▧tlan gr▧ W ponthalmaz (termin▧k) 鑊T p▧tlan ponthalmaz. Minden u \in W -re adott J_u ami T \Delta {u} -join gy, hogy ezek a J_u -k p▧nk鮴 囊iszjunktak. Ekkor l駅zik a gr▧an egy olyan v pont, amib▧▧inden W-beli pont el鯊et▧▧diszjunk utakon.

  Biz: Vegyunk fel egy plusz pontot, "t "-t, es kossuk ossze a W minden pontjaval, majd tegyuk bele T-be. Gt=(Vtn,Et) - vel illetve Tt-vel jeloljuk a kapott uj grafot es "T"-t.

  A feltetel szerint Gt-ben van k diszjunkt Tt-kotes, igy a minimalis Tt-vagas legalabb k elemu. Es akkor pontosan k elemu, mert \delta({t}) k-elemu Tt-vagas. Megmutatjuk, hogy *van olyan v \in V amitol "t"-t nem valasztja el k-nal kisebb vagas*. Ekkor kesz leszunk, mert akkor t es v kozott van k diszjunkt ut is (Menger) es ezek t-hez kapcsolodo elet elhagyva a v-bol a W kulonbozo pontjaiba mento k utat kapunk, vagyis egyet-egyet a W minden pontjaba. Sot, azt mutatjuk meg, hogy T-ben is van ilyen v pont:

- Lemma : G graf, T paros. Ha C min T-vagas akkor van olyan x, y\in T par amit C elvalaszt es C minimalis vagas x es y kozott.

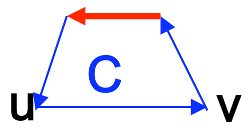
  Biz: Legyen F egy folyam-ekv fa es H a T-kotese, tehat C \cap H paratlan. *H eldiszjunkt utakra particionalhato, amelyek vegpontjai mind T-ben vannak.* Igy lesz olyan ut ami C-be paratlan sok ellel metsz, es igy az x, y vegpontjait a C elvalasztja, x,y\in T. KESZ

A particular-stable-set polytope

$\max \{1^T x : x(C) \le |E(C) \cap B| \ \forall \text{ cycle } C, \ x \ge 0\}$

Kathie and Jack : particular coflow polytope, so projection of TU, box-TDI, integer vertices. (Nicest proof for this case: by Pierre Charbit.)

Bessy-Thomassé-Knuth order ⇒ ∀ arc uv



$$x(u) + x(v) \le x(C) \le |E(C) \cap B| = 1$$

The integer vertices are stable sets.

Moreover the ID property (S. 2007 complicated).