A Genuine Rank-dependent Generalization of von Neumann-Morgenstern Expected Utility

Mohammed Abdellaoui

CNRS

France

Outline

I. Review and Motivation
II. Probability Tradeoffs
III. Probability Tradeoff Consistency
IV. Cumulative Prospect Theory
V. Concluding Remarks

I. Review and Motivation

Experimental Findings

Preston & Baratta (1948), ***AJP***

The authors explored whether subjects accounted for chance events at their true mathematical probabilities.

Edwards (1954), ***PR***

The author observed that subject's bets revealed preferences among probabilities.

I. Review and Motivation

The First Models

Handa (1977), *JPE*

$$\sum_i w(p_i)u(x_i)$$

$\Rightarrow$ Violation of First Order Stochastic Dominance

I. Review and Motivation

Kahneman & Tversky (1979)

Prospect Theory

The Fourfold Patern of Risk Attitudes cannot be explained by the utility function for money

	GAINS	LOSSES
Low Probability	Risk seeking	Risk aversion
High Probability	Risk aversion	Risk seeking

I. Review and Motivation

Quiggin (1982)

Rank-dependent EU Theory

An enhanced version of prospect theory avoiding violations of FSD.

It takes into account an additional behavioral rule:

The attention given to an outcome depends not only on the probability of the outcome but also on the favorability of the outcome in comparison to the other outcomes .

I. Review and Motivation

Illustration

Assume that a decision maker is a pessimist and evaluates the lottery

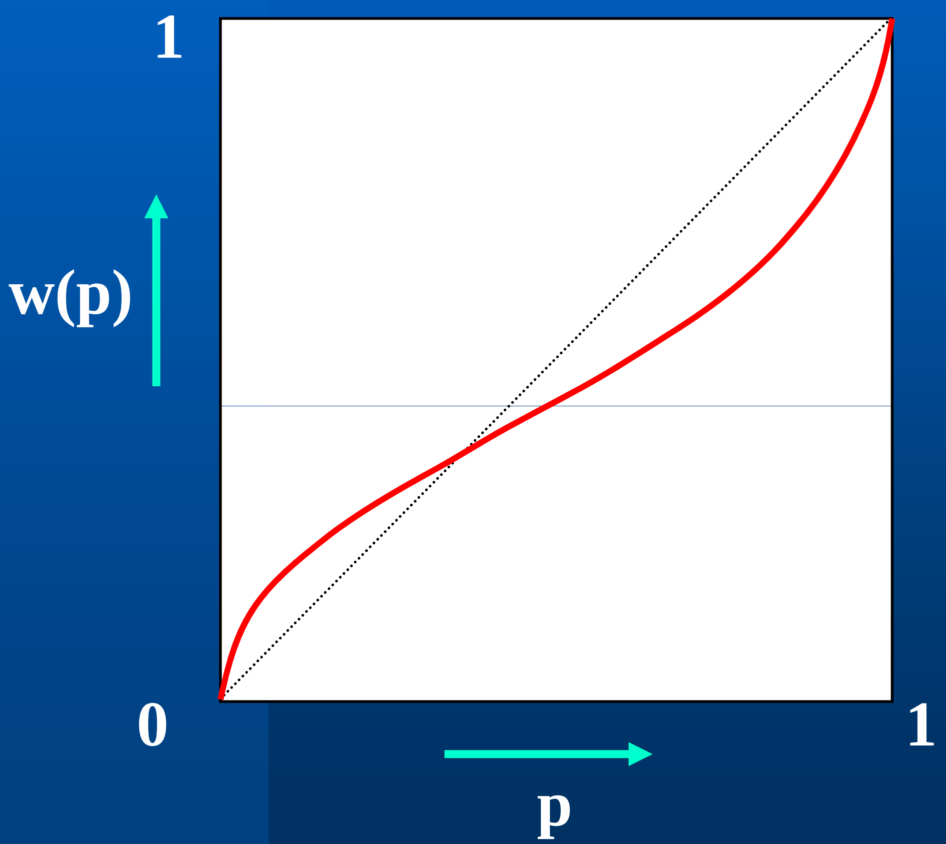
(30$, 1/3 ; 20$, 1/3 ; 10$, 1/3).

The DM will pay more attention (π_3) than 1/3 to the worst outcome 10. Say that the decision weight for outcome 10 is $\pi_3=1/2$.

The DM, accordingly, pays relatively less attention to the other outcomes ($\pi_1 + \pi_2=1/2$). Being a pessimist, he will pay more than helf of the remaining attention to outcome 20; say $\pi_2=1/3$. The remainder of the attention, devoted to the best outcome, is small (1/6).

I. Review and Motivation

The Common Probability Weighting Function



- Tversky & Kahneman (1992), *JRU*

- Wu & Gonzalez (1996), *Management Science*

- Bleichrodt & Pinto (2000)
- Abdellaoui (2000) *Management Science*

I. Review and Motivation

Existing Axiomatizations of RDEU

- Rank-dependent Expected Utility

$$P = (p_1 : x_1; \dots; p_n : x_n) \quad with \quad x_n \succcurlyeq x_{n-1} \succcurlyeq \dots \succcurlyeq x_1$$

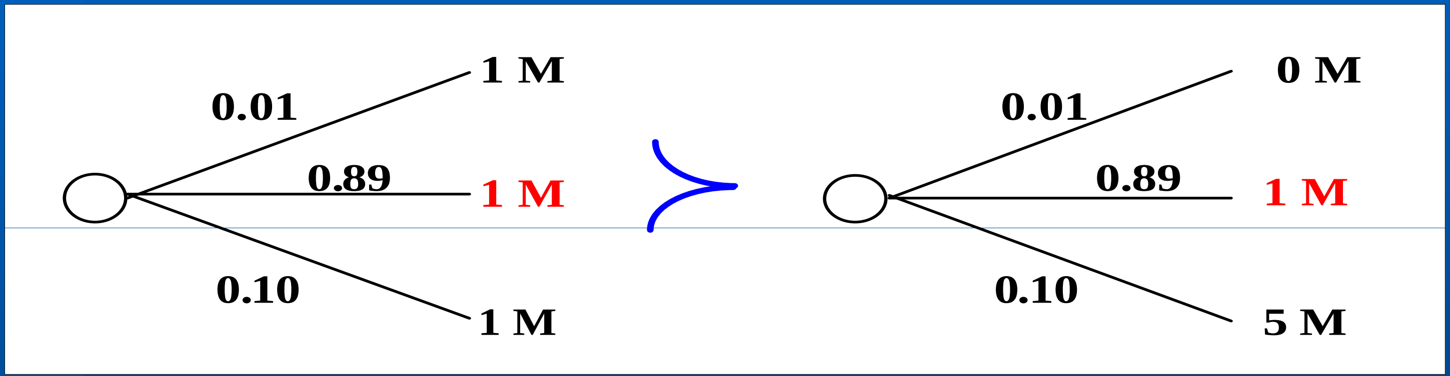
$$RDU(P) = \sum_i \pi_i u(x_i)$$

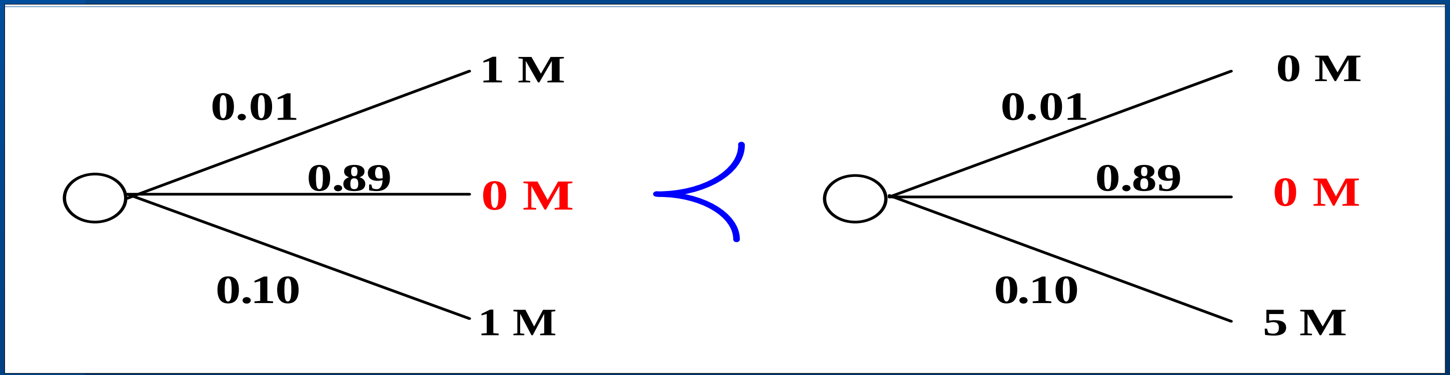
$$\pi_i = w(p_n + \dots + p_i) - w(p_n + \dots + p_{i-1})$$

- Outcome-oriented Axiomatizations of RDEU: Quiggin (1982), Segal (1989), Chew (1989), Wakker (1994), Chateauneuf (1995), Nakamura (1995)
- No obvious link with the von Neumann Morgenstern axiomatic set-up.

I. Review and Motivation

RDEU and the Allais Paradox





II. Probability Tradeoffs

Preliminaries

Measuring Utility from Mixtures

M is a mixture set

$$\forall \alpha \in [0,1], \forall x, y \in M : \alpha x + (1-\alpha) y \in M$$

x, y, z are three consequences in M such that

If $x \succ y \succ z$

Then …

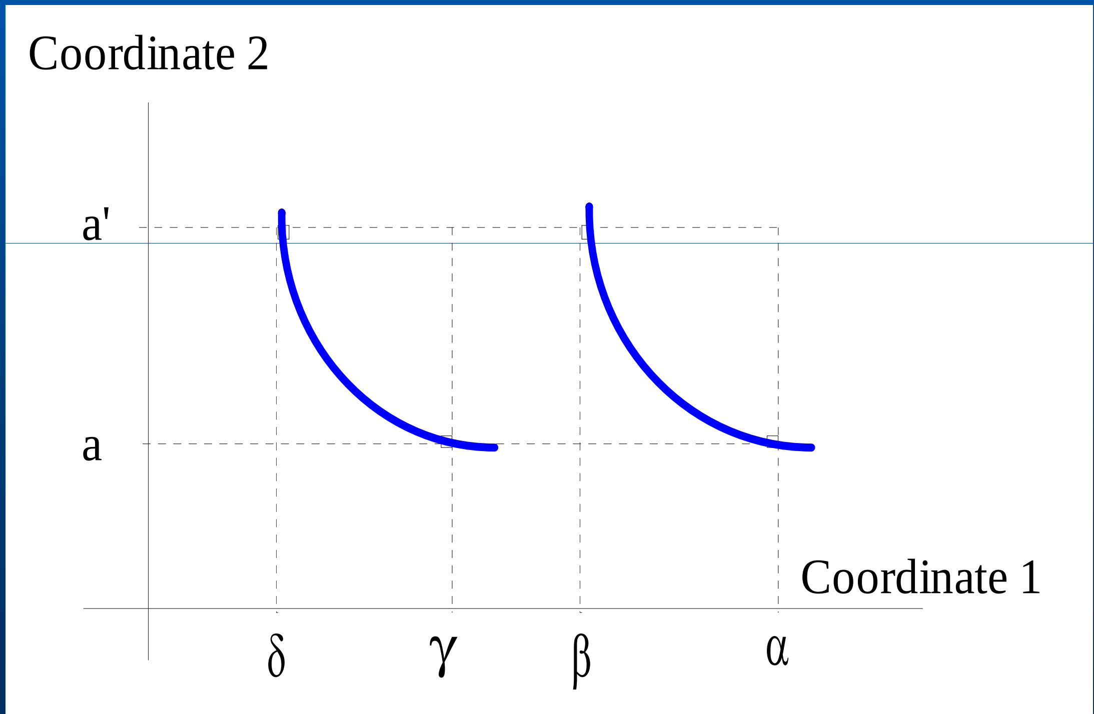
$$y \succ \tfrac{1}{2} x + \tfrac{1}{2} z$$

II. Probability Tradeoffs

Measuring Utility from Tradeoffs

Let $D = X^2$ a set of alternatives and $\precsim$ a preference relation.

$\sim$: indifference and $\prec$: strict preference



II. Probability Tradeoffs

Revealed Tradeoffs

$$\left\{\begin{matrix}(\gamma, a) \sim (\delta, a') \\ (\alpha, a) \sim (\beta, a')\end{matrix}\right\} \Rightarrow^{def} [\alpha\beta] \sim^{t} [\gamma\delta]$$

$$\left\{\begin{matrix}(\gamma, a) \preccurlyeq (\delta, a') \\ (\alpha, a) \succcurlyeq (\beta, a')\end{matrix}\right\} \Rightarrow^{def} [\alpha\beta] \succ^{t} [\gamma\delta]$$

II. Probability Tradeoffs

$$x_n \geqslant x_{n-1} \geqslant \ldots \geqslant x_1$$

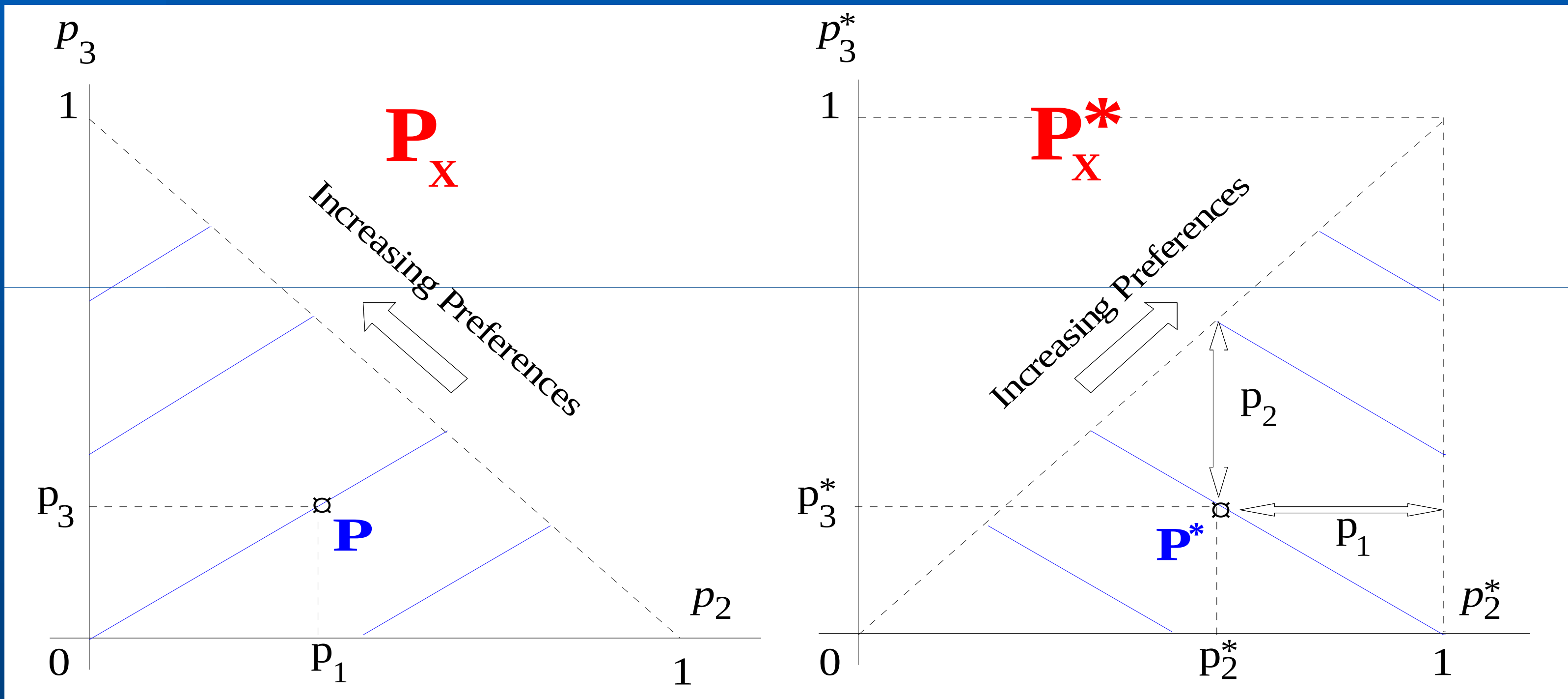
$$p_i^* = \sum_{j=i}^{n} p_j$$

Probability of receiving x_i or any better outcome.

II. Probability Tradeoffs

Probability/Rank-ordered Triangle

$$x_3 \succcurlyeq x_2 \succcurlyeq x_1 \qquad P = (p_1, p_3) \qquad P^* = (p_2^*, p_3^*)$$



II. Probability Tradeoffs

DEFINITION

For probabilities α, β, γ, δ we write $[\alpha\beta] \succcurlyeq_t [\gamma\, \delta]$ if

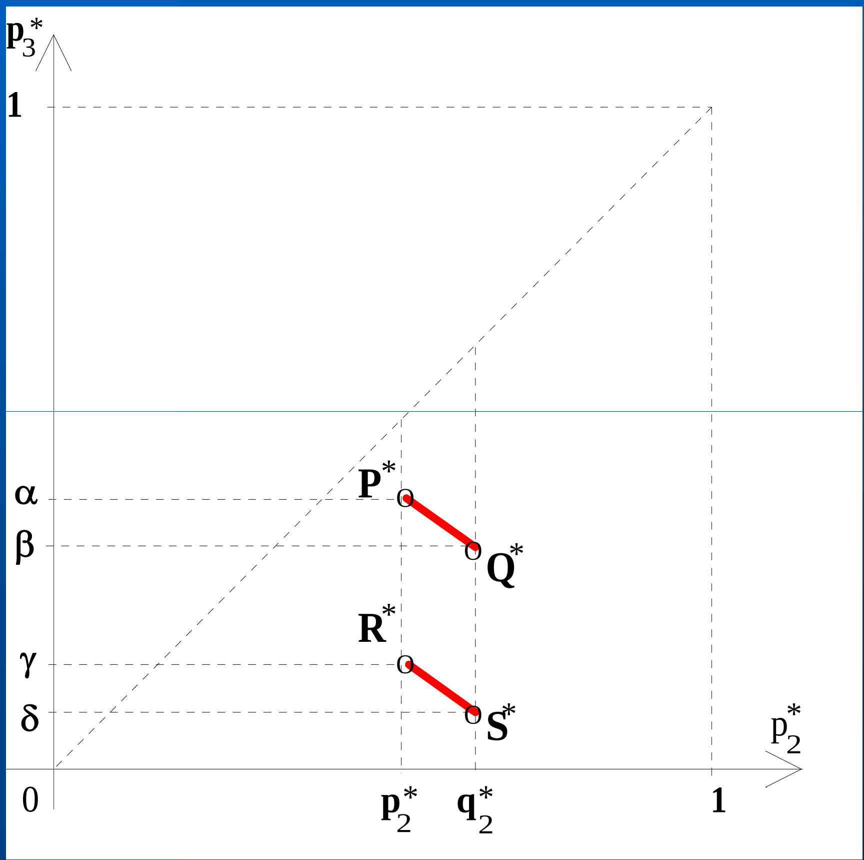
$(\alpha, P^*_{-i}) \succcurlyeq (\beta, Q^*_{-i})$ and

$(\gamma, P^*_{-i}) \preccurlyeq (\delta, Q^*_{-i})$

for some rank-ordered set $\{x_1, \ldots, x_n\}$ and $I \in \{2, \ldots, n\}$ such that $x_i \succ x_{i-1}$ and P^*, $Q^* \in \mathbb{P}$.

II. Probability Tradeoffs

Derived Probability Tradeoffs



- **OBSERVATION:**

Under EU we have

$$[\alpha\beta] \sim^{t} [\gamma\delta]$$

$$\Downarrow$$

$$\alpha - \beta = \gamma - \delta$$

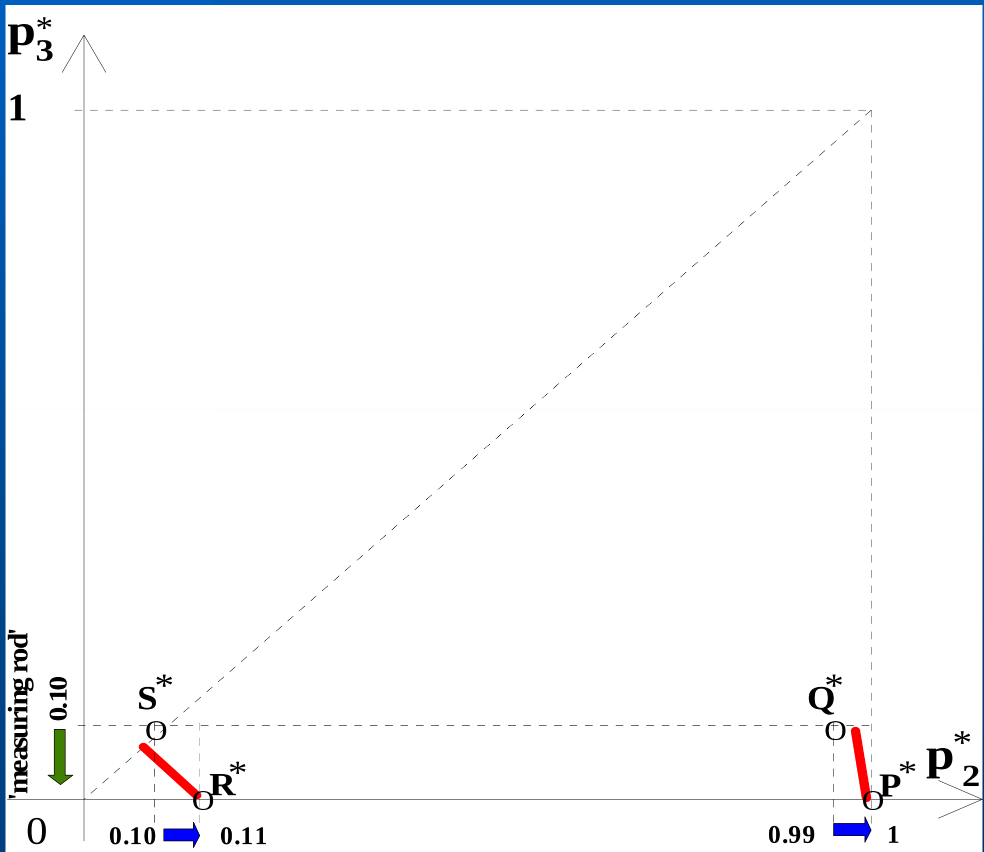
II. Probability Tradeoffs

Allais Paradox: $x_1 = 0$, $x_2 = 1M$, $x_3 = 5M$

	Alternatives in P	Alternatives in P*
Problem 1	P = (0, 1, 0) Q = (0.01, 0.89, 0.1)	P*=(1,0) Q* = (0.99, 0.10)
Problem 2	R = (0.89, 0.11, 0) S = (0.90, 0,0.10)	R* = (0.11, 0.10) S* = (0.10, 0.10)

II. Probability Tradeoffs

Allais Paradox (once again!)

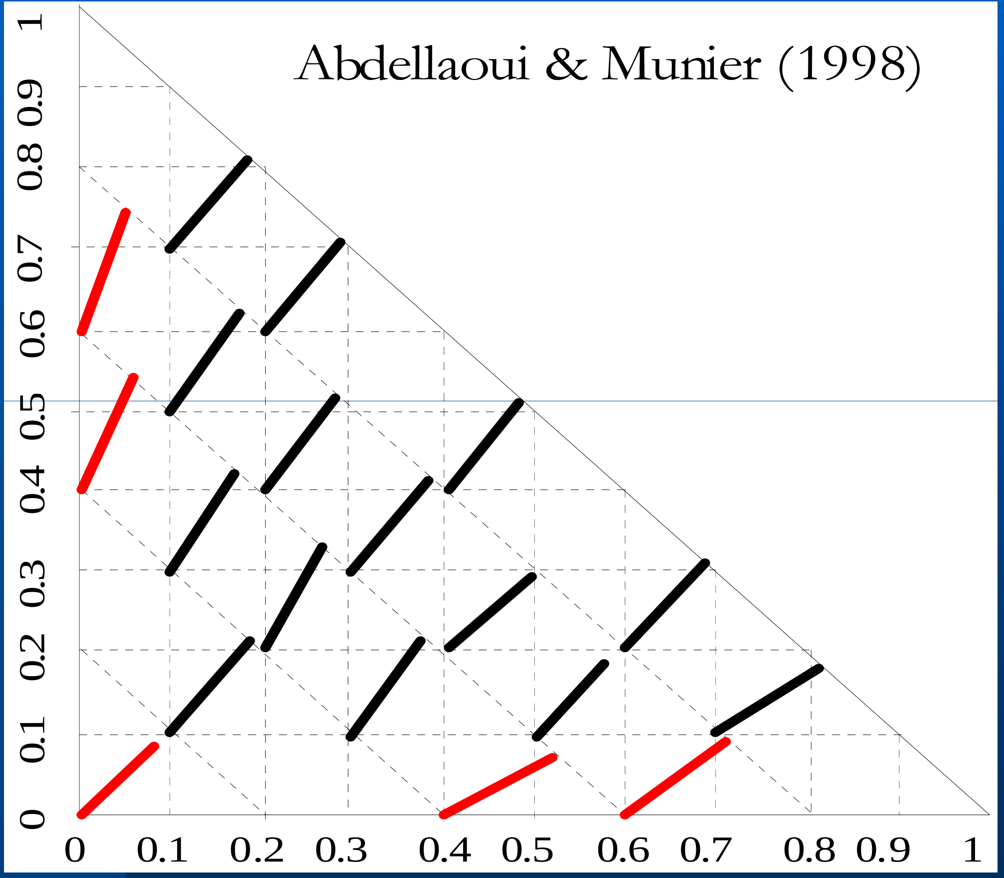


- **CONCLUSION:**

$$[1;0.99] \succ^t [0.11;0.10]$$

II. Probability Tradeoffs

Other Experimental Findings



III. Tradeoff Consistency

Rank-dependent Expected Utility

$$EU(P) = u_1 + [u_2 - u_1]p_2^* + [u_3 - u_2]p_3^*$$

$$\downarrow$$

$$RDU(P) = u_1 + [u_2 - u_1]w(p_2^*) + [u_3 - u_2]w(p_3^*)$$

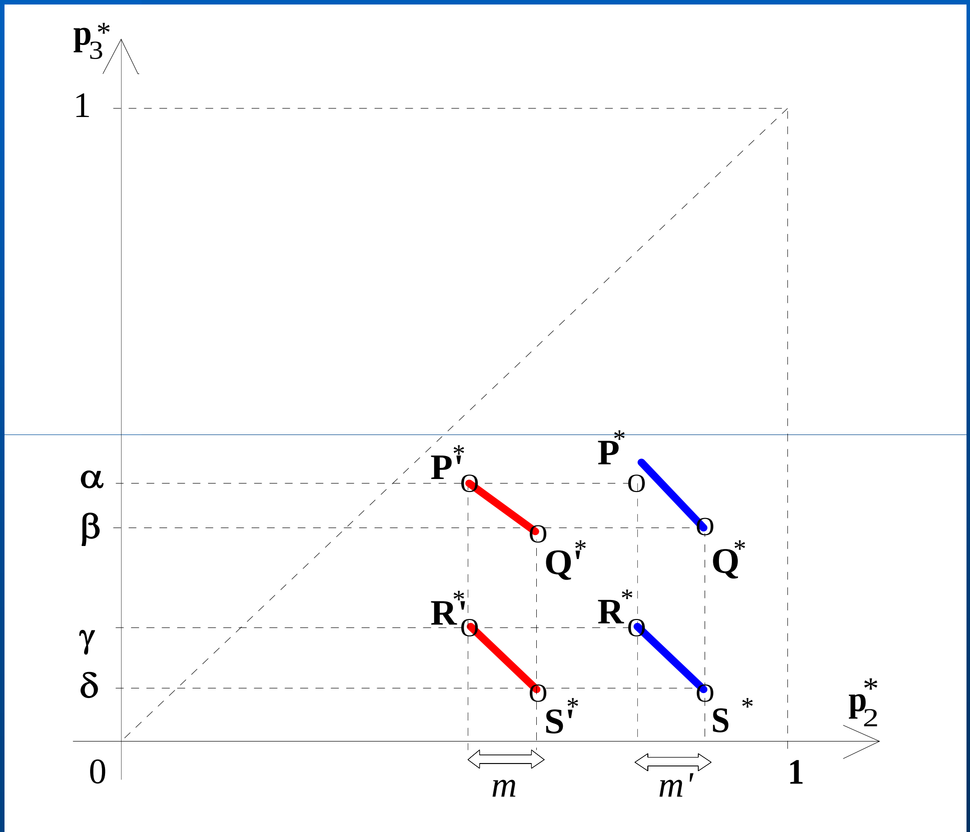
LEMMA: Under RDEU

$$[\alpha\beta] \sim^t [\gamma\delta] \Rightarrow w(\alpha) - w(\beta) = w(\gamma) - w(\delta)$$

$$[\alpha\beta] \succ^t [\gamma\delta] \Rightarrow w(\alpha) - w(\beta) > w(\gamma) - w(\delta)$$

III. Tradeoff Consistency

Probability Tradeoff Consistency



III. Tradeoff Consistency

MAIN THEOREM

Let $\succcurlyeq$ be a preference relation on $\mathbb{P}$. Then the following two statements are equivalent:

(i) RDU holds on $\mathbb{P}$;

(ii) The following conditions are satisfied

a. $\succcurlyeq$ is a weak order on $\mathbb{P}$;

b. $\succcurlyeq$ satisfies FSD;

c. $\succcurlyeq$ is Jensen continuous;

d. $\succcurlyeq$ satisfies tradeoff consistency.

III. Tradeoff Consistency

PROPOSITION

Let $\succcurlyeq$ be a vNM-independent weak order on $\mathbb{P}$.
Then:

(i) $\succcurlyeq$ satisfies FSD;

(ii) $[\alpha\ \beta] \succcurlyeq^t [\gamma\ \delta] \Rightarrow w(\alpha) - w(\beta) \geq w(\gamma) - w(\delta)$;

(iii) $[\alpha\ \beta] \succ^t [\gamma\ \delta] \Rightarrow w(\alpha) - w(\beta) > w(\gamma) - w(\delta)$;

(iv) $[\alpha\ \beta] \sim^t [\gamma\ \delta] \Rightarrow w(\alpha) - w(\beta) = w(\gamma) - w(\delta)$;

(v) $\succcurlyeq$ satisfies tradeoff consistency.

III. TradeoffConsistency

COROLLARY (vNM Theorem, 1944)

Let $\succcurlyeq$ be a preference relation on $\mathbb{P}$. Expected Utility holds if and only if:

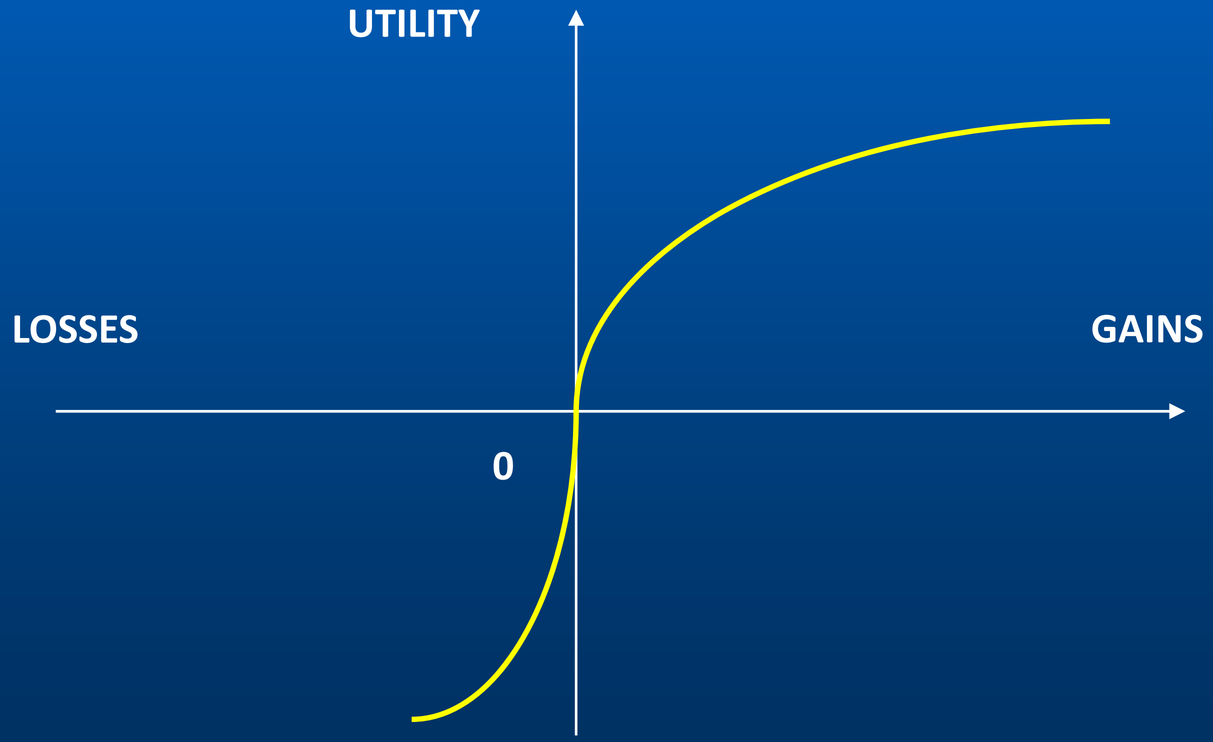
(i) $\succcurlyeq$ is a weak order;

(ii) $\succcurlyeq$ is Jensen continuous;

(iii) $\succcurlyeq$ is vNM-independent.

IV. Cumulative Prospect Theory

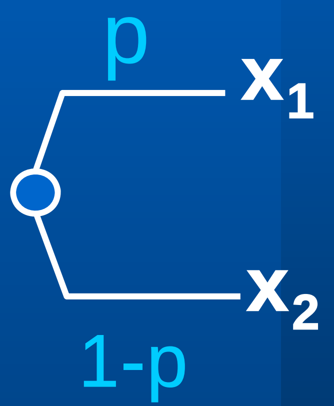
Utility under CPT (Diminishing sensitivity)



IV. Cumulative Prospect Theory

- Utility function for gains and losses;
- Utility is concave for gains and convex for losses with u(0)=0;
- Utility steeper forlosses than for gains (near 0);
- Two probability weighting functions: gainsand losses.

IV. Cumulative Prospect Theory



$x_1 \geq x_2 \geq 0 \quad \mapsto \quad w^+(p)U(x_1) + (1-w^+(p))U(x_2)$

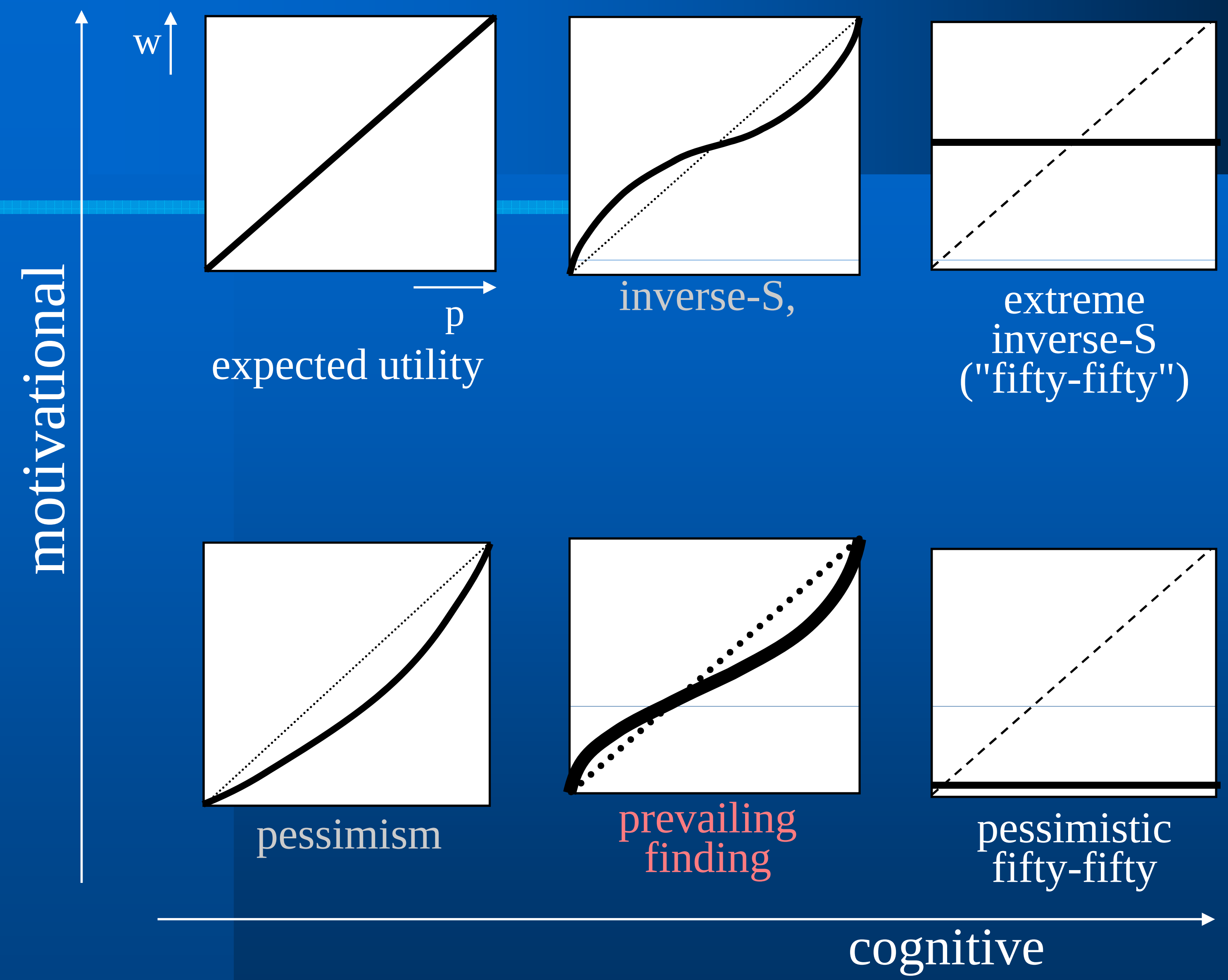
Gains

$0 \geq x_2 \geq x_1 \quad \mapsto \quad w^-(p)U(x_1) + (1-w^-(p))U(x_2)$

Losses

$x_1 > 0 > x_2 \quad \mapsto \quad w^+(p)U(x_1) + (w^-(1-p))U(x_2)$

Mixed



Abdellaoui (2000); Bleichrodt & Pinto (2000); Gonzalez & Wu 1999; Tversky & Fox, 1997.

IV. Probabilistic Risk Attitude

« Pratt-Arrow » for Probabilistic Risk

THEOREM

Suppose that RDEU holds for $\succcurlyeq_i$, w_i, u_i, $i = 1, 2$. Then the following two statements are equivalent :

(i) $w_2 = \varphi \circ w_1$ for a continuous, convex (respectively concave), strictly increasing $\varphi : [0,1] \to [0,1]$;

(ii) $\succcurlyeq_2$ is more averse (respectively prone) to probabilistic risk than $\succcurlyeq_1$.

Definition

$\succcurlyeq$ exhibits *probabilistic risk aversion* if [illegible] is excluded for all probabilities considered with [illegible] 0.

IV. Probabilistic Risk Attitude

COROLLARY

Under RDEU, *w* is convex (concave, linear) if and only if $\succcurlyeq$ exhibits probabilistic risk aversion (proneness, both aversion and proneness).

V. Concluding Remarks

Contributions:

This paper provides the first genuine generalization of the vNM EU theorem.

Its techniques are immediately directed towards the non-linear processing of probabilities.

Its techniques allow for straightforward testing and elicitation of RDEU.