Background

EXPECTED UTILITY AND ITS VIOLATIONS

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Abdellaoui Mohammed

1. Preliminaries

Representation of a Decision Problem under Uncertainty

- The DM is about making a choice from a set of possible actions;
- The consequence of any action is determined not just by the action itself but also by a number of external factors (beyond the control and unknown);
- These external factors are called states of the world. They are the carriers of uncertainty;
- The DM is assumed to have a complete description of these external factors through a set of states that are mutually exclusive and collectively exhaustive;
- A consequence results from the choice of a specific action and the occurrence of a specific state of the world.

Decision Table

		States of nature			
		s_1	s_2	...	s_n
Acts / Actions	f_1	x_{11}	x_{12}	...	x_{1n}
	f_2	x_{21}	x_{22}	...	x_{2n}
	.				
	.				
	.				
	f_m	x_{m1}	x_{m2}	...	x_{mn}

- The set of states of the world will be denoted by $\mathcal{S}$ (not necessarily finite);
- Subsets of $\mathcal{S}$ are called events; and an event A obtains if it contains the true state.
- The set of consequences is denoted by $\mathcal{X}$.
- f is constant if $f(\mathcal{S}) = \{x\}$ for some $x \in \mathcal{S}$; and f is simple if $f(\mathcal{S})$ is finite.
- Notation: $f_1 = (s_1: x_{11}; s_2: x_{12}; \ldots; s_n: x_{1n})$.

Risk versus Uncertainty

RISK

- The DM is in a context of decision under risk if the set of states of the world is exogenously given with a probability distribution P.

  - $\boldsymbol{f} \rightarrow \boldsymbol{P_f}$, where $\boldsymbol{P_f}$ is the probability distribution generated by act $\boldsymbol{f}$.
    If $\boldsymbol{f} = (\boldsymbol{E}: \boldsymbol{x}; \boldsymbol{S} - \boldsymbol{E}: \boldsymbol{y})$, then $\boldsymbol{P_f} = (\boldsymbol{p}: \boldsymbol{x}; \boldsymbol{1} - \boldsymbol{p}: \boldsymbol{y})$ with $\boldsymbol{p} = \boldsymbol{P}(\boldsymbol{E})$.
  - A <u>simple</u> act $\boldsymbol{f}$ such that $\boldsymbol{f}(\boldsymbol{S}) = \{\boldsymbol{x_1}, \dots, \boldsymbol{x_n}\}$ generates a <u>simple</u> probability distribution $\boldsymbol{P_f}$ satisfying $\boldsymbol{P_f}(\{\boldsymbol{x_1}, \dots, \boldsymbol{x_n}\}) = \boldsymbol{1}$. $\boldsymbol{P_f}$ is called a <u>simple lottery</u> (giving $\boldsymbol{x_i}$ with probability $\boldsymbol{p_i} = \boldsymbol{P_f}(\{\boldsymbol{x_i}\}), \boldsymbol{i} = \boldsymbol{1}, \dots, \boldsymbol{n}$).
  - We will assume that the set of alternatives is the set $\mathbb{P}_{\boldsymbol{X}}$ of simple probability distributions on $\boldsymbol{\mathcal{X}}$.

UNCERTAINTY (Subjective)

- Most uncertainties in decision making concern one-shot events for which no exogenously (objective) given probabilities are available.
- De Finetti (1931), Ramsey (1931), and Savage (1954) subsequently showed that probabilities can still be defined for one-shot events.
- They suggest inferring probabilities or degrees of belief from the DM's willingness to bet (on events).

  - **Example:**

    Event $\boldsymbol{A}$ will be considered as more likely than event $\boldsymbol{B}$ for the decision maker if she / he prefers act $\boldsymbol{f} = (\boldsymbol{A}: \mathbf{100€};\ \boldsymbol{S} - \boldsymbol{A}: \mathbf{0})$ to act $\boldsymbol{g} = (\boldsymbol{B}: \mathbf{100€};\ \boldsymbol{S} - \boldsymbol{B}: \mathbf{0})$.

2. Formal Representation of the DM Preferences

- The DM preferences and tastes are represented by means of a binary relation $\succcurlyeq$ on the set $\boldsymbol{E}$ of alternatives.

- $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ means that the DM weakly prefers object $\boldsymbol{x}$ to object $\boldsymbol{y}$; the DM holds $\boldsymbol{x}$ to be at least as good as $\boldsymbol{y}$.

Strict Preference	Indifference
• $\boldsymbol{x} \succ \boldsymbol{y}$ if $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{not}(\boldsymbol{y} \succcurlyeq \boldsymbol{x})$.	• $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{x}$.

Non-triviality

- $\boldsymbol{x} \succ \boldsymbol{y}$ for some $\boldsymbol{x}, \boldsymbol{y}$.

Weak Order

- $\succcurlyeq$ is a weak order if it is
  - transitive ($\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{z} \Rightarrow \boldsymbol{x} \succcurlyeq \boldsymbol{z}$) and
  - complete (for all $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ or $\boldsymbol{y} \succcurlyeq \boldsymbol{x}$ or both).

Numerical Representation

- $\boldsymbol{V}: \boldsymbol{E} \rightarrow \mathbb{R}$ represents $\succcurlyeq$ if : $\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Leftrightarrow \boldsymbol{V}(\boldsymbol{x}) \geq \boldsymbol{V}(\boldsymbol{y})$.

Observation

- If $\boldsymbol{V}$ represents $\succcurlyeq$ on $\boldsymbol{E}$, then $\succcurlyeq$ is a weak order.

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- If $\boldsymbol{V}$ represents $\succcurlyeq$ on E, then:
  - (i) $\boldsymbol{x} \succ \boldsymbol{y} \Leftrightarrow \boldsymbol{V}(\boldsymbol{x}) > V(y)$;
  - (ii) $\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{y})$.

Fundamental Properties

Assume that $\succcurlyeq$ is a weak order. Then:

a) $\succcurlyeq$ and $\sim$ are reflexive.

b) $\succ$ is transitive.

c) For no $\boldsymbol{x}$ and $\boldsymbol{y}$ we have $\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{y} \succ \boldsymbol{x}$ ($\succ$ is asymmetric).

d) $[\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{y} \succ \boldsymbol{z} \Rightarrow \boldsymbol{x} \succ \boldsymbol{z}]$ and $[\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{z} \Rightarrow \boldsymbol{x} \succ \boldsymbol{z}]$.

e) $\sim$ is an <u>equivalence relation</u>, i.e. reflexive, transitive and symmetric ($\boldsymbol{x} \sim \boldsymbol{y} \Rightarrow \boldsymbol{y} \sim \boldsymbol{x}$).

f) If $\boldsymbol{y} \sim \boldsymbol{x}$ then $\boldsymbol{y}$ is substitutable for $\boldsymbol{x}$ in every preference.

g) $\boldsymbol{x} \succ \boldsymbol{y} \Leftrightarrow \boldsymbol{not}(\boldsymbol{y} \succcurlyeq \boldsymbol{x})$.

3. Expected Utility with Known Probabilities

- Let $\boldsymbol{\mathcal{X}}$ be a set of outcomes and $\mathbb{P}_X$ the set of simple lotteries on $\boldsymbol{\mathcal{X}}$.

- $\succcurlyeq$ denotes the weak preference relation on $\mathbb{P}_X$. Strict preference and indifference are defined as usual.

- $\succcurlyeq$ satisfies <u>first order stochastic dominance</u> on $\mathbb{P}_X$ if for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{P}, \boldsymbol{P} \succ \boldsymbol{Q}$ whenever $\boldsymbol{P} \neq \boldsymbol{Q}$ and for all $\boldsymbol{x} \in \boldsymbol{X}, \boldsymbol{P}(\{\boldsymbol{y} \in \boldsymbol{\mathcal{X}}: \boldsymbol{y} \succcurlyeq \boldsymbol{x}\}) \geq \boldsymbol{Q}(\{\boldsymbol{y} \in \boldsymbol{\mathcal{X}}: \boldsymbol{y} \succcurlyeq \boldsymbol{x}\})$.

- For $\boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1}]$, the combination $\boldsymbol{\alpha P} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{Q}$ of lotteries $\boldsymbol{P}$ and $\boldsymbol{Q}$ is a lottery.

- $\boldsymbol{\alpha P} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{Q}$ can be interpreted as a compound (two-stage) prospect giving $\boldsymbol{P}$ with probability $\boldsymbol{\alpha}$ and $\boldsymbol{Q}$ with probability $\mathbf{1} - \boldsymbol{\alpha}$.

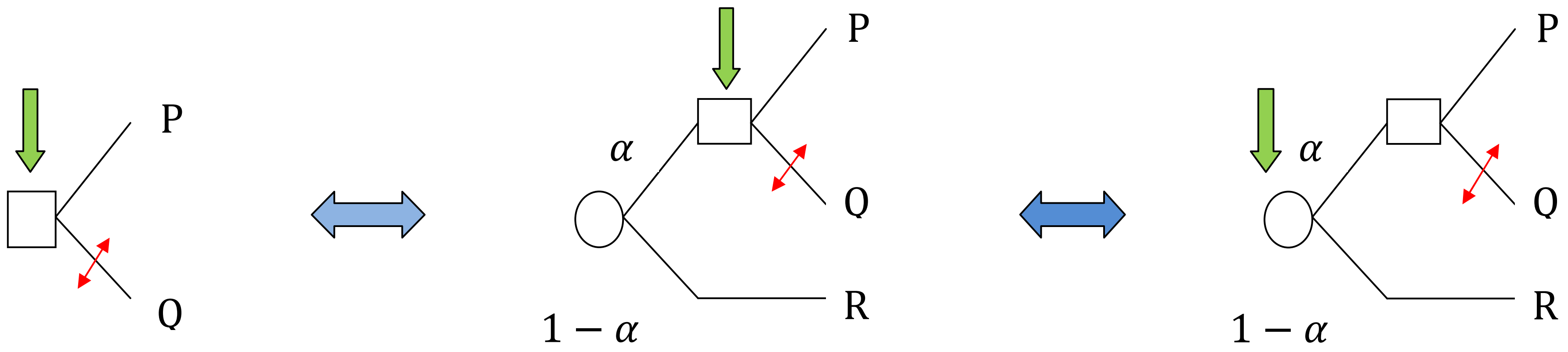
- $\succcurlyeq$ is <u>Jensen Continuous</u> if for all prospects $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathbb{P}$, **if** $\boldsymbol{P} \succ \boldsymbol{Q}$ then there exist $\boldsymbol{\lambda}, \boldsymbol{\mu} \in (\mathbf{0}, \mathbf{1})$ such that $\boldsymbol{\lambda P} + (\mathbf{1} - \boldsymbol{\lambda})\boldsymbol{R} \succ \boldsymbol{Q}$ and $\boldsymbol{P} \succ \boldsymbol{\mu R} + (\mathbf{1} - \boldsymbol{\mu})\boldsymbol{Q}$.

- The key axiom of Expected utility theory with known probabilities is called vNM-independence.

vNM-independence

- For all $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R} \in \mathbb{P}, \forall \boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{1})$: $\boldsymbol{P} \succcurlyeq \boldsymbol{Q} \Leftrightarrow \boldsymbol{\alpha P} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{R} \succcurlyeq \boldsymbol{\alpha Q} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{R}$.

- This axiom says that, if a decision maker has to choose between prospects $\boldsymbol{\alpha P} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{R}$ and $\boldsymbol{\alpha Q} + (\mathbf{1} - \boldsymbol{\alpha})\boldsymbol{R}$, her choice does not depend on the ‘common consequence’ $\boldsymbol{R}$.



The Expected Utility Theorem

- A Jensen-continuous weak order satisfying vNM-independence on the set $\mathbb{P}$ is necessary and sufficient for the existence of a utility function $\boldsymbol{u}: \boldsymbol{\mathcal{X}} \to \mathbb{R}$ such that

$$\forall \boldsymbol{P}, \boldsymbol{Q} \in \mathbb{P}, \boldsymbol{P} \succcurlyeq \boldsymbol{Q} \Leftrightarrow \boldsymbol{E}(\boldsymbol{u}, \boldsymbol{P}) \geq \boldsymbol{E}(\boldsymbol{u}, \boldsymbol{Q}),$$

where $\boldsymbol{E}(\boldsymbol{u}, \boldsymbol{R}) = \sum_{x \in X} \boldsymbol{r}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})$ for any prospect $\boldsymbol{R}$. $\boldsymbol{u}$ is unique up to a positive affine transformation (i.e. unique up to level and unit).

4. Expected Utility with Unknown Probabilities

- An *act* is a function from $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{\mathcal{X}}$, the set of outcomes. The set of acts is denoted by $\boldsymbol{\mathcal{A}}$.
- For outcome $\boldsymbol{x}$, event $\boldsymbol{A}$, and acts $\boldsymbol{f}$ and $\boldsymbol{g}$:
- $\boldsymbol{fAg}$ denotes the act resulting from $\boldsymbol{g}$ if all outcomes $\boldsymbol{g}(\boldsymbol{s})$ on A are replaced by the corresponding outcomes $\boldsymbol{f}(\boldsymbol{s})$ (by consequence $\boldsymbol{x}$).
- $\boldsymbol{xAg}$ denotes the act resulting from $\boldsymbol{g}$ if all outcomes $\boldsymbol{g}(\boldsymbol{s})$ on $\boldsymbol{A}$ are replaced by consequence $\boldsymbol{x}$.
- $\boldsymbol{xAy}$ denotes the act giving consequence $\boldsymbol{x}$ if $\boldsymbol{A}$, and consequence $\boldsymbol{y}$ otherwise.
- The set of simple acts $\boldsymbol{\mathcal{A}}$ is provided with a (non-trivial) weak order $\succcurlyeq$.
- The preference relation on acts is extended to the set of consequences by the means of constant acts.
- An event $\boldsymbol{A}$ is said to be *null* if the decision maker is indifferent between any pair of acts differing only on $\boldsymbol{A}$.

- Small event Continuity Axiom: For any non-indifferent acts ($\boldsymbol{f} \succ \boldsymbol{g}$), and any outcome ($\boldsymbol{x}$), the state space can be (finitely) partitioned into events ($\{\boldsymbol{A_1}, \dots, \boldsymbol{A_n}\}$) small enough so that changing either act to equal this outcome over one of these events keeps the initial indifference unchanged ($\boldsymbol{xA_if} \succ \boldsymbol{g}$ and $\boldsymbol{f} \succ \boldsymbol{xA_jg}$ for all $\boldsymbol{i}, \boldsymbol{j} \in \{\boldsymbol{1}, \dots, \boldsymbol{n}\}$).

Sure-thing Principle

- For all events $\boldsymbol{A}$ and acts $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ and $\boldsymbol{h}'$, $\boldsymbol{fAh} \succcurlyeq \boldsymbol{gAh} \Leftrightarrow \boldsymbol{fAh'} \succcurlyeq \boldsymbol{gAh'}$.

- The sure-thing principle (Axiom P2) states that if two acts $\boldsymbol{f}$ and $\boldsymbol{g}$ have a common part over ($\boldsymbol{S} - \boldsymbol{A}$), then the ranking of these acts will not depend on what this common part is.

Eventwise Monotonicity

- For all non-null events $\boldsymbol{A}$, and outcomes $\boldsymbol{x}, \boldsymbol{y}$ and acts $\boldsymbol{f}$,
- $\boldsymbol{xAf} \succcurlyeq \boldsymbol{yAf} \Leftrightarrow \boldsymbol{x} \succcurlyeq \boldsymbol{y}$.

Likelihood Consistency

- For all events $\boldsymbol{A}, \boldsymbol{B}$ and outcomes $\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{x}' \succ \boldsymbol{y}'$,

  $\boldsymbol{xAy} \succcurlyeq \boldsymbol{xBy} \Leftrightarrow \boldsymbol{x}'\boldsymbol{A}\boldsymbol{y}' \succcurlyeq \boldsymbol{x}'\boldsymbol{B}\boldsymbol{y}'$.
- Likelihood consistency (axiom P4) states that the revealed likelihood binary relation $\succcurlyeq^{\star}$ (read 'weakly more likely than') defined over events by

$$\boldsymbol{A} \succcurlyeq^{\star} \boldsymbol{B} \text{ if for some } \boldsymbol{x} \succ \boldsymbol{y}, \boldsymbol{xAy} \succcurlyeq \boldsymbol{xBy}$$

is independent of the specific outcomes $\boldsymbol{x}, \boldsymbol{y}$ used.

- The likelihood relation $\succcurlyeq^{\star}$, represents the DM beliefs.

Savage's Subjective Expected Utility

Subjective Probabilities from Preferences

Savage axioms (P1 to P6) are sufficient for the existence of a unique subjective probability measure $P^\star$ on $2^{\mathcal{S}}$, preserving likelihood rankings

$$\boldsymbol{A} \succcurlyeq^\star \boldsymbol{B} \Leftrightarrow \boldsymbol{P}^\star(\boldsymbol{A}) \geq \boldsymbol{P}^\star(\boldsymbol{B}),$$

and satisfying convex-rangeness

$$\boldsymbol{A} \subset \boldsymbol{\mathcal{S}}, \boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1}] \Rightarrow (\boldsymbol{P}^\star(\boldsymbol{B}) = \boldsymbol{\alpha}\boldsymbol{P}^\star(\boldsymbol{A}) \text{ for some } \boldsymbol{A} \subset \boldsymbol{B}).$$

Savage's Theorem

Under Savage's axioms (P1 to P6), there exists a vNM utility function on $\boldsymbol{\mathcal{X}}$ such that the decision maker ranks simple acts $\boldsymbol{f}$ on the basis of $\boldsymbol{E}(\boldsymbol{P_f}, \boldsymbol{u})$.

5. Violations of Expected Utility

Three important Experimental Results

1. The Allais Paradox
2. The Ellsberg Paradox
3. The Fourfold Pattern of Risk Attitudes

The Allais Paradox

	Probabilities		
	$p = 0.01$	$p = 0.01$	$p = 0.89$
A	\$1M	\$1M	\$1M
B	0	\$5M	\$1M
A’	\$1M	\$1M	0
B’	0	\$5M	0

The most frequent choice pattern is $\boldsymbol{AB'}$.

- Let $\boldsymbol{C} = \left(\frac{\mathbf{10}}{\mathbf{11}}\right)\$\mathbf{5}\boldsymbol{M} + \left(\frac{\mathbf{1}}{\mathbf{11}}\right)\mathbf{0}$ and $\boldsymbol{D} = \mathbf{0}$ two lotteries. We have

  $\boldsymbol{A} = \mathbf{0.11}\boldsymbol{A} + \mathbf{0.89}\boldsymbol{A}$ and $\boldsymbol{B} = \mathbf{0.11}\boldsymbol{C} + \mathbf{0.89}\boldsymbol{A}$

  $\boldsymbol{A'} = \mathbf{0.11}\boldsymbol{A} + \mathbf{0.89}\boldsymbol{D}$ and $\boldsymbol{B'} = \mathbf{0.11}\boldsymbol{C} + \mathbf{0.89}\boldsymbol{D}$.

The Allais Paradox

	States ($S = A \cup B \cup C$)		
	A	B	C
$\boldsymbol{f_{A \cup B} h}$	\$1M	\$1M	\$1M
$\boldsymbol{g_{A \cup B} h}$	0	\$5M	\$1M
$\boldsymbol{f_{A \cup B} h'}$	\$1M	\$1M	0
$\boldsymbol{g_{A \cup B} h'}$	0	\$5M	0

- $\boldsymbol{f_{A \cup B} h \succ g_{A \cup B} h}$ and $\boldsymbol{g_{A \cup B} h' \succ f_{A \cup B} h'}$ violate the sure-thing principle.

The Ellsberg Paradox

	30 balls	60 balls	
	Red	Black	Yellow
f	$1000	0	0
g	0	$1000	0
f'	$1000	0	$1000
g'	0	$1000	$1000

- Ellsberg claimed that many reasonable people will exhibit the choice pattern $\boldsymbol{f}\boldsymbol{g}'$. He suggested that preferring $\boldsymbol{f}$ to $\boldsymbol{g}$ is motivated by ambiguity aversion: the DM has more precise knowledge of the probability of the 'winning event' in act $\boldsymbol{f}$ than in act $\boldsymbol{g}$.

The Ellsberg Paradox

	30 balls	60 balls	
	Red	Black	Yellow
f	\$1000	0	0
g	0	\$1000	0
f'	\$1000	0	\$1000
g'	0	\$1000	\$1000

- In the second choice situation, the choice of act $\boldsymbol{g'}$ can be explained by the absence of precise knowledge regarding the probability of event **Y.**

The Ellsberg Paradox

	30 balls	60 balls	
	Red	Black	Yellow
f	$1000	0	0
g	0	$1000	0
f'	$1000	0	$1000
g'	0	$1000	$1000

- In terms of likelihood relation $\succcurlyeq^{\star}$, it can easily be shown that, under expected utility, the choice pattern $\boldsymbol{f g'}$ implies two contradictory likelihood statements, namely $\boldsymbol{R} \succ^{\star} \boldsymbol{B}$ and $\boldsymbol{R} \cup \boldsymbol{Y} \succ^{\star} \boldsymbol{B} \cup \boldsymbol{Y}$.