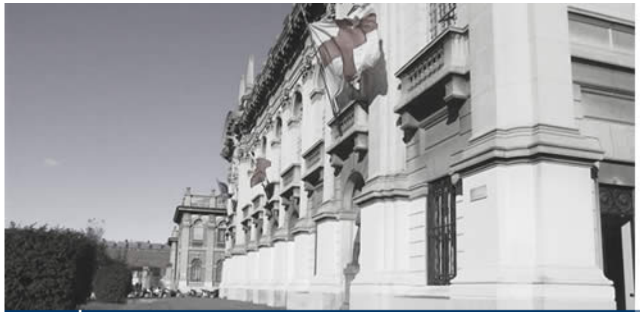

POLITECNICO DI MILANO

4°

Methods and Models for Decision Making

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God in 7 steps:

- Four classes of decision problems
- The main two (in this context) → ranking, rating
- Binary relations (A > B, A ≥ B, A ~ B, A ? B)
- Ranking-1 → the risk analysis
- Non-deterministic environment (random outcomes)
- Lotteries to measure the risk attitude of the DM
- Utility function (one for each indicator) of *this* DM

Index:

Ranking-2:
multicriteria analysis

MultiObjective / MultiCriteria

- Decision problem with one DM and full information
- Different points of view (objectives or criteria)
- Final solution = a good trade-off between the criteria
- Various phases →
  - ph1: from indicators to utilities
  - ph2: subset of efficient solutions
  - ph3: preference and final solution
- Two cases (one continuous, one discrete) for understanding

The various phases

- A decision problem with different (conflicting) objectives/criteria
- Objectives = continuous case // Criteria = discrete case
- The need of a synthesis (considering different points of view)

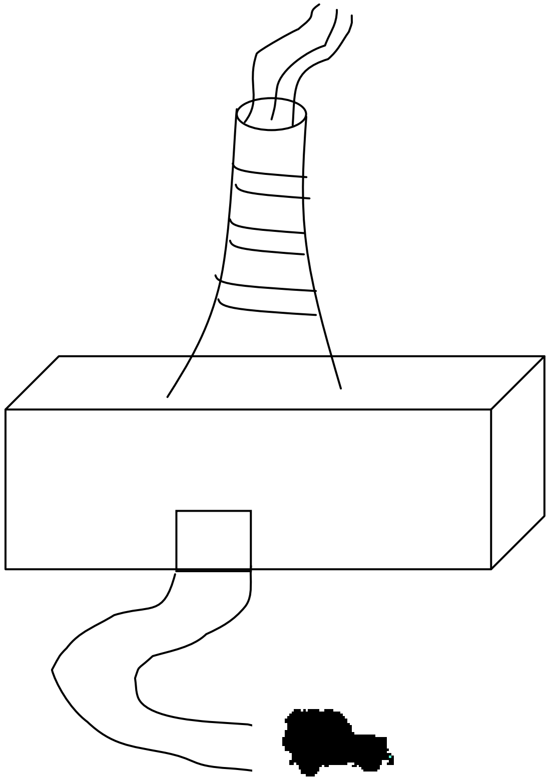
Ph1 – The treatment of different data (from indicators to utilities)

Ph2 – The search of efficient (or non-dominated or Pareto) solutions

Ph3 – The final (best trade-off solution) choice and the sensitivity

- The procedure is not “objective”, but the analysis can point out the crucial aspects of subjectivity (what influence, where, …)

Example – The incinerator project



*There is an air standard quality Q**

- **Variables of decision:**

  D = plant dimension,

  H = smokestack height,

  P = % of pollutant eliminated

- **Sectors of attention:**

  - economics,
  - waste service,
  - fly sefety (the smokestack),
  - local viability (congestion),
  - environment.

Indicators (measures of the effects)

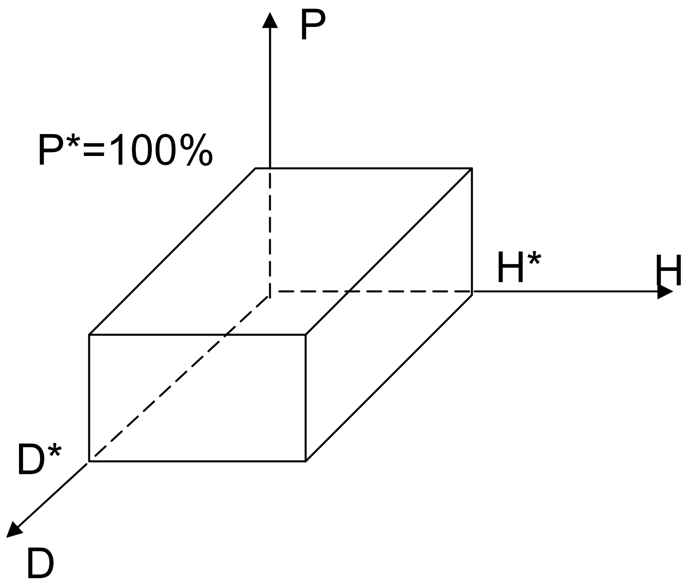
Sector	Indicator	Constraint	Objective
Economics	R (benefits-costs)	-	max R
Service	D (smaltiti wastes)	$D \leq D^*$	-
Security	H (air trafic)	$H \leq H^*$	-
Viability	D (number of vehicles)	$D \leq D^*$	-
Environm.	P (% removed particules)	$P \leq 100\%$	max Q/Q*

- Indicators →
  - directly in the constraints
  - directly in the obj. functions
  - undirectly in the o.f. (i.e. particules)

- Sector models (to supply the measures)

The variable space (decisions)

- Three (continuous) variables



$D \leq D^*$

$H \leq H^*$

$P \leq P^* = 100\%$

$\Downarrow$

The feasible region X
is a parallelepiped

- Each point **x** $\in$ X is a feasible solution (∞ solutions)
- For each point **x** it is possible to compute the values of R & Q (sector models)

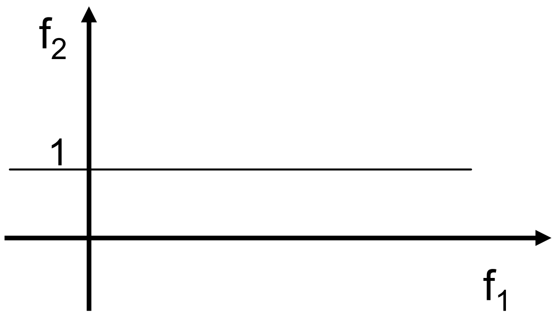
The objective space (results)

- How many objectives ?
  - max R → max f_1
  - max Q/Q* → max f_2

[attention: it must be $f_2 = Q/Q^* \geq 1$ → why ?]

Because it must be $Q \geq Q^*$

- Two dimensions

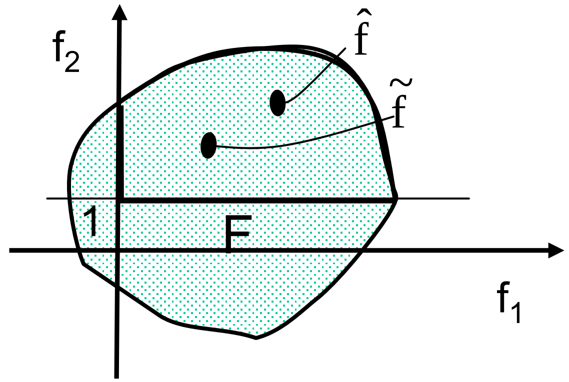


$f_1 = R$
$f_2 = Q/Q^*$

- Each vector **x** (a tern of decision variables) corresponds to a vector **f** (a cople of results)

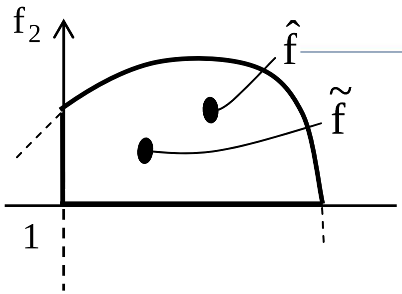
How obtaining F from X

- Region X is known (you can explore it)
- For each $\bar{x} \rightarrow$ the corresponding $\bar{f}$
- In general: $x \in X \rightarrow f \in F$
- So, you have F



- Question: given two vectors of results, is it better $\hat{f}$ or $\tilde{f}$?

Dominance



- Comparison bertween $\hat{f}$ and $\tilde{f}$
- $\hat{f}$ dominates $\tilde{f}$ (and the solution $\hat{x}$ dominates the solution $\tilde{x}$): why ?
- Definition (1), ***dominance*** → in a decision problem with m objectives (to be maximized) max $f_1(x)$, …, max $f_m(x)$, a solution **x** dominates a solution **y** if $f_1(x) \geq f_1(y)$, …, $f_m(x) \geq f_m(y)$, that is the solution **x** obtains better (or equivalent) results with respect to the solution **y**, for all the objectives.
- Definition (2), ***efficient solution*** → a solution **x** non dominated by any other solution is called efficient (or parentian).

Example2 – A sabbatical year

- Professor C. has to decide
  where going for a sabbatical year

- Data are the following:

	Rome	Berlin	Geneva	Moscow	Tokio
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

- Qualitative scales, converted in numerical [0, 10] ones
- Search for the best choice, between the 5 alternatives
- A multi-criteria (discrete set of options) decision problem

More about dominance

- In this context it is still valid the concept of dominance ?
- There are → 2 dominated solutions / 3 efficient (non dominated) solutions
- If ithe data are correct and if the teacher is rational, he must choose only between → Rome – Berlin – Geneva
- So he has reduced the options, but he doesn't already chosen the final solution
- What option ? It depends on the importance that the teacher acknowledges to the various criteria: economics, working place, environment
- The preference structure of the DM could be very complex; but in the simpler case it is a vector with dimension equal to the number of criteria (3 in this case)

Common & different features

- Commun elements:

  - deterministic problems (all the data are known)

  - multi objective/criteria — 2 in the case of incineretor; 3 in the case of sabbatical

  - only one DM

- Decision problems → **1/m/d** (1 dec.maker / m criteria / det. info)

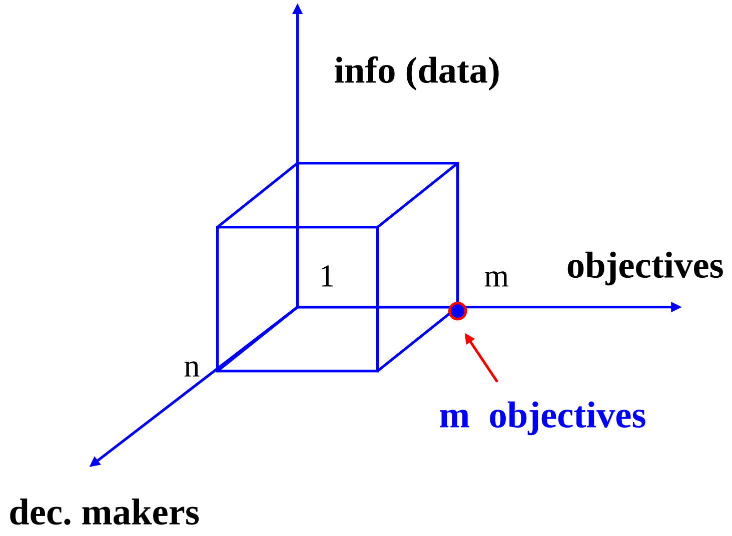
- Different elements:

  - continuous problema with ∞ solutions (MODM), discrete problem with only 5 alternatives (MCDM)

  - in one case (incineretor) we have done only definitions, in the other (sabbatical) we obtained the efficient solutions

- MODM (or MCDM) → trade-off → subjectivity

The reference frame



- Three axis

- The 1/m/d case → **Decision with m objectives**

- Formulation → **Min or max with $x \in X$** → $\begin{vmatrix} f_1(x) \\ f_2(x) \\ : \\ f_m(x) \end{vmatrix}$

  (a vector of obj. functions)

- Problems
  - **continuous case → multi-objective analys**
  - **discrete case → multi-criteria analys**

Three phases of the choice

- **Phase 1 → Data analysis**
  - the objectives of the decision maker are measured by functions
  - each function shows the value of an indicator
  - each indicator has his own unit
  - to compare a common scale is needed
  - the scale is the measure of the utilities perceived by the decision maker

- **Phase 2 → Efficient solutions**
  - are there some dominated solutions among the others (infinite or prearranged) ?
  - elimination of the dominated solutions
  - not dominated or efficient or Pareto solutions (synonyms) remain

- **Phase 3 → Final choice**
  - analysis of the preferences structure of the decision maker
  - vector of weights (pair comparison)
  - weighted sum of the utility of each alternative
  - ranking, final choice, sensitivity

Phase 1 – Indicators (and their units of measure)

- **Example of the incinerator :**

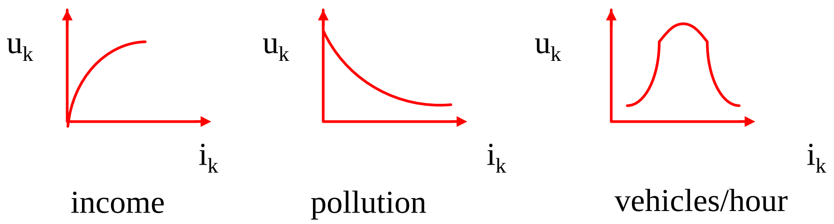
  max f_1 (profit) → millions €/year

  max f_2 (air quality) → fraction between 2 values in mg/m^3

- **What:** to analyze the link between a certain indicator and **utility** perceived by the decision maker → a function u_k (i_k), where i_k represent the value of the indicator related to the objective-function $f_k(x)$

- **Why:** the **utility function** u_k allows to affirm that the solution $\hat{x}$ is better than the solution $\bar{x}$ (following that objective or criteria) if $u_k(\hat{x}) > u_k(\bar{x})$; while there is no preference if $u_k(\hat{x}) = u_k(\bar{x})$

- **Examples of utility functions**



Estimation of the utility functions

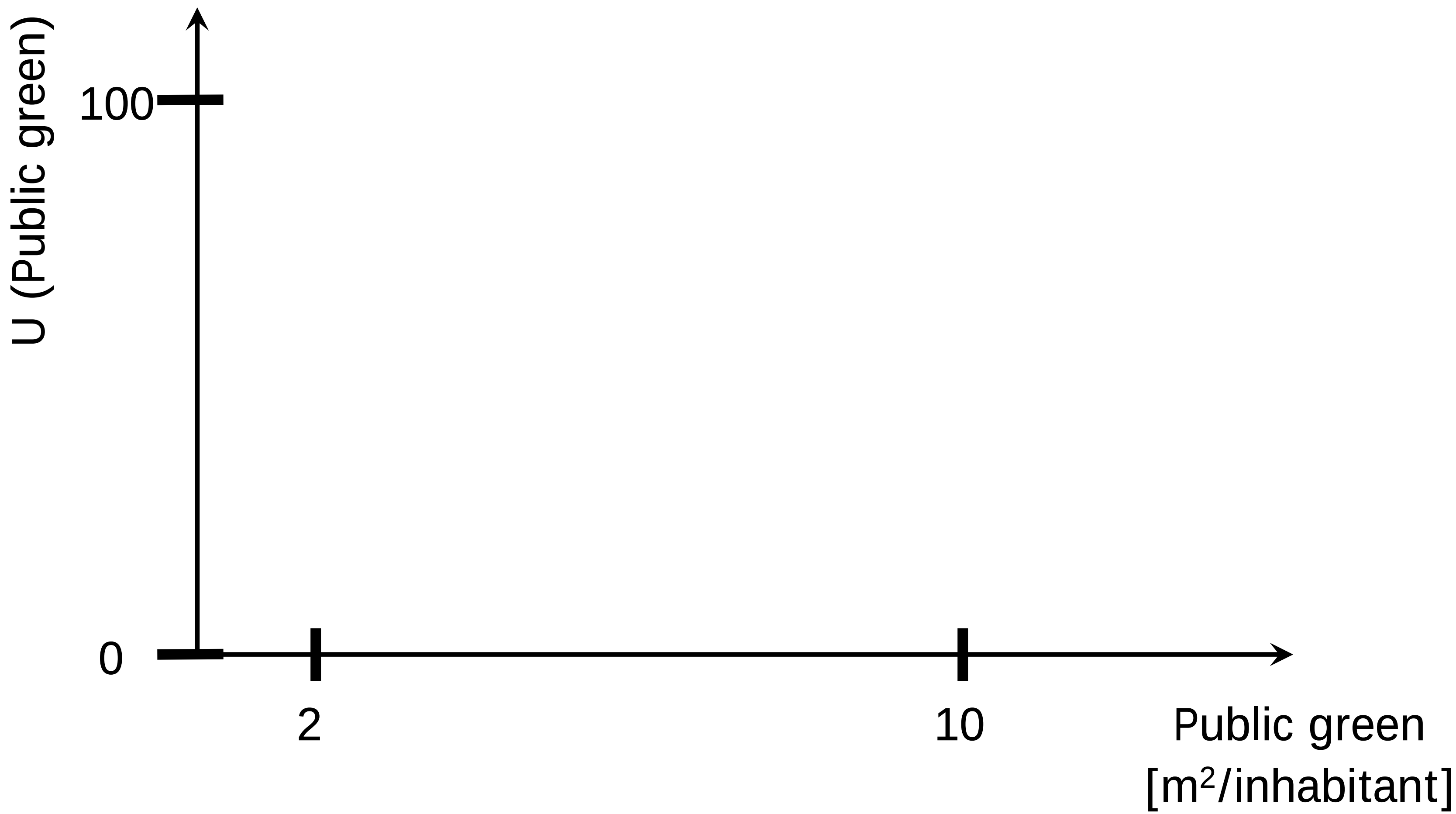
- By the literature
- By an empirical procedure (points):

↓

1. To define the range of admissible value for the considered attribute (wide? narrow?)
2. To state the shape of the utility function (increasing? decreasing? Non-monotonic?)
3. To estimate the function

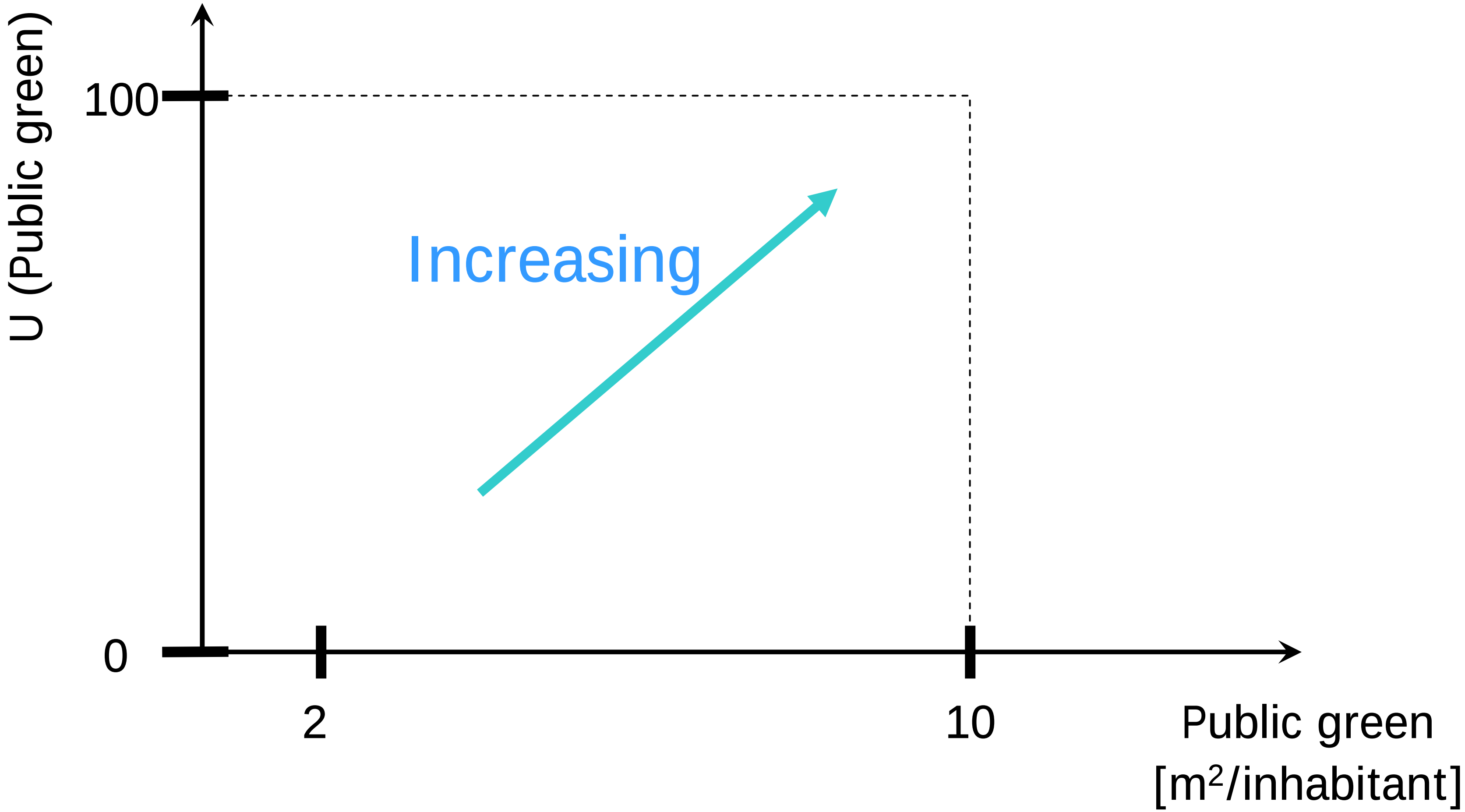
The mean fraction: step 1

Fix the min & max values

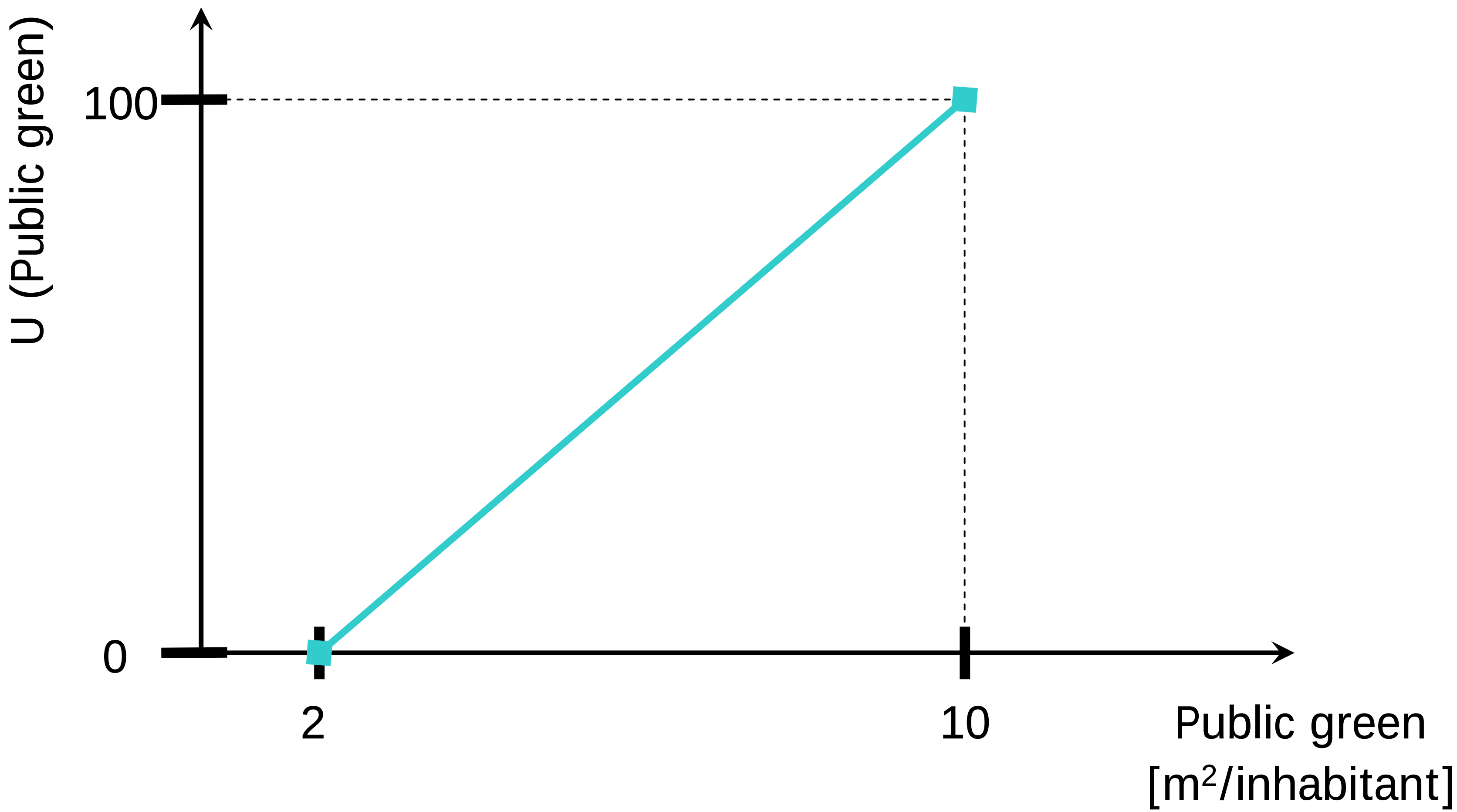


The mean fraction : step 2

Increasing or decreasing ?

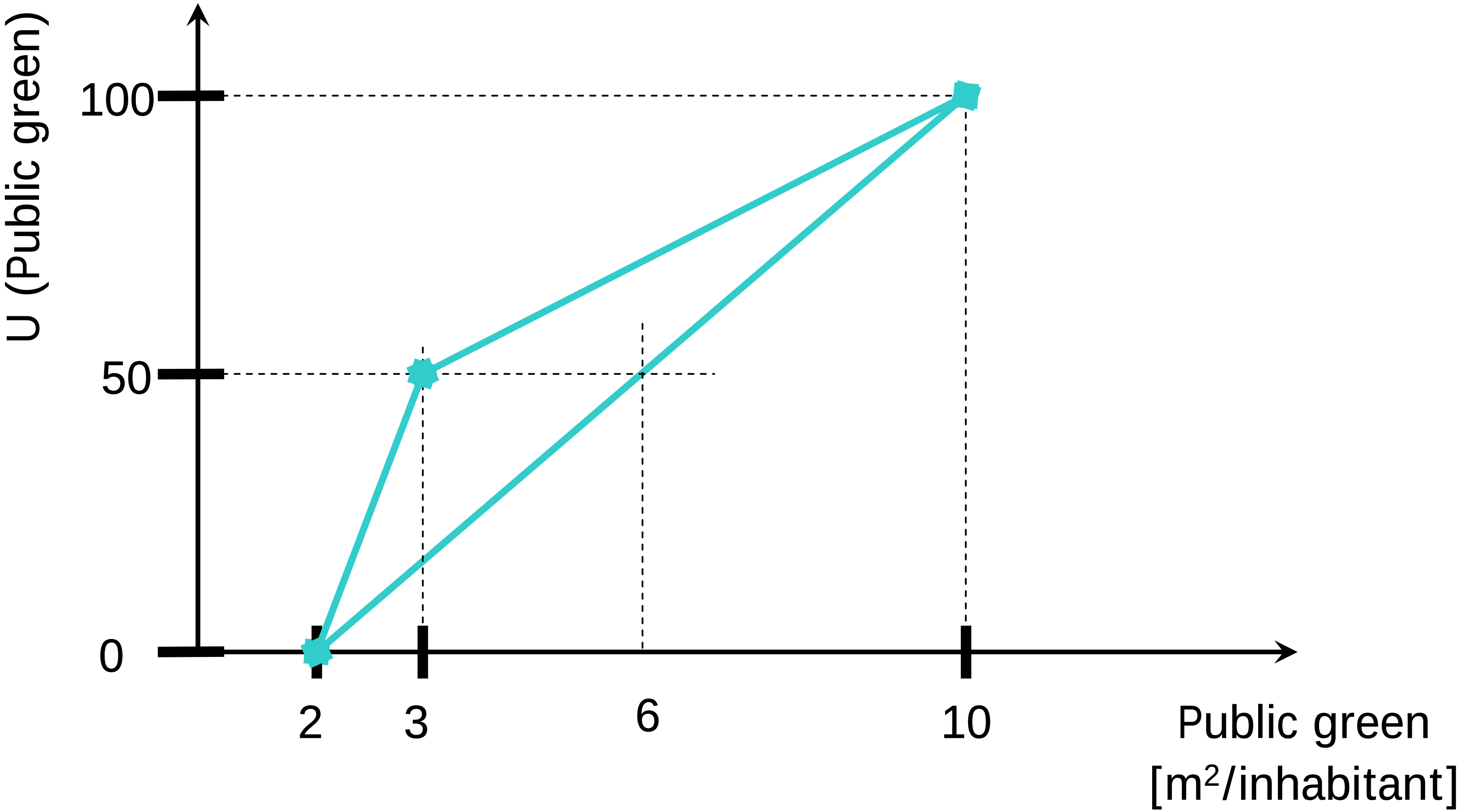


The mean fraction : step 3

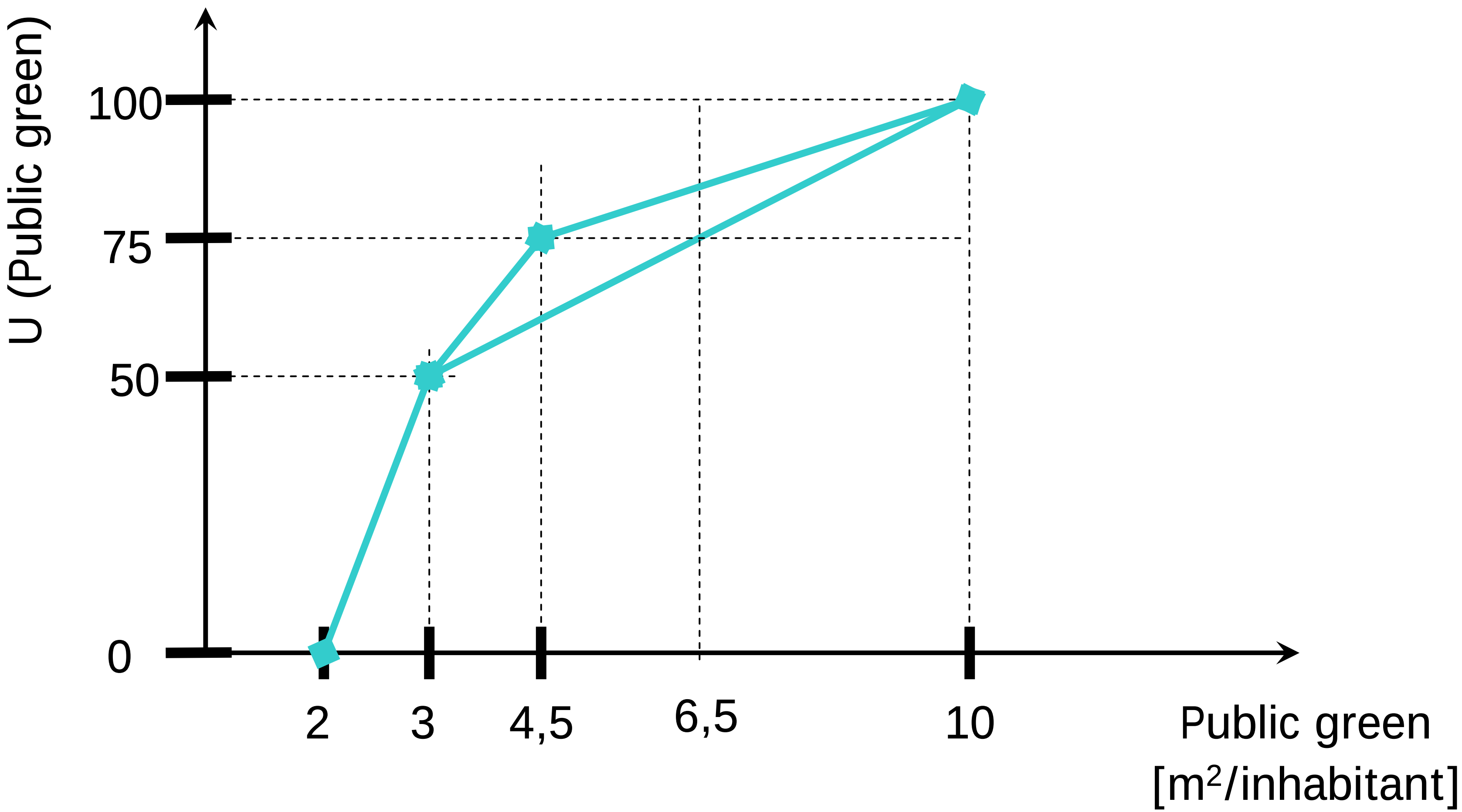


The mean fraction : step 3

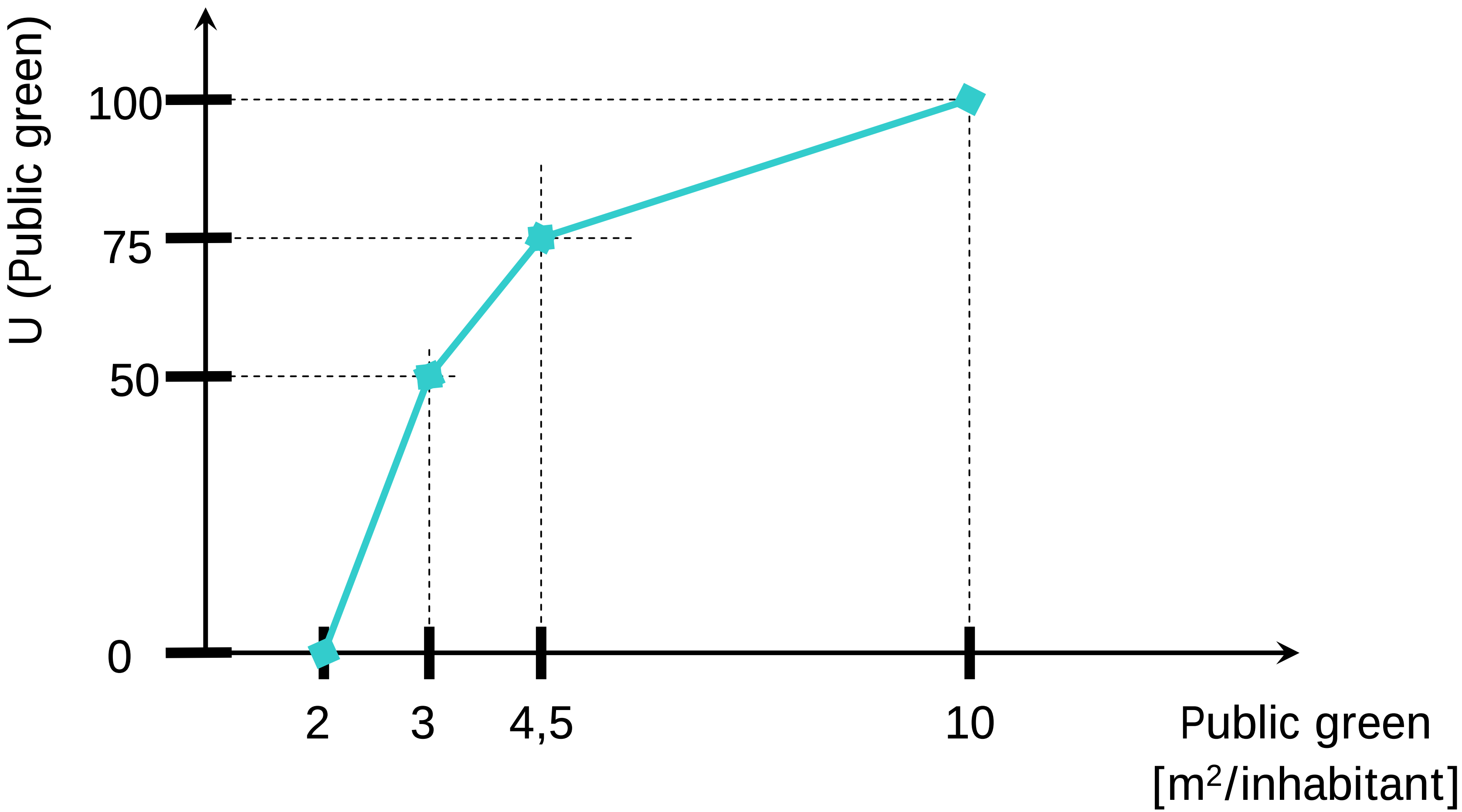
Some intermediate values



The mean fraction : step 3



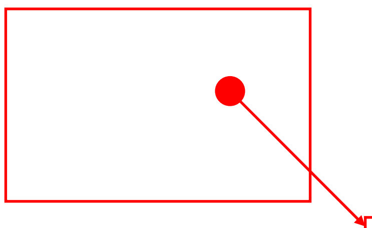
The mean fraction : step 3



Phase 2 – Evaluation matrix

- **Discrete case:** **Multi Criteria Analysis (MCA)**
  - a finite number (usually small) of alternatives
  - a finite number of criteria (m)

- **Evaluation matrix**



rows (m) → criteria

columns (n) → alternatives

u_{kj} = utility with respect to criterion k of the alternative j

- **Example (sabbatical):**

	R	B	G	M	T
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

Values are in the conventional scale [0, 10]

Phase 2 – Efficient solutions

- **Are there cities in which the teacher will not (…) in the future ?**

  ⇨ **Phase 2**

  Search of the dominated alternatives

  (and then of the efficient alternatives)

- **Dominance** → alternative A dominates alternative B if:

  $u_{1A} \geq u_{1B}$, $u_{2A} \geq u_{2B}$, …., $u_{mA} \geq u_{mB}$
  (and if for at least an attribute there is >)

- **Search of efficient solutions** ⇨ comparison between r columns

  (how many comparisons?)

- **Example** R dominates B, or viceversa ? NO → B dominates M; B dominates T

  . R dominates G, or viceversa ?

  M dominates T, or viceversa ?

Efficient solutions are → Z R, B

Phase 3 – The final choice

- One more element → the preferences structure

- Matrix

	Rome	Berlin	Genève	Moscow *	Tokyo *
Reward	5	7	10	2	7
University prestige	3	9	4	6	5
Quality of life	10	4	5	3	3

Evaluation matrix

weights
0.3
0.6
0.1

(*) dominated alternative

- The vector of the weights measures the importance that the decision maker gives to the criteria (objectives)

- Weighted sum:

Rome	Berlin	Geneva	Moscow	Tokyo
4.3	7.9	5.9	4.5	5.4
(5°)	(1°)	(2°)	(4°)	(3°)

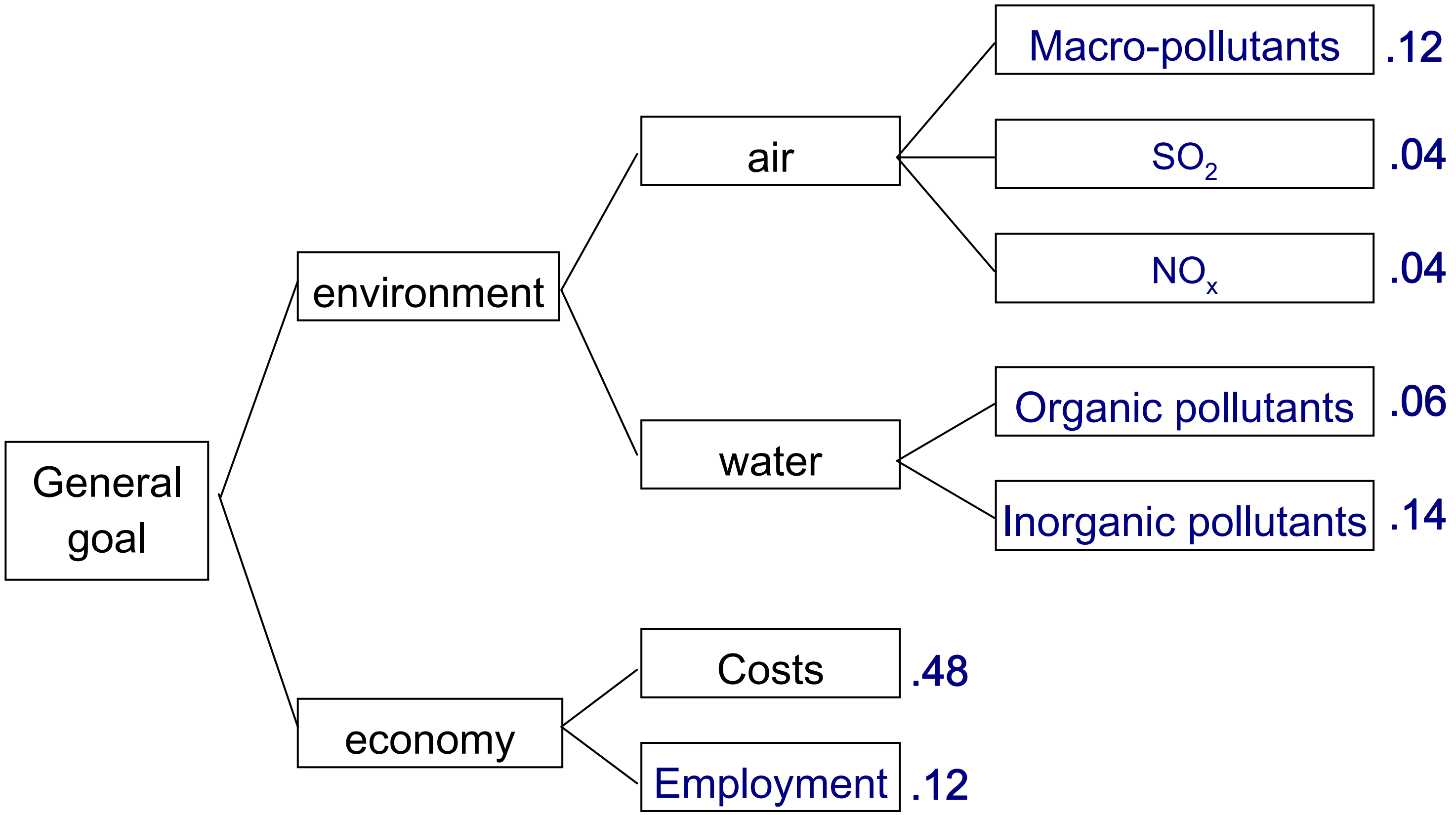
These values (total utility) are calculated as sum of the products of the rows and the weights

- What does it mean ? What is his use?

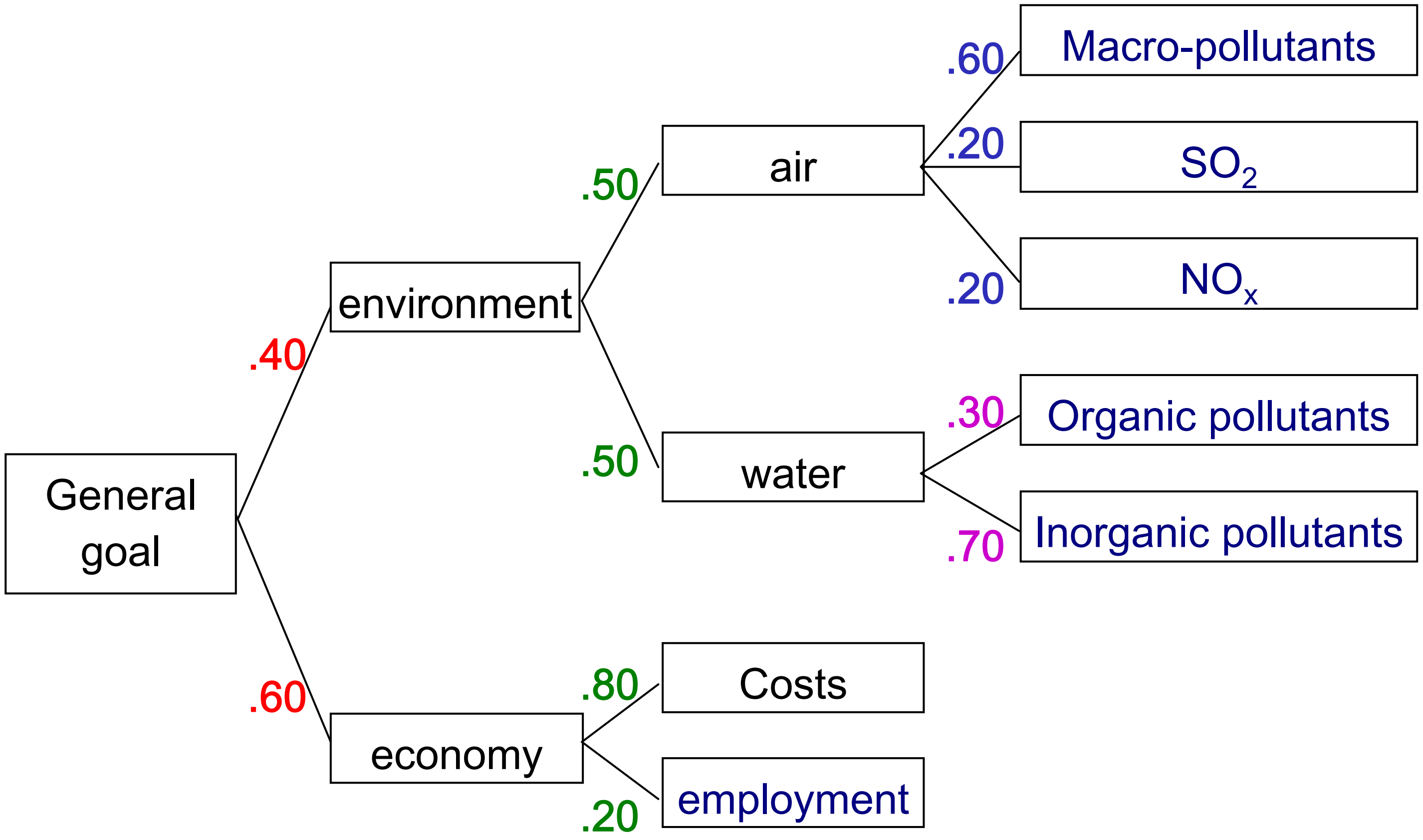
  → The satisfaction related to each alternative

  → To rank the alternatives giving the choice → Berlin

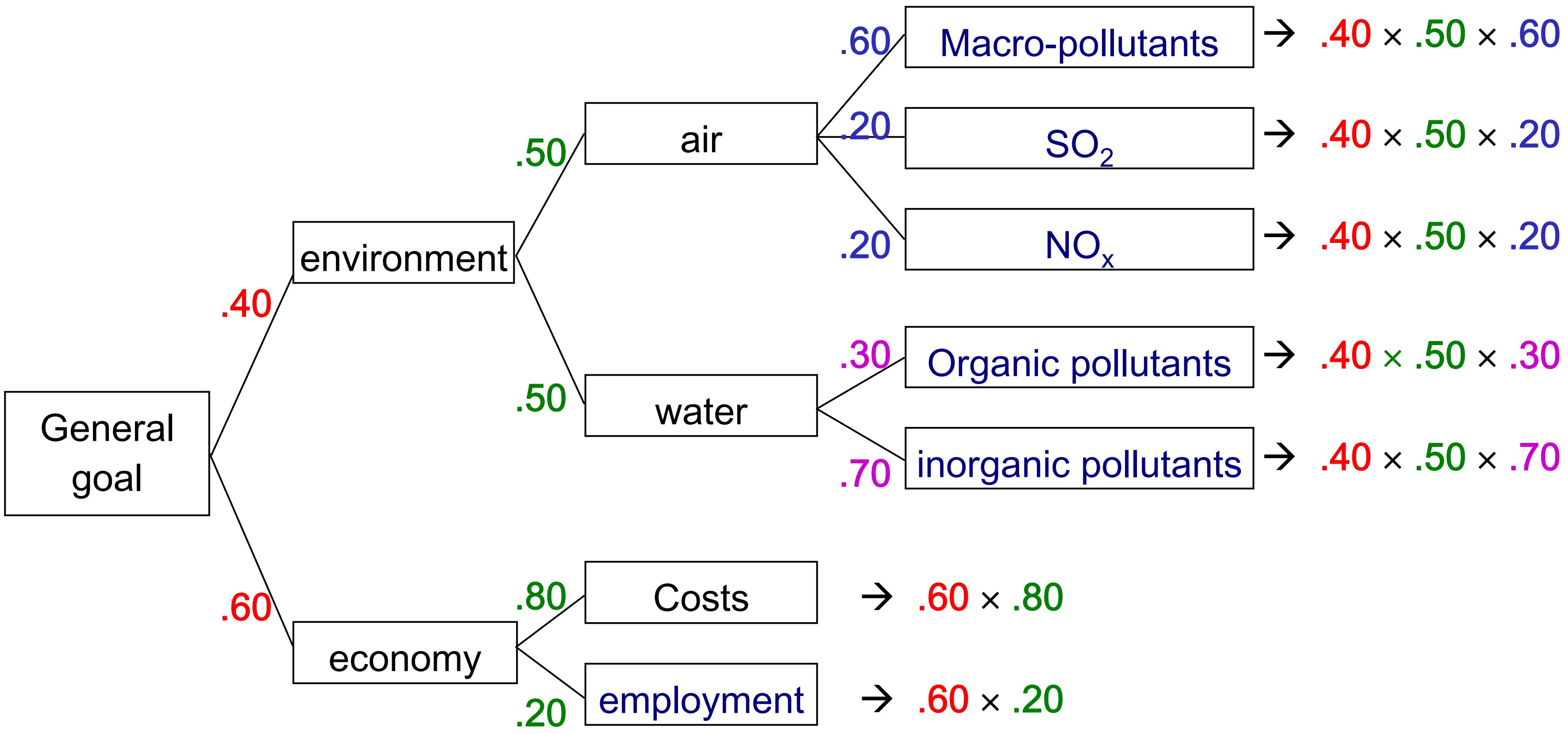
Weight assignment: list



Weight assignment: hierarchy



Weights on the hierarchy



Pair comparison

- **How to obtain the vector of the weights?**
  - Thanks to many pair comparison between criteria
- **Example:**

$$\begin{array}{c} \\ \end{array}\begin{matrix} c_1 & c_2 & c_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 1/2 & 3 \\ 2 & 1 & 6 \\ 1/3 & 1/6 & 1 \end{pmatrix} \begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix} \qquad \begin{pmatrix} * & * & \bullet \\ * & * & * \\ * & * & * \end{pmatrix}$$

a_{ij} = how criterion c_i is more important than criterion c_j

  - Responses of the decision maker:
    - c_2 is 2 times more important than c_1
    - c_1 is 3 times more important than c_3
    - c_2 is 6 times more important than c_3
- **Substitution rate**
  - To a worsening of 1 unit as regards c_2 must correspond an improvement of 2 units as regard c_1 so that the DM considers equally (indifferently) the two alternatives
  - The same for the other pair comparisons: c_1 in comparison with c_3, c_2 in comparison with c_3

Consistency (internal coherence of judgements)

- **Consistent DM:** $a_{ij} = a_{ik} \cdot a_{kj}$
  - In this case each column of the matrix, after normalization, (dividing by the sum of the values of the column), gives the vector of the weights

$$\begin{pmatrix} 1 & 1/2 & 3 \\ 2 & 1 & 6 \\ 1/3 & 1/6 & 1 \end{pmatrix} \Rightarrow \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} = w$$

- **Non-consistent DM:**
  - An ad hoc procedure of the matrix calculation is needed (calculation of eigenvalue-eigenvector) to obtain the w vector of the weights

- **From the vector of the weights:**
  - i) weighted sum of the columns of the evaluation matrix
  - ii) calculation of utility u_j (j=1,2,...,r) and ranking of alternatives

Phase 3 – Subjectivity (the wife decision)

- Another possible Decision Maker → the wife

- Her structure of preferences

  The wifes gives much more importance to the life quality (and much less importance to the university prestige)

0.4
0.1
0.5

- Wife weighted sum and ranking

Roma	Berlin	Geneva	Moscow	Tokyo
7.3	5.7	6.9	2.9	4.8
(1°)	(3°)	(2°)	(5°)	(4°)

the choice of the wife would be for Roma

- Conclusion: → Subjectivity

  Though the use of the same data (evaluation matrix) different DM can make different choice → it depends on the structure of preference (vectors of weight)

The dominated alternatives cannot win given any preference structure

Dependance by weights

Overall Function: weighted sum of the utilities: $\max_j f_j = \sum_{i=1}^{p} w_i z_{ij}$

$z_{ij} = u_i(x_j)$

Example

	A1	A2	A3	w
Ob. 1	90	100	80	.20
Ob. 2	100	70	40	.20
Ob. 3	60	80	100	.50 ← w_3
Ob. 4	80	100	90	.10
(u_j)	76	84	83	

Alt. 2

The last row (overall utility of each alternative) determine the ranking:
the best alternative is A2 (utility = 84/100), followed by A3 and then by A1.

How the final choice depend on the weights ? (i.e. if w_3 changes …)

Sensitivity

The result depend on the weights w_i (and on something else …) →

$$\max_j f_i = \sum_{i=1}^{p} w_i z_{ij}$$

	A1	A2	A3		w
Ob. 1	90	100	80		.20
Ob. 2	100	70	40		.20
Ob. 3	60	80	100		.50
Ob. 4	80	100	90		.10

→ .51 → .52 → …

→ .49 → .48 → …

76	84	83	(f_j)

(best)

Changing the w_3 value:

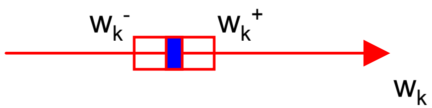
.50 → .51 …→ .55 | 79 | 88 | 88 | if $w_3 > 0.55$ the best is A3

.50 → .49 …→ .10 | 52 | 52 | 43 | if $w_3 < 0.10$ the best is A1

Sensitivity and RR (Rank Reversal)

- **Goal:**
  - To find the variations w_k^+ (increasing) e w_k^- (decreasing) of the weight of the k^{th} criteria w_k within which the choice doesn't change (cioè l'alternativa in 1ª posizione)
- **Method:**
  - keep all the weights w_i (i=1,...,m; $i \neq k$) except w_k with the values given by the DM and calculate the overall utilities of the alternatives as functions of w_k
  - calculate the values of w_k given which the alternative ranked first keep having the higher utility
- **Result:**
  - "narrow" range, little changes in the weight w_k

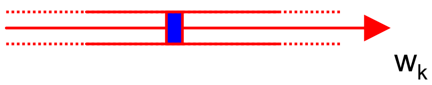
    would cause a different choice of the alternative



Narrow range

  - "wide" range, big changes in the weight w_k

    wouldn't cause a different choice of the alternative



Wide range

An example of sensitivity

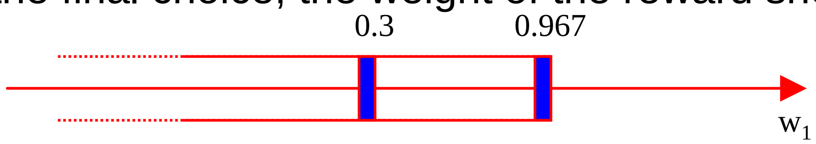
- **Does the choice of the professor change, if the weight w_1 change ?**
  - Vector of weights (non-normalized) $\begin{bmatrix} w_1 \\ 0.6 \\ 0.1 \end{bmatrix}$

- **Comparison of the utility when w_1 changes**

$$\left.\begin{array}{lcrcl} u_R & = & 5w_1 & + & 2.8 \\ u_B & = & 7w_1 & + & 5.8 \\ u_G & = & 10w_1 & + & 2.9 \\ u_M & = & 2w_1 & + & 3.9 \\ u_T & = & 7w_1 & + & 3.3 \end{array}\right\} \text{looking for the values } u \text{ so that} \quad \begin{array}{l} u_B \geq u_R \; ? \;\rightarrow \text{ALWAYS} \\ u_B \geq u_G \; ? \;\rightarrow \; . \quad . \quad . \\ u_B \geq u_M \; ? \;\rightarrow \text{USELESS} \\ u_B \geq u_T \; ? \;\rightarrow \text{USELESS} \end{array}$$

  - The choice (B) doesn't change for → $w_1 \leq 0.967$

- **Result**
  - To modify the final choice, the weight of the reward should be bigger then triple



Summary

- We discussed the decisional problem in a more **general frame**

- We saw the **three phases** needed to solve a multi-objectives or a multi-criteria problem, analyzing their own aspects

- We obtained a different result depending on the DM (the professor or his wife) → **subjective evaluation**

MultiCriteria Decision Making (MCDM)

Relevant characters of a MCDM problem:

- analyze the model of the specific application
  as a multi criteria analysis problem
- build utility functions (asking to the DM)
- build the vector of the weights (asking to the DM)
- document the subjectivity in the choice
  (it can not be removed, only documented);
- be supported by specific software

Test-1

In a multi objectives (criteria) problem:

- the 2 phase is the only one not dependant on the DM true / false

What does it mean to pass from indicators to objectives?

- to correct the results of the measurements true / false
- to modify the values of the indicators so that the maximum value become 1 and the minimum become 0 true / false
- to modify the indicators in utility value, in a conventional scale, i.e. from 0 (worst case) to 1 (best case) true / false

Test-2: sabbatical year

A inequality shows that the utility of Berlin is higher than the one of Genève: which one ?

1. $$7w_1 + 5.8 \geq 2w_1 + 3.9$$

2. $$7w_1 + 5.8 \geq 10w_1 + 2.9$$

3. $$7w_1 + 5.8 \geq 7w_1 + 3.3$$

Test-3: Pair comparison

The following matrix of pair comparisons is consistent.

$$\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 4 \\ 1/5 & 1/4 & 1 \end{pmatrix}$$

true / false

Test-4: Sensitivity

The sensitivity analysis consists in changing simultaneously all the weights in a multi criteria problem to check if some dominated solutions become efficient.

true / false