International Doctoral School Algorithmic Decision Theory: MCDA and MOO

Lecture 1: Multiobjective Optimization Theory

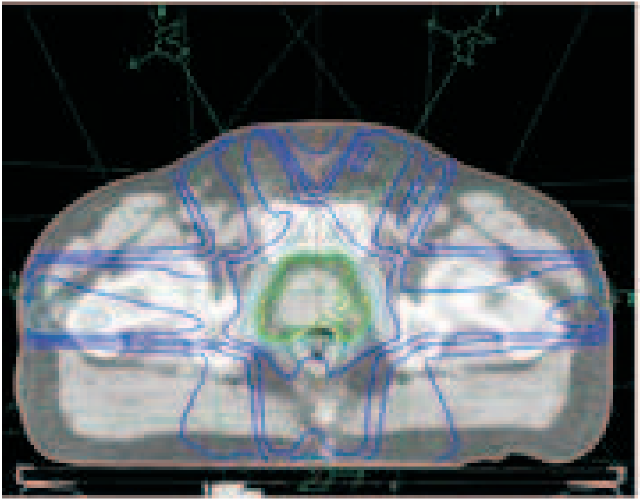
Matthias Ehrgott

Department of Engineering Science, The University of Auckland, New Zealand
Laboratoire d'Informatique de Nantes Atlantique, CNRS, Université de Nantes, France

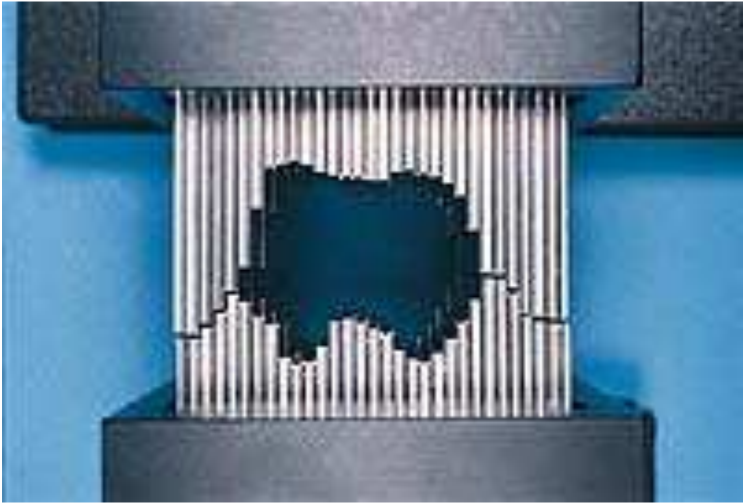
MCDA and MOO, Han sur Lesse, September 17 – 21 2007

Overview

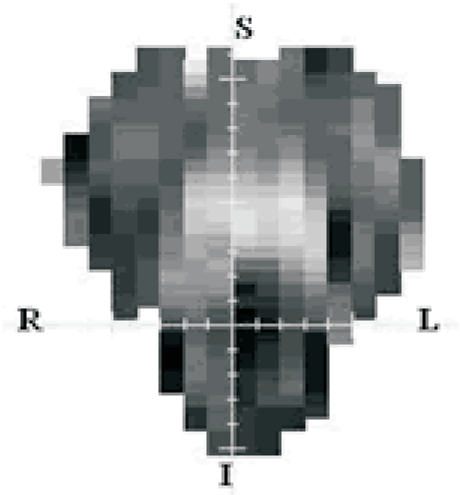
Irradiation from *a* directions or angles



Discretization of body into m voxels



Discretization of beam into b bixels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Modelling Intensity Optimization

- Given: a beam directions, b bixels per beam, dose deposition matrix $A \in \mathbb{R}^{m \times ab}$ with a_{ji} dose delivered to voxel j at unit intensity for bixel i
- Wanted: $x = (x_i : i = 1, \ldots, ab = n)$ intensity profiles such that dose $d = Ax$ satisfies the treatment goals
- Goal 1: Destroy the tumour, phyician prescribes lower and upper bound l_T and u_T for dose in tumour
- Goal 2: Avoid damage to healthy tissue, physician prescribes upper bounds u_C for critical organs and u_N for other normal tissue

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$$
\begin{array}{lrcl}
\min & (y_T, y_C, y_N) & & \\
\text{s.t.} & A_T x + y_T e & \geqq & l_T \\
 & A_T x & \leqq & u_T \\
 & A_C x - y_C e & \leqq & u_C \\
 & A_N x - y_N e & \leqq & u_N \\
 & y_C & \geqq & -u_C \\
 & y_N & \geqq & 0 \\
 & x & \geqq & 0
\end{array}
$$

$e = (1, \ldots, 1)$ is a vector of ones

Overview

BBC NEWS

Sunday, 4 August, 2002, 20:29 GMT 21:29 UK

Delays as Easyjet cancels 19 flights

Passengers with low-cost airline Easyjet are suffering delays after 19 flights in and out of Britain were cancelled.

The company blamed the move - which comes a week after passengers staged a protest sit-in at Nice airport - on crewing problems stemming from technical hitches with aircraft.

Crews caught up in the delays worked up to their maximum hours and then had to be allowed home to rest.

Mobilising replacement crews has been a problem as it takes time to bring people to airports from home. Standby crews were already being used and other staff are on holiday.

- Given: Schedule of m flights
- Wanted: Tours of duty, such that each flight is contained in exactly one tour of duty
- Goal: Small cost for operation

Assume all possible ToDs $i = 1, \ldots, n$ are known

$$a_{ji} = \begin{cases} 1 & \text{flight } j \text{ contained in ToD } i \\ 0 & \text{otherwise} \end{cases}$$

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$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad Ax &= e \\ x &\in \{0,1\}^n \end{aligned}$$

$$\begin{aligned} \min \quad & (c^T x, r^T x) \\ \text{s.t.} \quad Ax &= e \\ x &\in \{0,1\}^n \end{aligned}$$

Encourages tight connections, not robust

r_i is penalty for tight connection/expected delay in ToD i

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Overview

Mathematical Formulation

$$\begin{aligned} \min f(x) & \\ \text{subject to } g(x) & \leqq 0 \\ x & \in \mathbb{R}^n \end{aligned}$$

$x \in \mathbb{R}^n \longrightarrow n$ variables, $i = 1, \ldots, n$

$g : \mathbb{R}^n \to \mathbb{R}^m \longrightarrow m$ constraints, $j = 1, \ldots, m$

$f : \mathbb{R}^n \to \mathbb{R}^p \longrightarrow p$ objective functions, $k = 1, \ldots, p$

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Feasible Sets

- $X = \{x \in \mathbb{R}^n : g(x) \leqq 0\}$
  feasible set in decision space
- $Y = f(X) = \{f(x) : x \in X\}$
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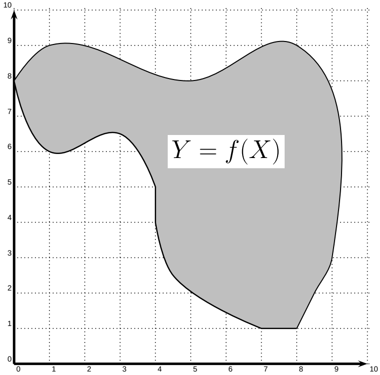

Let $y^1, y^2 \in \mathbb{R}^p$

- $y^1 \leqq y^2 \Leftrightarrow y^1_k \leqq y^2_k$ for $k = 1, \ldots, p$
  y^1 weakly dominates y^2
- $y^1 < y^2 \Leftrightarrow y^1_k < y^2_k$ for $k = 1, \ldots, p$
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- $y^1 \leq y^2 \Leftrightarrow y^1 \leqq y^2$ and $y^1 \neq y^2$
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- $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$
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Example

$$y^1 = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad y^2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \quad y^3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad y^4 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

- $y^3 < y^1$
- $y^4 \leq y^2$ but $y^4 \not< y^2$
- $y^4 \leqq y^1$
- y^3 and y^4 are incomparable

The fundamental difference between single and multiple objective optimization: Vectors cannot always be compared

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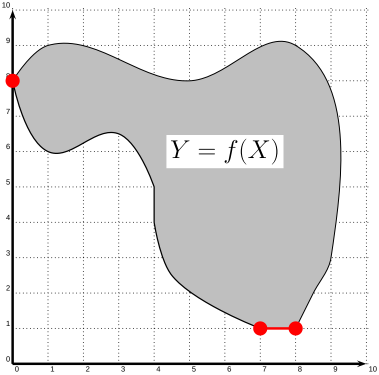
Lexicographic Optimality

- Individual minima
  $f_k(\hat{x}) \leqq f_k(x)$ for all $x \in X$
- Lexicographic optimality (1)
  $f(\hat{x}) \leqq_{lex} f(x)$ for all $x \in X$
- Lexicographic optimality (2)
  $f^\pi(\hat{x}) \leqq_{lex} f^\pi(x)$ for all $x \in X$
  and some permutation f^π of $(f_1, \dots, f_p)$



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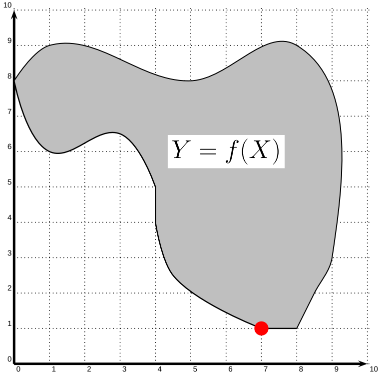

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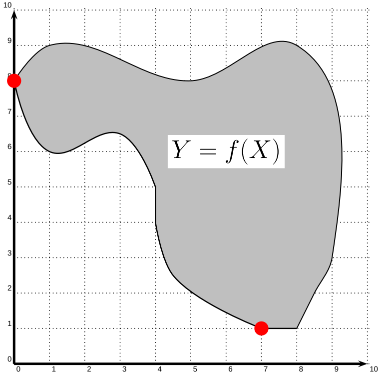

(Weakly) Efficient Solutions

- Weakly efficient solutions X_{wE}
  There is no x with $f(x) < f(\hat{x})$
  $f(\hat{x})$ is weakly nondominated
  $Y_{wN} := f(X_{wN})$
- Efficient solutions X_E
  There is no x with $f(x) \leq f(\hat{x})$
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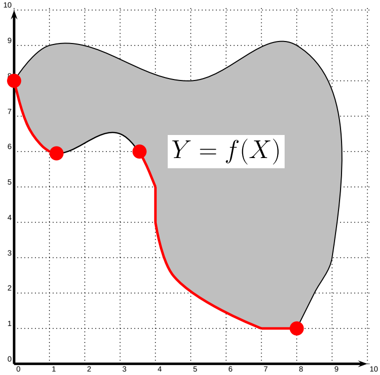
  $Y_N := f(X_E)$



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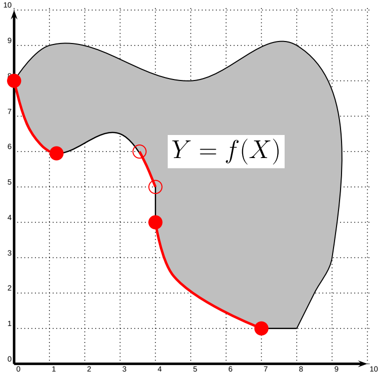

Properly Efficient Solutions

- Properly efficient solutions X_{pE}
  - $\hat{x}$ is efficient
  - There is $M > 0$ such that for each k and x with $f_k(x) < f_k(\hat{x})$ there is l with $f_l(\hat{x}) < f_l(x)$ and

$$\frac{f_k(\hat{x}) - f_k(x)}{f_l(x) - f_l(\hat{x})} \leqq M$$

$f(\hat{x})$ is properly nondominated
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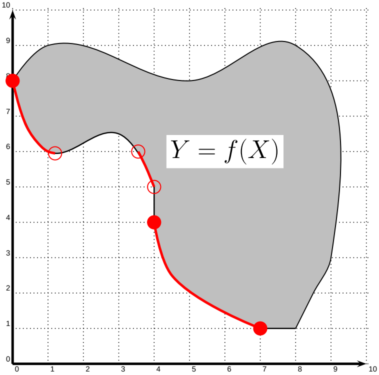

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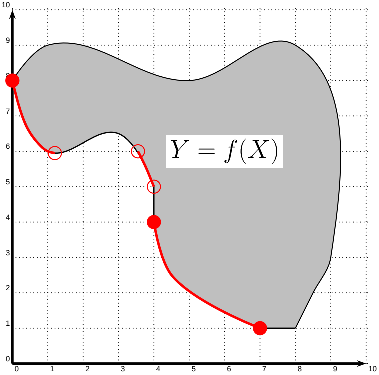
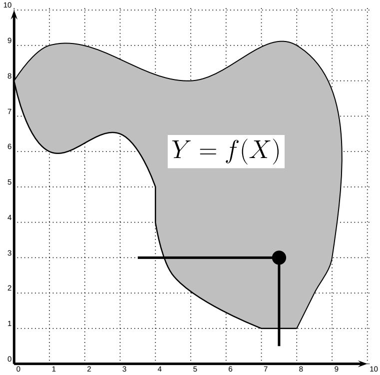
$f(\hat{x})$ is properly nondominated $Y_{pN} := f(X_{pE})$



Existence

- $Y_N \neq \emptyset$ if for some $y^0 \in Y$ the section $\left(y^0 - \mathbb{R}_{\geqq}\right) \cap Y \neq \emptyset$ is compact
- $X_E \neq \emptyset$ if X is compact and f is ($\mathbb{R}_{\geqq}$-semi-)continuous



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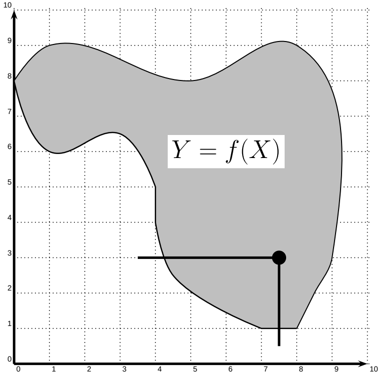
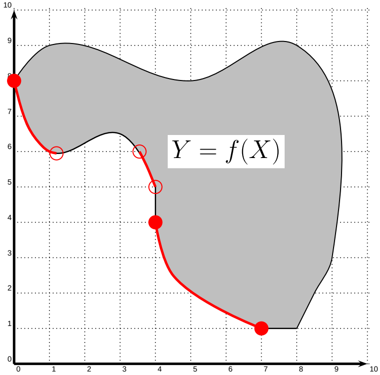


Relationships of Solution Sets

$X_{pE} \subseteq X_E \subseteq X_{wE}$
$Y_{pN} \subseteq Y_N \subseteq Y_{wN}$

It is possible that
$Y_N = Y$ but $Y_{pN} = \emptyset$
$Y = \left\{ (y_1, y_2) : y_2 = \frac{1}{y_1}, y_1 < 0 \right\}$



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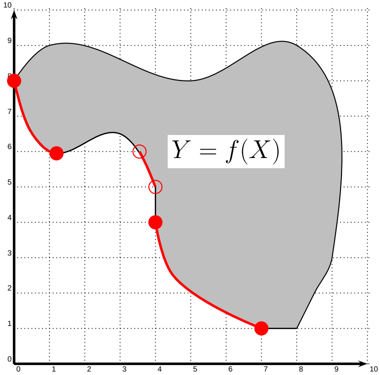

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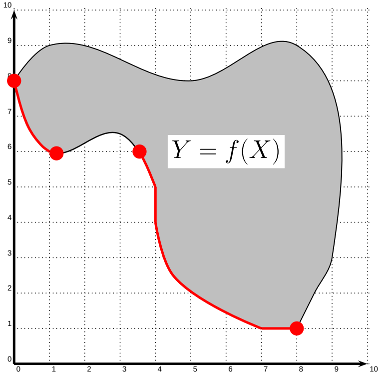

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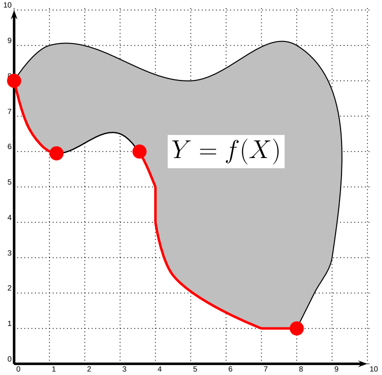

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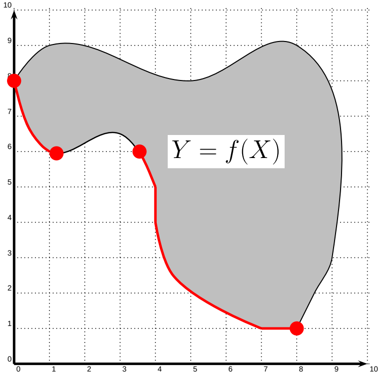

Ideal and Nadir Points

Ideal point y^I

- $y^I_k = \min\{y_k : y \in Y\}$

Nadir point y^N

- $y^N_k = \min\{y_k : y \in Y_N\}$

Anti-ideal point y^{AI}

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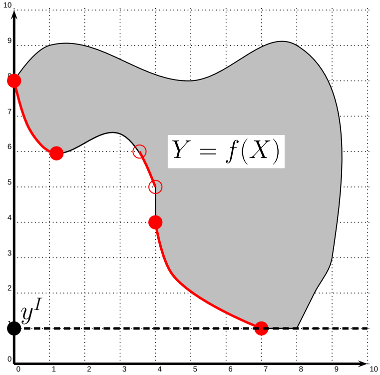
Utopia point y^U

- $y^U_k = y^I_k - \varepsilon_k$



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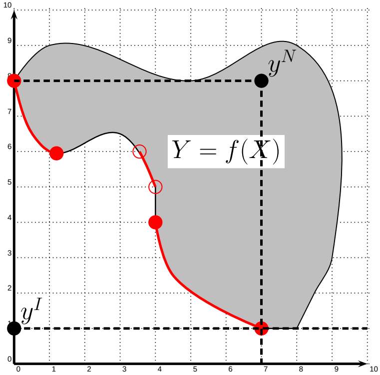

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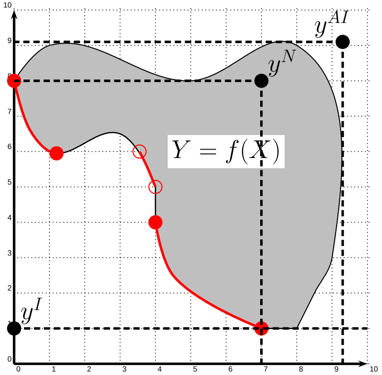

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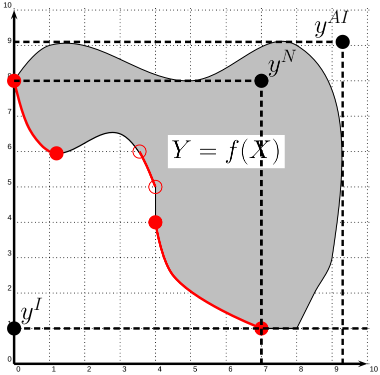
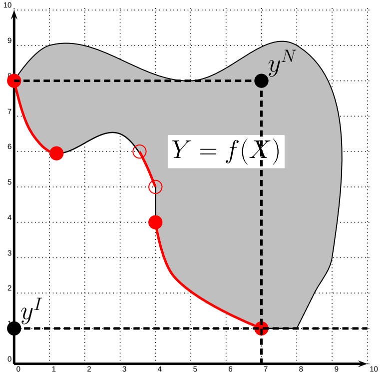


General Assumptions

- X_E is non-empty
- $y^I \neq y^N$



Overview

Principle of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations

- Correctness: Optimal solutions are (weakly, properly) efficient
- Completeness: All (weakly, properly) efficient solutions can be found

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Let $\lambda \geq 0$

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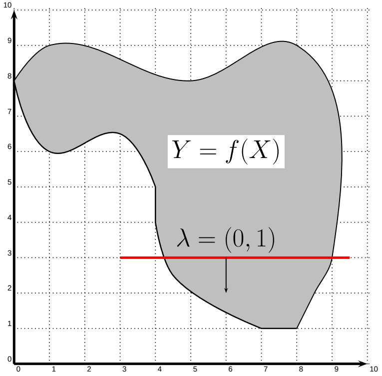

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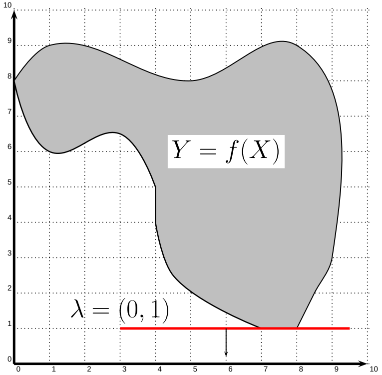

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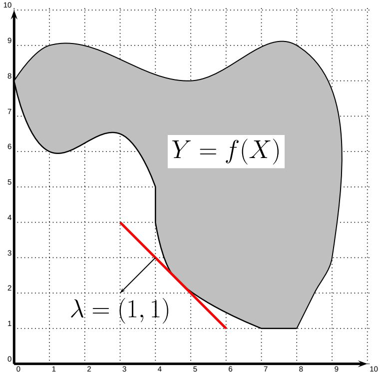

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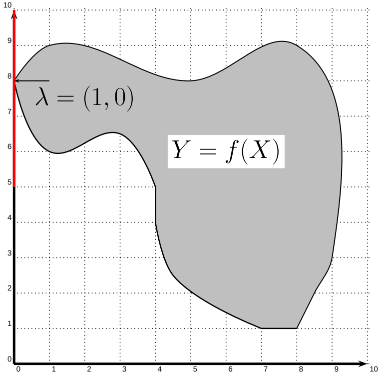

The Weighted Sum Method: Results

Theorem

Let $\hat{x}$ be an optimal solution of (1).

1. *If $\lambda \geq 0$ then $\hat{x} \in X_{wE}$.*
2. *If $\lambda \geq 0$ and $\hat{x}$ is unique then $\hat{x} \in X_E$.*
3. *If $\lambda > 0$ then $\hat{x} \in X_{pE}$.*

Proof.

1. By contradiction
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3. Construct M so that larger tradeoff would contradict optimality of $\hat{x}$

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The Weighted Sum Method: Results

Theorem (Geoffrion 1968)

Let X and f be such that $Y = f(X)$ is convex.

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Proof.

1. Apply separation theorem to $(Y + \mathbb{R}^p_{\geqq} - \hat{y})$ and $-\mathbb{R}^p_{>}$
2. Apply separation theorem to $\text{cl}\left(cone(Y + \mathbb{R}^p_{\geqq} - \hat{y})\right)$ and $-\mathbb{R}^p_{>}$ to show that weights are positive
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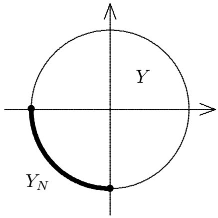
Nondominated and Properly Nondominated Points



$X_{sE} := \{x \in X : x \text{ is optimal solution to (1) for some } \lambda > 0\}$

Theorem

Assume that $Y + \mathbb{R}^p_{\geqq}$ is closed and convex. Then

$$Y_{pN} = f(X_{sE}) \subseteq Y_N \subseteq \operatorname{cl}(f(X_{sE})) = \operatorname{cl}(Y_{pN})$$

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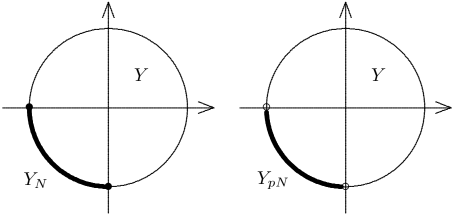

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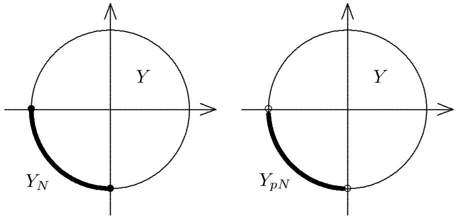

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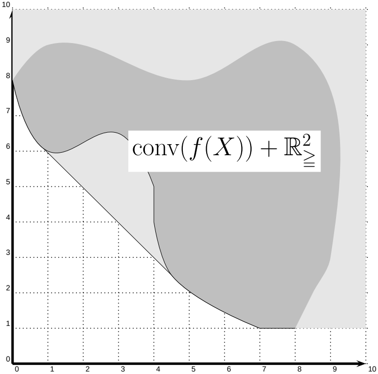
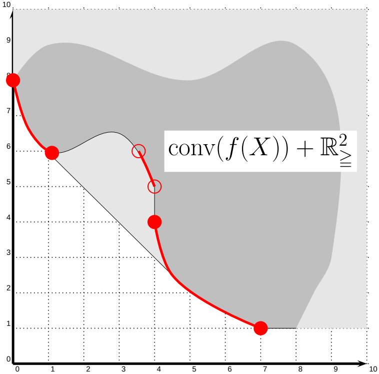
Supported Efficient Solutions

Supported efficient solutions are efficient solutions with $f(x)$ on the convex hull of Y

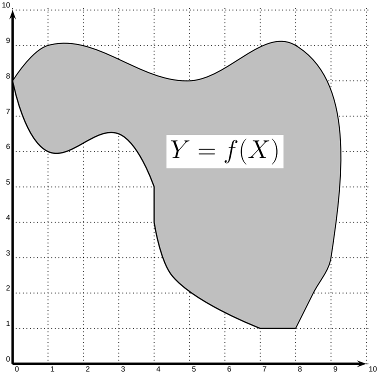




The ε-constraint Method

Let $\varepsilon \in \mathbb{R}^p$

$$\begin{aligned} \min \ & f_l(x) \\ \text{s.t.} \ & f_k(x) \leqq \varepsilon_k \quad k \neq l \\ & g_j(x) \leqq 0 \quad j = 1, \ldots, m \end{aligned} \tag{2}$$



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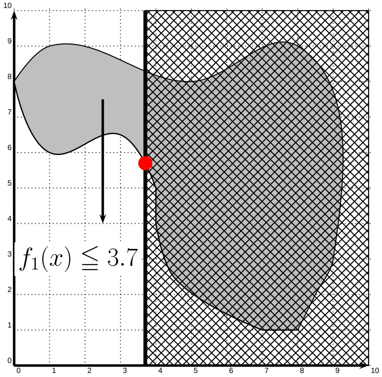

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1. *If $\hat{x}$ is an optimal solution to (2) then $\hat{x} \in X_{wE}$.*
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3. *$\hat{x} \in X_E$ if and only if there is $\hat{\varepsilon} \in \mathbb{R}^p$ such that $\hat{x}$ is an optimal solution to (2) for all $l = 1, \ldots, p$.*

Proof.

By contradiction and using $\varepsilon_l = f_l(\hat{x})$ □

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Let $\lambda \in \mathbb{R}^p_{\geq}$ and $\varepsilon \in \mathbb{R}^p$

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\min \sum_{k=1}^{p} \lambda_k f_k(x) & & & \qquad (3)\\
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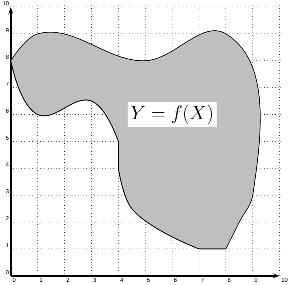

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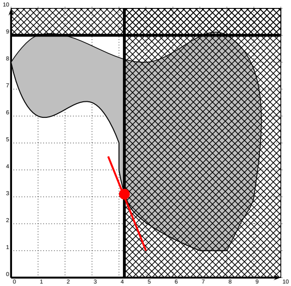

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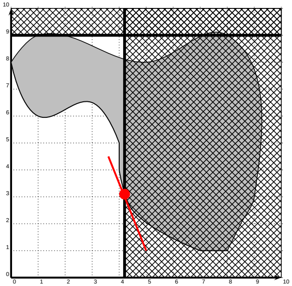

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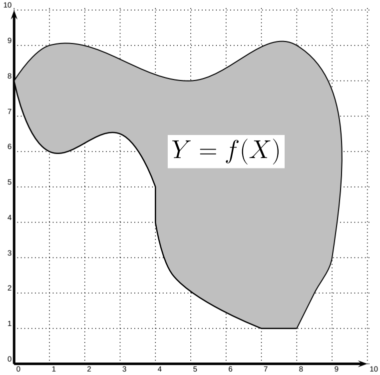
Compromise Solutions

Let $\lambda \in \mathbb{R}^p_{\geq}$ and $1 \leq q < \infty$

$$\min_{x \in X} \left(\sum_{k=1}^{p} \lambda_k (f_k(x) - y_k^I)^q \right)^{\frac{1}{q}} \qquad (4)$$

Let $\lambda \in \mathbb{R}^p_{\geq}$

$$\min_{x \in X} \max_{k=1,\dots,p} \lambda_k (f_k(x) - y_k^I) \qquad (5)$$



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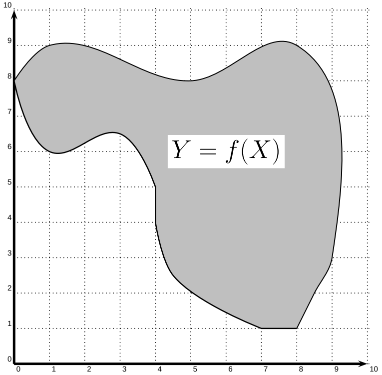

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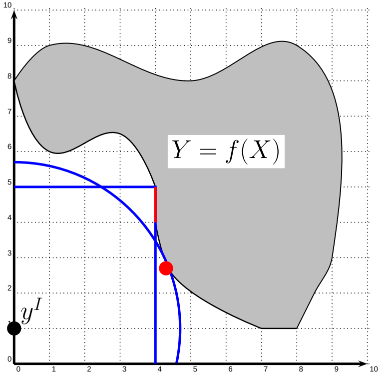

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1. *If $\hat{x}$ is a unique optimal solution to (4) or if $\lambda > 0$ then $\hat{x}$ is efficient.*
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Compromise Solutions

- For $q = 1$ (4) is the weighted sum scalarization
- If y^I is replaced by y^U in (4) stronger results follow: Solutions obtained are properly efficient, and Y_N is contained in the closure of the set of all solutions obtained (Sawaragi et al. 1985)
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