Uniform Sources of Uncertainty for Subjective Probabilities and Ambiguity

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(joint with Aurélien Baillon and Peter Wakker)

“Informal”

- Central in this work will be the recent finding of the home bias, with investors preferring domestic stocks to foreign stocks in ways beyond uncertainty or utility.
- Here decisions depend on the source of uncertainty (Fox & Tversky, 1995)

Introduction

- Since Keynes (1921) and Knight (1921), economist has argued that in most situations of interest to economics, uncertainty concern one-shot events.
- We owe to de Finetti (1931) and Ramsey (1931) that subjective probabilities can be defined for one-shot events from choice.
- Savage formalized the work of de Finetti and Ramsey providing a *subjective theory of rational choice / probability*.

Introduction

- Two main criticisms against SEU MSOffice1
  - Allais paradox

Table 1: Allais paradox

	100 Balls		
	# 1	#2 - #11	#12 - #100
A	$1M	$1M	$1M
B	0	$5M	$1M
A'	$1M	$1M	0
B'	0	$5M	0

Diapositive 4

MSOffice1 I claimed here that this violation was due to "absolute non-ambiguity pronness".

Abdellaoui Mohammed; 03/04/2007

Introduction

- Ellsberg (three-color) paradox

Table 2: Ellsberg (three-color) paradox

	30 balls	60 balls	
	Red	**Black**	**Yellow**
→ ***f***	**$1000**	**0**	**0**
g	**0**	**$1000**	**0**
f'	**$1000**	**0**	**$1000**
→ ***g'***	**0**	**$1000**	**$1000**

Introduction

MSOffice2

- As a reaction to these criticisms
  - Generalizations of Expected Utility
    - CEU (Gilboa, 1987; Schmeidler 1989)
    - MEU (Gilboa & Schmeidler 1989)
    - CPT (Tversky & Kahneman 1992)

    The 'common denominator' of these models is a biseparable preference model (Ghirardato & Marinacci 2001).
  - 'Liberation' of subjective probability theory from Specific Preference Functionals (Probabilistic Sophistication)
    - Machina & Schmeidler (1992)
    - Chew & Sagi (2006)

Diapositive 6

MSOffice2 I claimed here that we finally combined the two criticisms to end up with a tool allowing quantitative ambiguity analysis: "Common denominator" model + "Exchangeability-based PS"

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Preliminaries

☐ Savagean setup

- S denotes a state space
- X the outcome set (non-negative real numbers)
- $(E_1: x_1, \ldots, E_n: x_n)$ denotes a simple act
- xEy denotes (E:x, S-E:y)
- $\succcurlyeq$ is a weak order on the set of acts

Preliminaries

MSOffice3

- We assume a biseparable representation of $\succcurlyeq$
  - Unknown probabilities
    - For $x \geq y$, $E \subset S$:

      $$V(xEy) = W(E)u(x) + (1-W(E))u(y)$$

    - W is a weighting function (a capacity)
    - $u: X \rightarrow IR$
  - Known probabilities
    - $V(xpy) = w(p)u(x) + (1-w(p))u(y)$
    - w: $[0,1] \rightarrow [0,1]$ is the probability weighting function

Diapositive 8

MSOffice3 I insisted here on the neutrality of using the dichotomy: "known" vs. "unknown" probabilities; to avoid the trap of the use of words like ambiguity or uncertainty.

This seems simple, but i think that it is pedagogically more helpful. A reading of the introduction of Epstein & Zhang shows how the early use of the word ambiguity is problematic.

Abdellaoui Mohammed; 03/04/2007

Preliminaries

- ☐ Equally likely relation
  - o A $\sim^*$ B if xAy $\sim$ xBy for some x $\succ$ y.
  - ☐ Exchangeability
  - o A and B are exchangeable if exchanging the outcomes under events A and B in an act does not affect its preference value.

    (xAyBf $\sim$ xByAf)
  - o A partition is exchangeable if all of its elements are mutually exchangeable.

Preliminaries

- Probabilistic sophistication
  - PS holds if there exists a probability measure P on S such that for each act f the only relevant aspect for its preference value is the induced probability distribution.

    $(P_f = P_g) \Rightarrow (f \sim g)$
  - Consequence: The preference of an event captures everything relevant for preference evaluation.

Preliminaries

- Probabilistic sophistication

  Under PS:

  - two events are exchangeable iff they have the same probability

  - All events in an exchangeable partition (E_1, …,E_n) have probability 1/n.

Bayesian Beliefs for Ellsberg?

- Ellsberg two-color paradox
  - Two urns:
    - Known Urn contains 50 R^k (Red from known) and 50 B^k (Black from known);
    - Unknown Urn containing 100 R^u (Red from unknown) and B^u (Black from unknown).
  - Common preferences (under SEU):
    - $1000R^k0 > 1000R^u0 \Rightarrow P(R^k) > P(R^u)$
    - $1000B^k0 > 1000B^u0 \Rightarrow P(B^k) > P(B^u)$

Bayesian Beliefs for Ellsberg?

- Reconciliation with Bayesian Beliefs
  - Distinguishing two sources (small worlds) of uncertainty.
    - Decision makers has a general dislike of the unknown Urn (source 1) relative to the known Urn (source 2).
    - Similarly, the performance of the Dow Jones index tomorrow can be one source of uncertainty, and the performance of the Nikkei index tomorrow is another.

  A source is a collection (an algebra) of events that pertain to a particular mechanism of uncertainty

Bayesian Beliefs for Ellsberg?

- Reconciliation with Bayesian Beliefs
  - The two-color paradox concerned a between source comparison
  - Events R^k and B^k are exchangeable and events R^u and B^u are also exchangeable:
    - $(R^k{:}\$100, B^k{:}0) \sim (B^k{:}\$100, R^k{:}0)$
    - $(R^u{:}\$100, B^u{:}0) \sim (B^u{:}\$100, R^u{:}0)$
  - The above exchangeabilities suggest that the events in question have subjective probability 1/2 (Chew and Sagi 2006a, 2006b).

Uniform Sources

- PS and Uniformity
  - We call a source *uniform* if PS holds with respect to that source.
  - Chew and Sagi (2006a, b) showed that, under some regularity conditions, a source is uniform iff the following conditions hold.
    i. Events that are equally likely are exchangeable.
    ii. For each pair of disjoint events, one contains a subset that is exchangeable with the other.
    iii. For each n there exists an exchangeable n-fold partition.

Uniform Sources

- Consequence:
  - For a rich uniform source, we can elicit subjective probabilities to any degree of precision using a bisection method and using condition ii (comparability).
    - For example we can partition S into two equally likely events E^1_1 and E^1_2 that then must have probability ½.
    - Next we partition E^1_1 into two equally likely events E^2_1 and E^2_2 that must both have probability ¼, and we partition E^1_2 into E^2_3 and E^2_4 that also have probability ¼.

Uniform Sources

- Uniformity and Ellsberg three-color paradox
  - Assume that an urn contains 30 R balls, and 60 B and Y balls in unknown proportion.
    - People prefer betting on R to betting on B (P(R) > P(B));
    - People prefer betting on [B or Y] to betting on [R or Y], which contradicts the inequality derived before.
  - The (ambiguity) of the urn is not uniform
    - Events have different effects and interactions in different configurations, with the weight of Y high in the presence of B but low in the absence of B.

Uniform Sources

- Combining *biseparability* and PS
  - Unknown Probabilities
    - Assume that For $x \geq y$, $E \subset S$:

      $V(xEy) = W(E)u(x) + (1-W(E))u(y)$
    - Under PS, there exists a transformation function w_S such that $W(E) = w_S[P(E)]$.
  - Known Probabilities
    - $V(xpy) = w(p)u(x) + (1-w(p))u(y)$

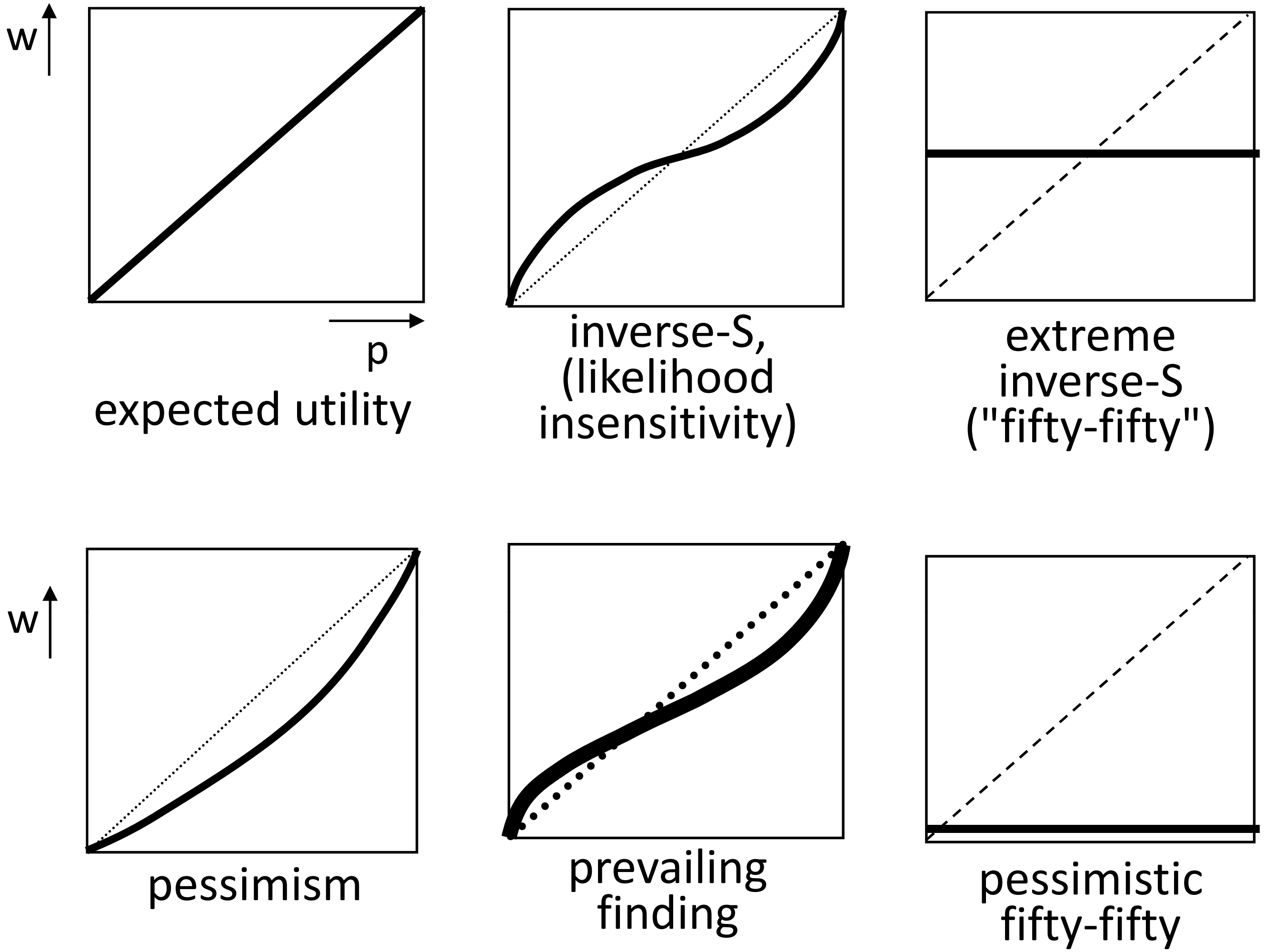
      Function w, called the weighting function, is specific to the case of known probabilities.

Ambiguity Attitudes

- Quantifying attitudes towards uncertainty
  - The weighting functions w_S and w provide convenient tools for expressing various attitudes towards uncertainty and risk.
  - Consider two sources of uncertainty S_1 and S_2 and two (corresponding) exchangeable partitions (A^1_i, i=1,…,n) and (A^2_i, i=1,…,n). A^j (k) is the cumulative union of exch. events in source j.
  - Comparison of willingness to bet on events generated by S1 and willingness to bet on events generated by S2, could quantitatively analyzed through the decision weights

$$w_1(k/n) \text{ and } w_2(k/n),\ i= 1,\dots,n-1.$$

□ Shapes of probability transformations



Abdellaoui (2000); Bleichrodt & Pinto (2000); Gonzalez & Wu 1999; Tversky & Fox, 1997.

Ambiguity Attitudes

- Tractable quantifications of ambiguity attitudes:
  - The use of probability transformation functions (resulting from the combination of biseparability and PS) allows for easy and intuitive within subject comparisons of attitudes towards uncertainty.
  - Comparative concepts can be defined, with one weighting function being more convex or more inverse-S shaped than another one.

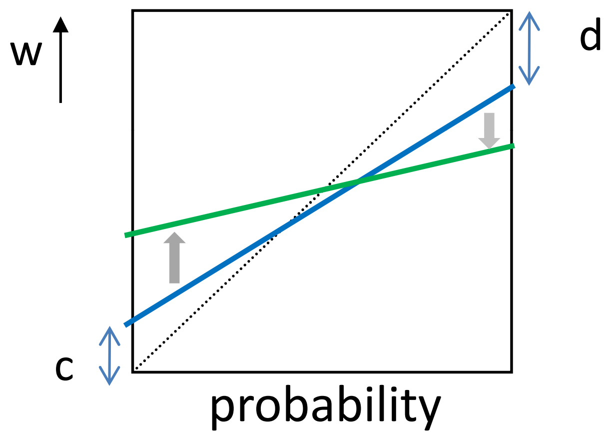
Ambiguity Attitudes

- For empirical purposes, we consider two simple indexes for likelihood insensitivity and pessimism.
  - Assume that

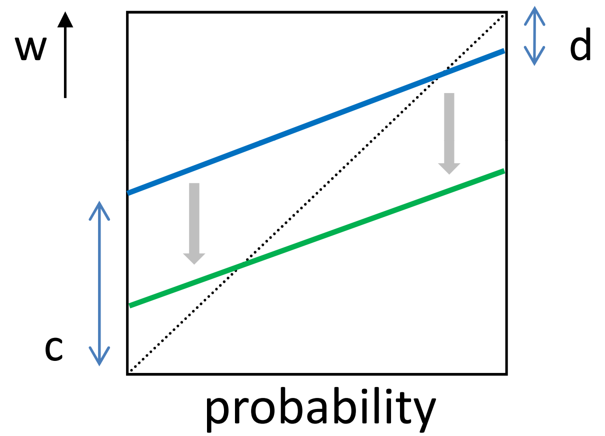
    $w(p) = c + sp$ on the open interval (0,1) with c the intercept and s the slope.
  - Let $d = 1 – c – s$ be the distance from 1 of the line at $p = 1$ (dual intercept).

Ambiguity Attitudes

- Likelihood Sensitivity and Pessimism
  - Index of Likelihood insensitivity: a = c + d,
  - Index of pessimism: b = d – c.



Likelihood (in-)Sensitivity



Pessimism

Ambiguity Attitudes

- Examples

Quantitative indexes of pessimism and likelihood insensitivity

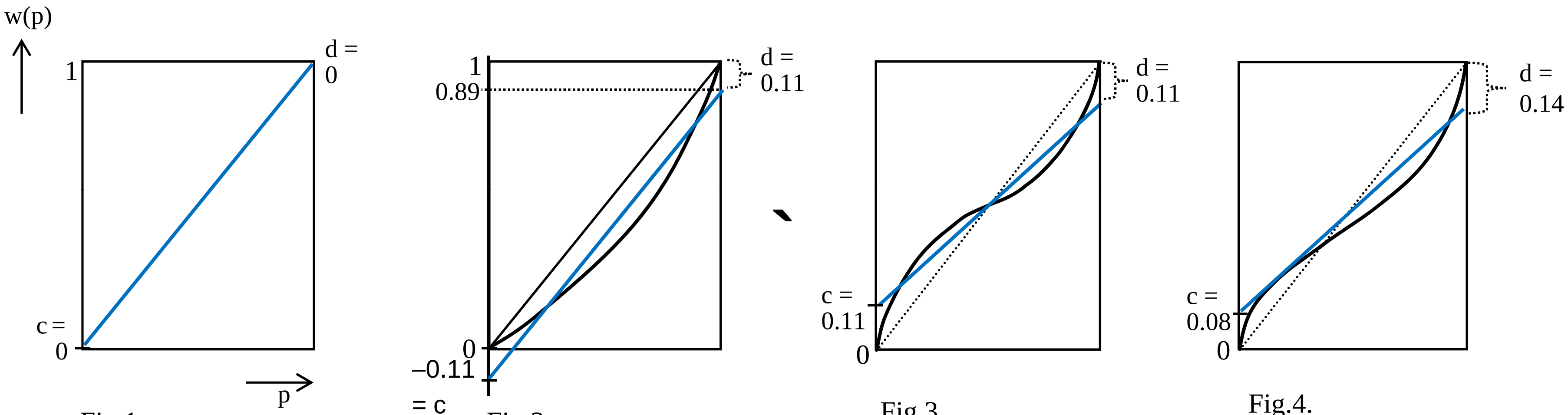


Fig.1. Insensitivity index a: 0; pessimism index b: 0.

Fig.2. Insensitivity index a: 0; pessimism index b: 0.22.

Fig.3. Insensitivity index a: 0.22; pessimism index b: 0.

Fig.4. Insensitivity index a: 0.22; pessimism index b: 0.06.

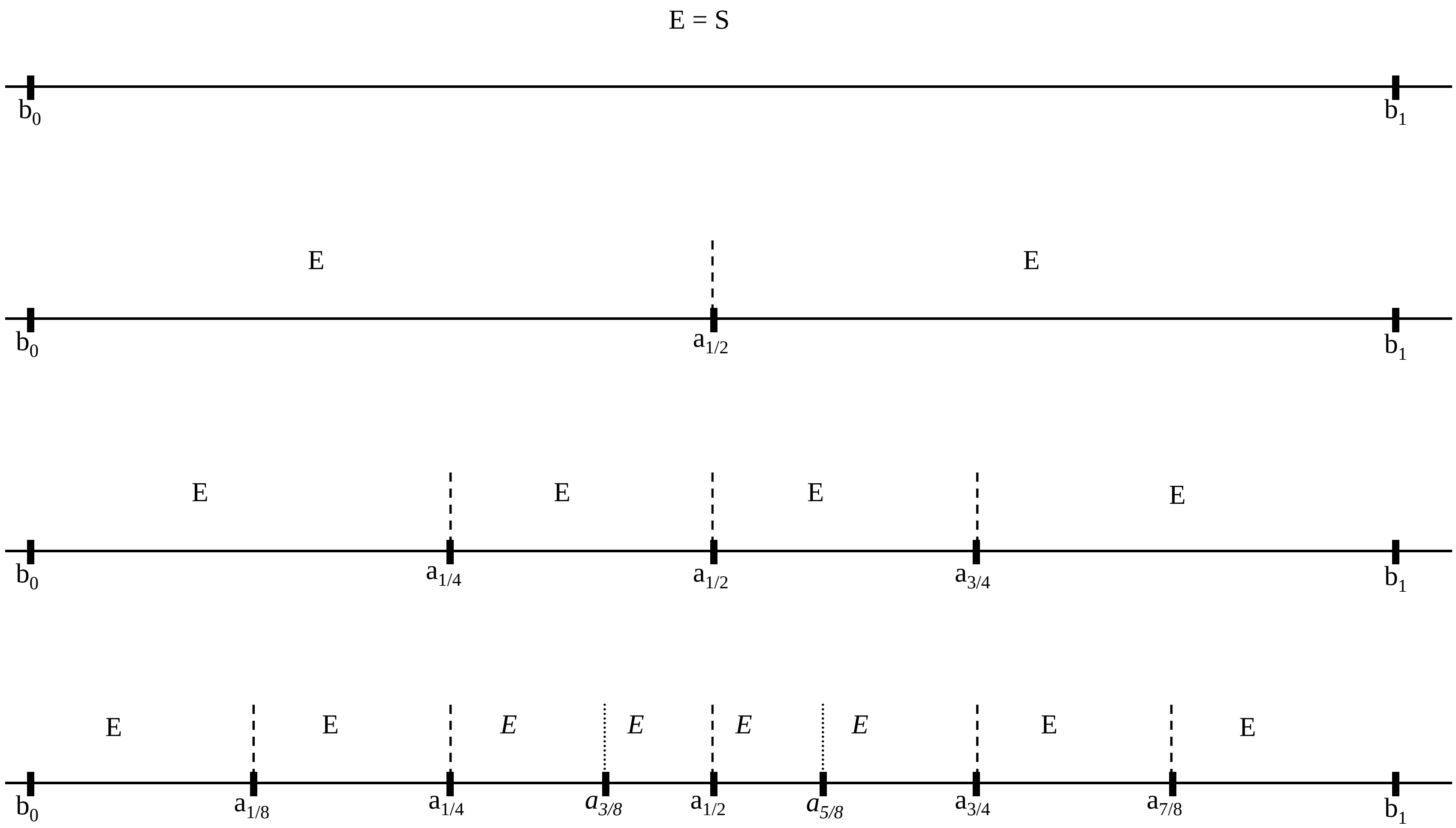
Ambiguity Attitudes

- Ellsberg two-color paradox continued
  - Let us remember that
    $P(R^k) = P(B^k) = P(R^u) = P(B^u) = ½$.
  - For the events R^k and B^k, the weight is $w_k(1/2) = 0.4$, and for events R^u and B^u, the weight $w_u(1/2) = 0.3$.
  - Different weighting explains the different preferences.

Experimental Method

- Participants:
  - 62 students from the ENSAM-Paris. They were mathematically sophisticated and well acquainted with probability theory, but had no training in economics or decision theory.
- Stimuli; unknown probabilities
  - Three sources of uncertainty:
    - I. French Stock Index (CAC40) [how much it would change in a given day]
    - II. Temperature in Paris [on May 31, 2005]
    - III. Temperature in the capital of a randomly drawn remote country [on May 31, 2005].

□ Decomposition of the universal event



The italicized numbers and events in the bottom row were not elicited.

Experimental Method

- Stimuli; known probabilities
  - Utility u(.) and probability weighting function w(.) were elicited through certainty equivalents.

- Procedure and Motivation
  - Each participant was interviewed individually (95 minutes with a break of few minutes).
  - All participants received a flat payment of 20 €. For 31 subjects, real incentives were implemented through the random lottery incentive system (higher prize: 1000 €) in addition to flat payment.

Results: Subjective Probabilities

- Uniformity confirmed for 5 out of 6 cases

Figure 1. Probability distributions for CAC40

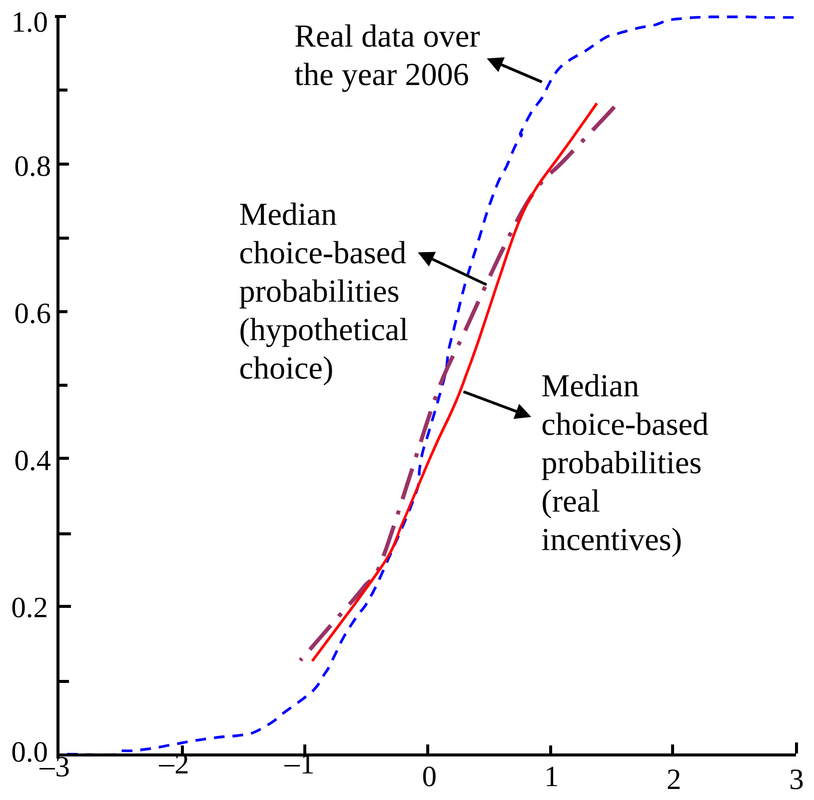
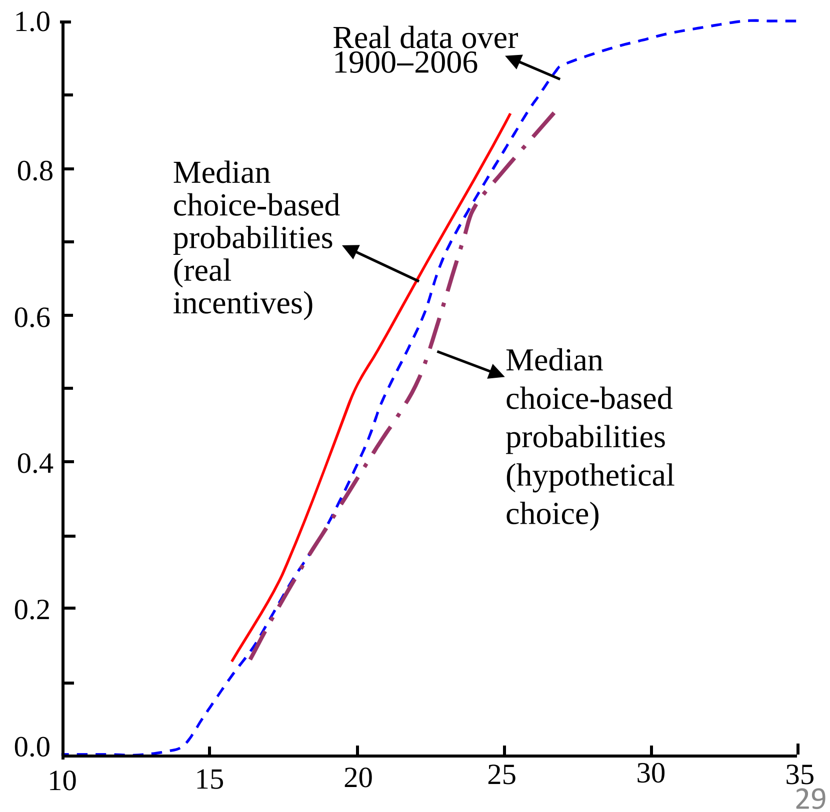


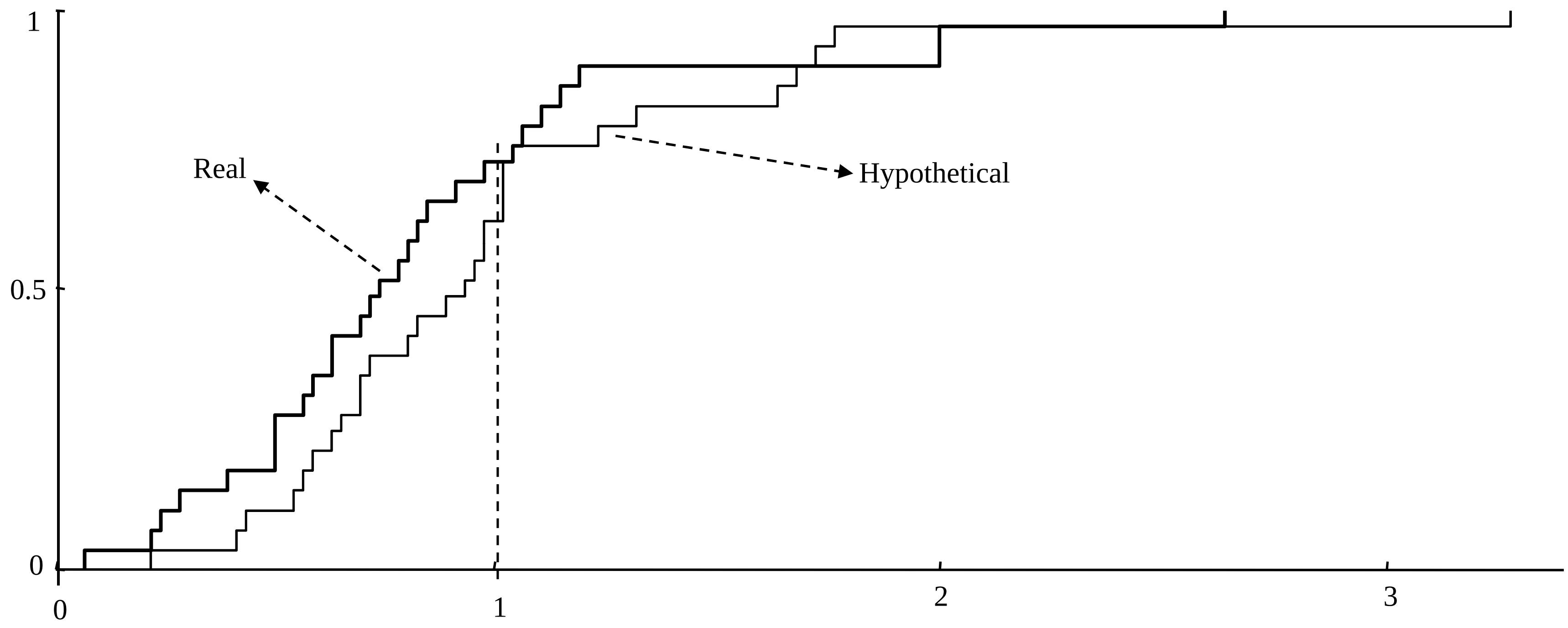
Figure 2. Probability distributions for Paris temperature



Results: Utility

- Power Utility

Figure 3. Cumulative distribution of powers



Results: Ambiguity for U. Sources

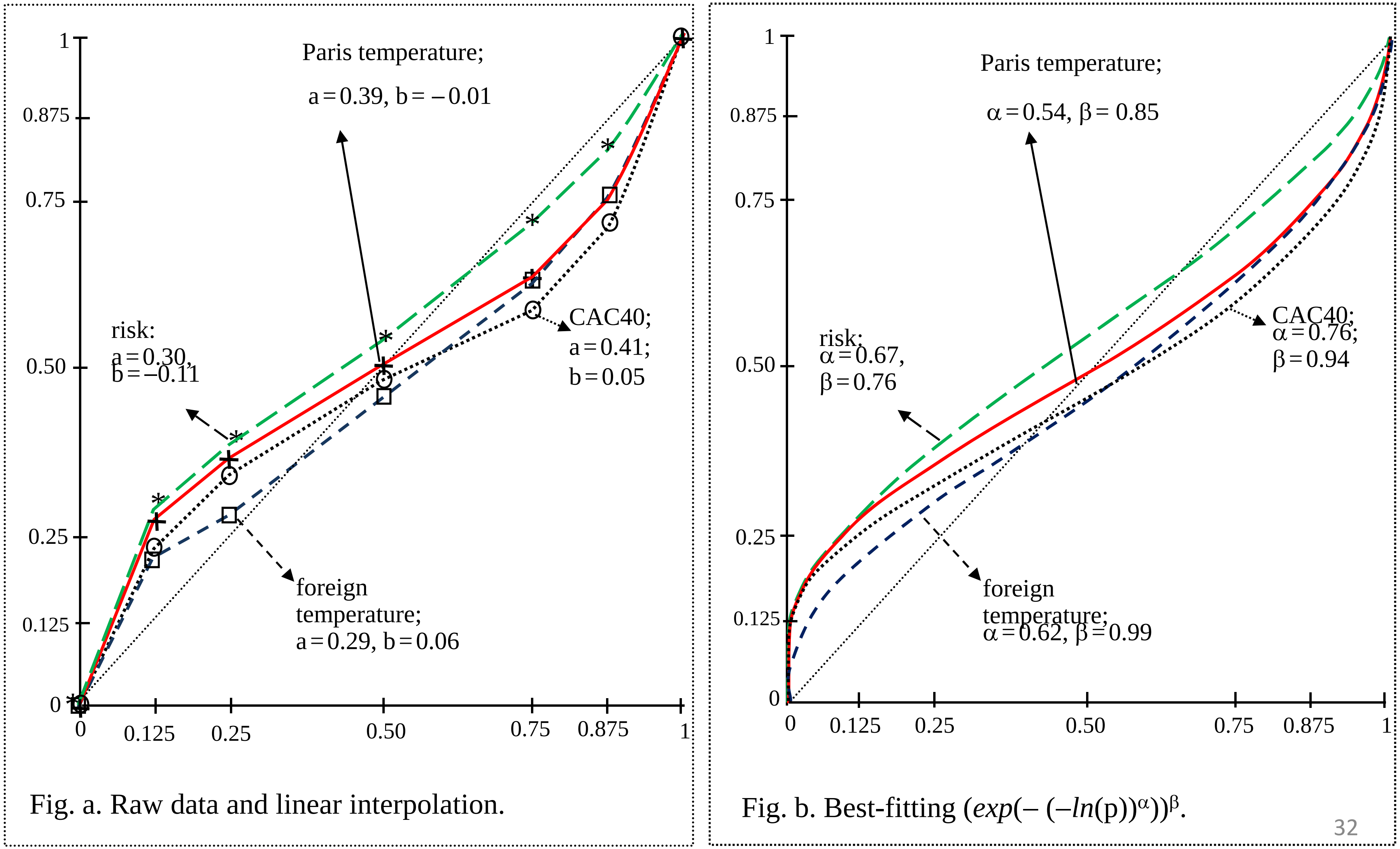
- Method for Measuring Ambiguity Attitude
  - Certainty equivalents were measured for gambles on events. Knowing utility, we could calculate $w_S[P(E)]$ for events E, and then knowing P(E), infer w_S.
  - Consider a source S_1 and the corresponding exchangeable partitions (A_i, i=1,…,n).
  - Assume that $A_i(k)$ is the union of k exchangeable events
  - Assume that

    $CE \sim (A_i(k){:}\$100, S- A_i(k){:}\ 0)$.
  - We have

    $u(CE) = w_1[P(A_i(k)]u(\$100) + (1- w_1[P(A_i(k)])u(0)$

□ Overall Results

Figure 4. Average probability transformations for real payment



□ Results at individual level

Figure 8.3. Probability transformations for participant 2

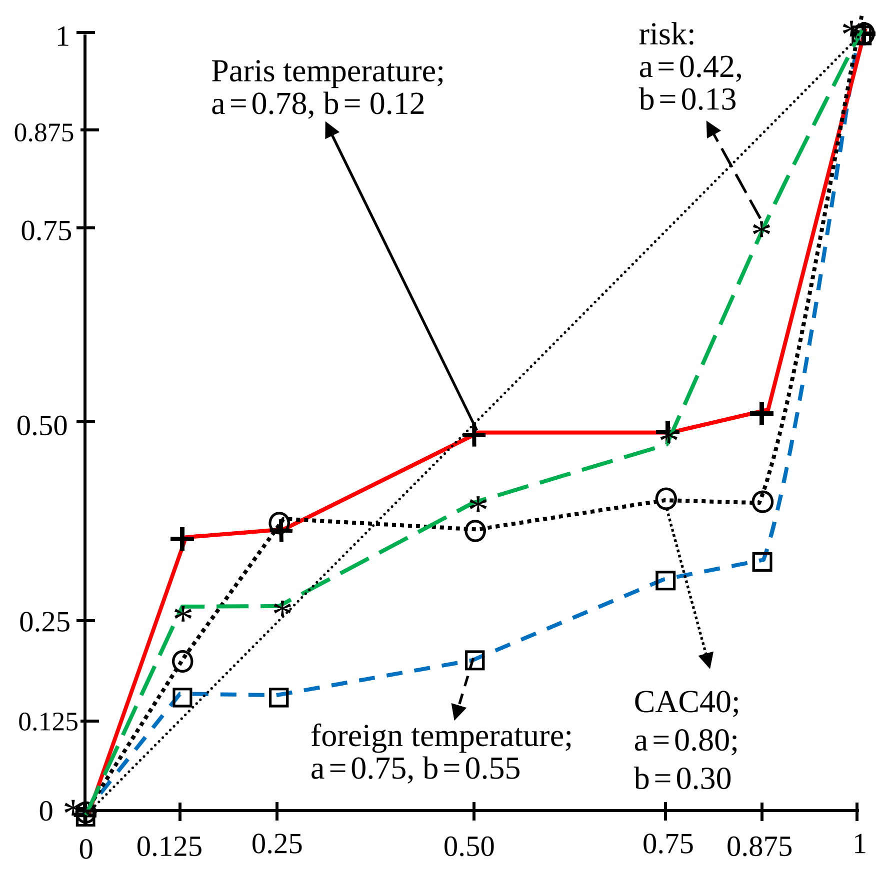


Fig. a. Raw data and linear interpolation.

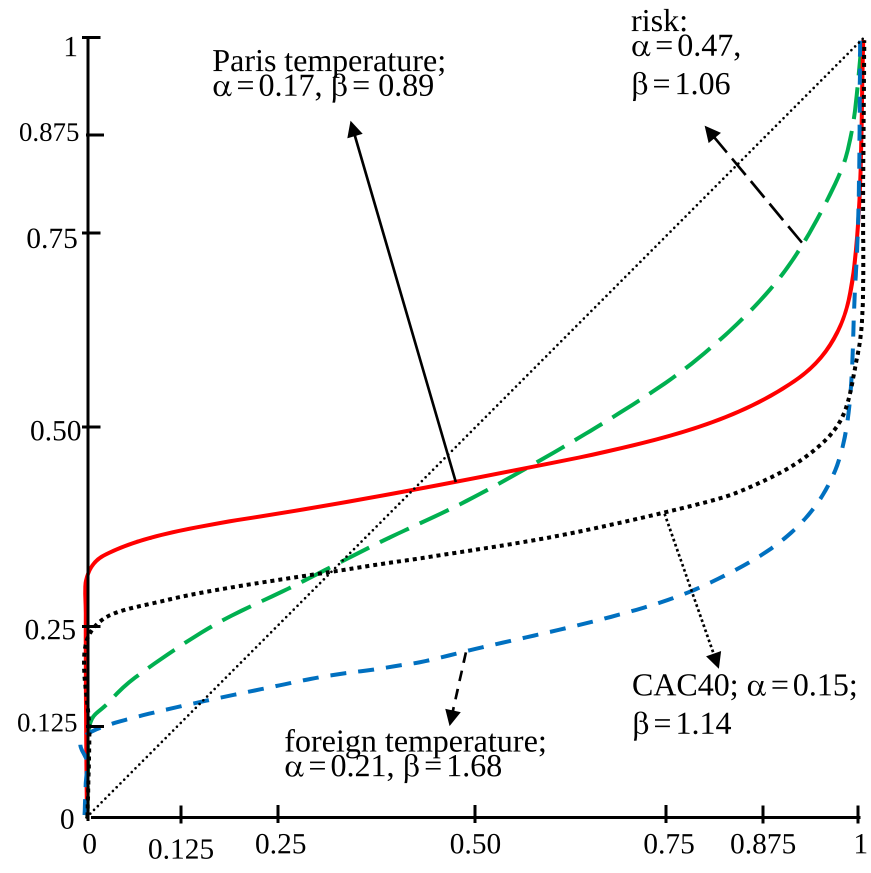
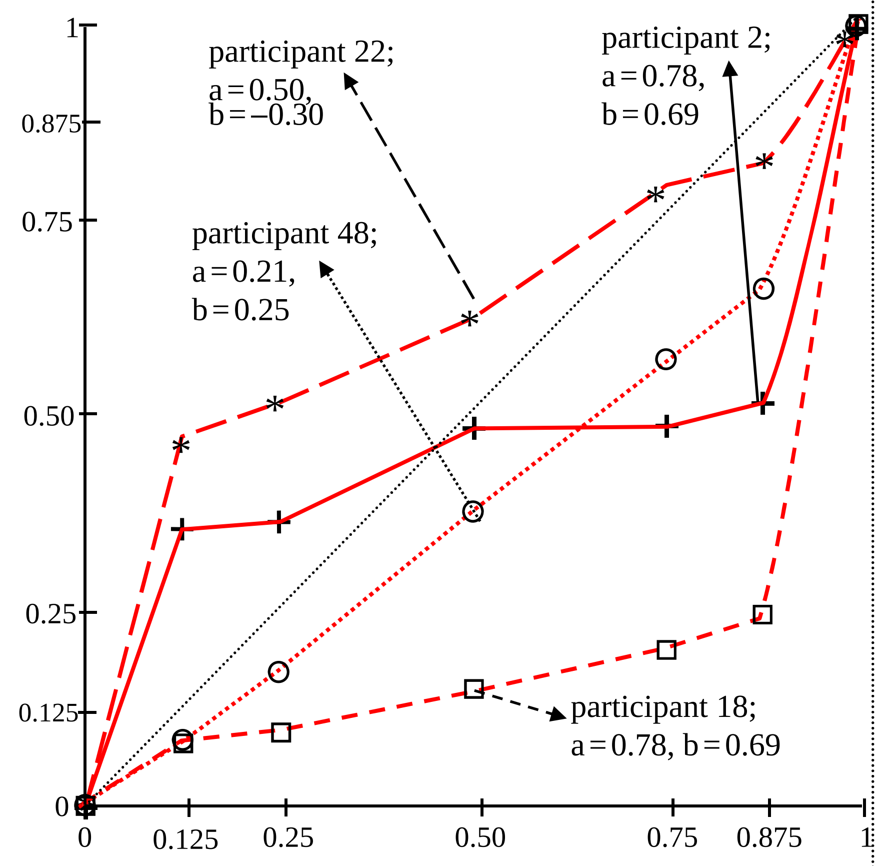


Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

□ Results at individual level

Figure 5. Probability transformations for Paris temperature and 4 participants



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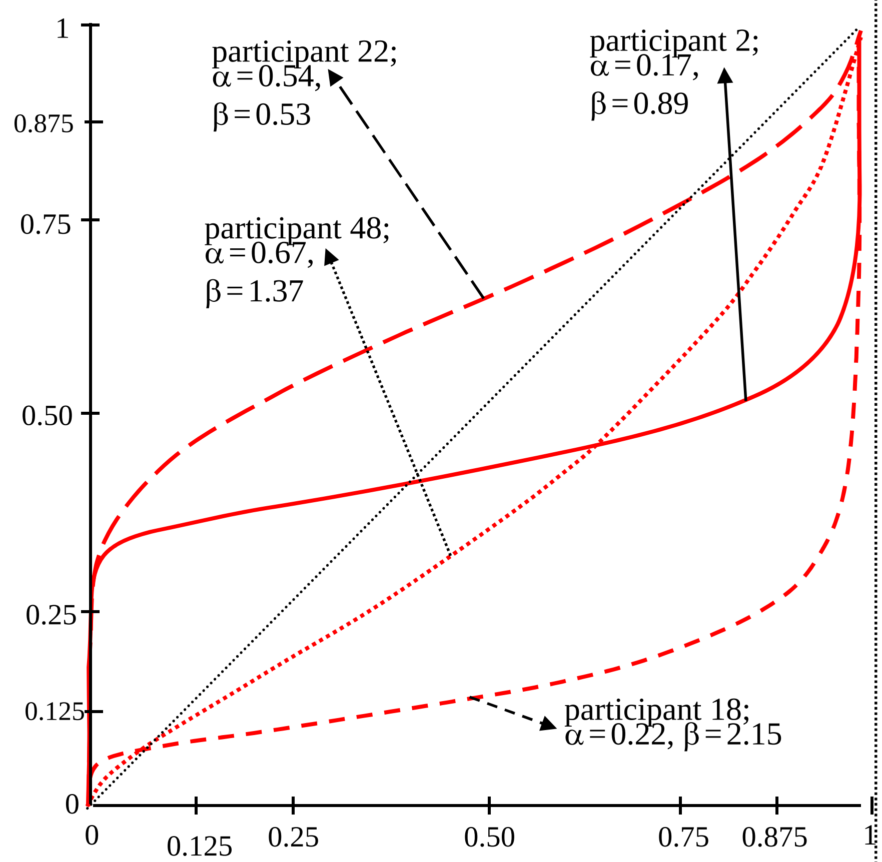

Fig. b. Best-fitting $(exp(-(-ln(p))^{\alpha}))^{\beta}$.

Concluding Remarks

- We have demonstrated that a biseparable representation of preferences combined with PS can be used to analyze ambiguity for uniform sources of uncertainty.
- The Ellsberg two-color example was reconciled with consistent subjective probabilities.
- We introduced a new method for deriving subjective probabilities and demonstrated its validity (good calibrations were achieved).
- We demonstrated the feasibility of complete quantifications of ambiguity attitudes.