International Doctoral School Algorithmic Decision Theory: MCDA and MOO

Lecture 2: Multiobjective Linear Programming

Matthias Ehrgott

Department of Engineering Science, The University of Auckland, New Zealand
Laboratoire d'Informatique de Nantes Atlantique, CNRS, Université de Nantes, France

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Overview

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- Variables $x \in \mathbb{R}^n$
- Objective function Cx where $C \in \mathbb{R}^{p \times n}$
- Constraints $Ax = b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

$$\min \{Cx : Ax = b, x \geqq 0\} \tag{1}$$

$$X = \{x \in \mathbb{R}^n : Ax = b, x \geqq 0\}$$

is the feasible set in decision space

$$Y = \{Cx : x \in X\}$$

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Example

$$\min \begin{pmatrix} 3x_1 + x_2 \\ -x_1 - 2x_2 \end{pmatrix}$$

$$\begin{aligned} \text{subject to } x_2 &\leqq 3 \\ 3x_1 - x_2 &\leqq 6 \\ x &\geqq 0 \end{aligned}$$

$$C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

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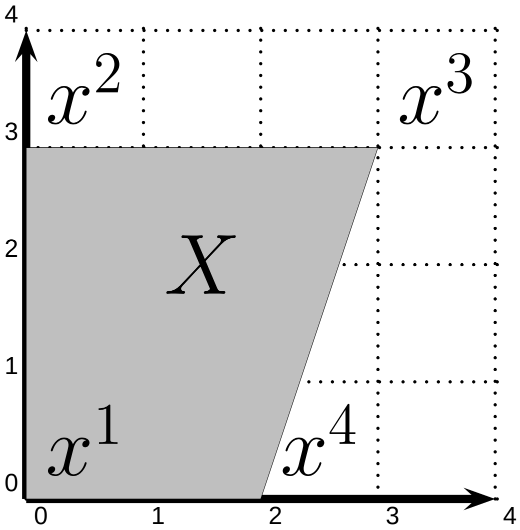
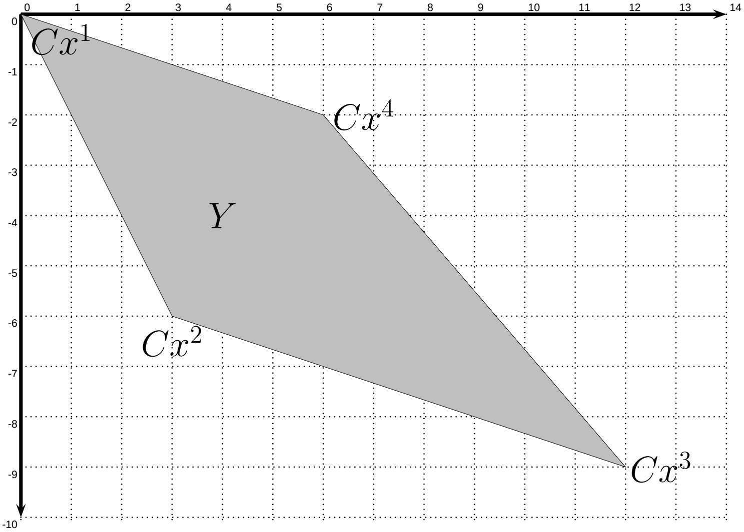


Feasible set in decision space



Feasible set in objective space

Definition

Let $\hat{x} \in X$ be a feasible solution of the MOLP (1) and let $\hat{y} = C\hat{x}$.

- $\hat{x}$ is called weakly efficient if there is no $x \in X$ such that $Cx < C\hat{x}$; $\hat{y} = C\hat{x}$ is called weakly nondominated.
- $\hat{x}$ is called efficient if there is no $x \in X$ such that $Cx \leq C\hat{x}$; $\hat{y} = C\hat{x}$ is called nondominated.
- $\hat{x}$ is called properly efficient if it is efficient and if there exists a real number $M > 0$ such that for all i and x with $c_i^T x < c_i^T \hat{x}$ there is an index j and $M > 0$ such that $c_j^T x > c_j^T \hat{x}$ and

$$\frac{c_i^T \hat{x} - c_i^T x}{c_j^T x - c_j^T \hat{x}} \leq M.$$

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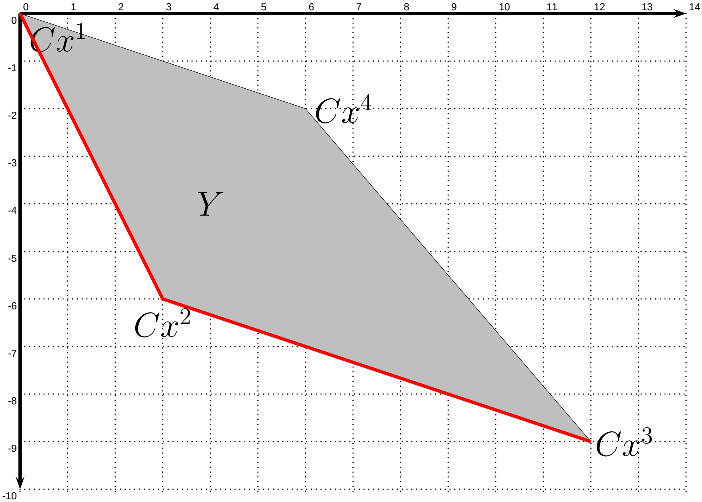
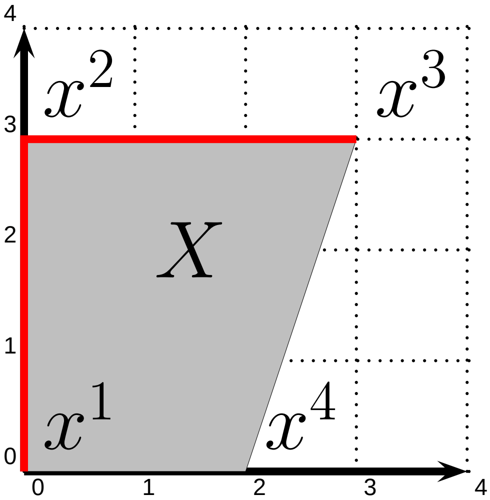


Nondominated set



Efficient set

Overview

- Let $\lambda_1, \ldots, \lambda_p \geqq 0$ and consider

$$LP(\lambda) \quad \min \sum_{k=1}^{p} \lambda_k c_k^T x = \min \lambda^T Cx$$
$$\text{subject to } Ax = b$$
$$x \geqq 0$$

with some vector $\lambda \geq 0$ (Why not $\lambda = 0$ or $\lambda \leq 0$?)

- $LP(\lambda)$ is a linear programme that can be solved by the Simplex method
- If $\lambda > 0$ then optimal solution of $LP(\lambda)$ is properly efficient
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- Converse also true, because Y convex

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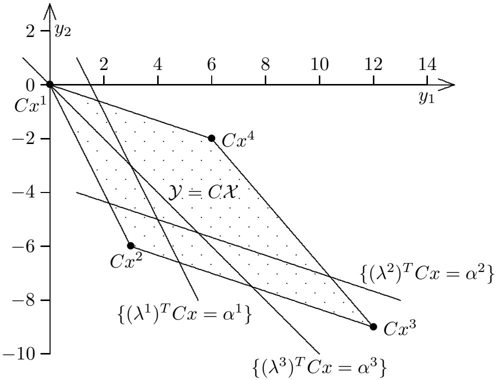
Illustration in objective space



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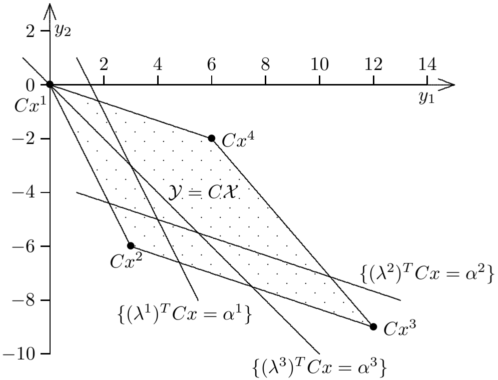

$\lambda^1 = (2,1), \lambda^2 = (1,3), \lambda^3 = (1,1)$

- $y \in \mathbb{R}^p$ satisfying $\lambda^T y = \alpha$ define a straight line (hyperplane)
- Since $y = Cx$ and $\lambda^T Cx$ is minimised, we push the line towards the origin (left and down)
- When the line only touches Y nondominated points are found
- Nondominated points Y_N are on the boundary of Y
- Y is convex polyhedron and has finite number of facets. Y_N consists of finitely many facets of Y. The normal of the facet can serve as weight vector λ

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Question: Can all efficient solutions be found using weighted sums?

If $\hat{x} \in X$ is efficient, does there exist $\lambda > 0$ such that $\hat{x}$ is optimal solution to

$$\min\{\lambda^T Cx : Ax = b, x \geqq 0\}?$$

Lemma

A feasible solution $x^0 \in X$ is efficient if and only if the linear programme

$$\begin{array}{lrcl} \max & e^T z & & \\ \text{subject to} & Ax & = & b \\ & Cx + Iz & = & Cx^0 \\ & x, z & \geqq & 0, \end{array} \qquad (2)$$

where $e^T = (1, \ldots, 1) \in \mathbb{R}^p$ and I is the $p \times p$ identity matrix, has an optimal solution $(\hat{x}, \hat{z})$ with $\hat{z} = 0$.

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Proof.

- LP is always feasible with $x = x^0, z = 0$ (and value 0)
- Let $(\hat{x}, \hat{z})$ be optimal solution
- If $\hat{z} = 0$ then $\hat{z} = Cx^0 - C\hat{x} = 0 \Rightarrow Cx^0 = C\hat{x}$
- There is no $x \in X$ such that $Cx \leq Cx^0$ because $(x, Cx^0 - Cx)$ would be better solution $\Rightarrow x^0$ efficient
- If $\hat{x}^0$ efficient there is no $x \in X$ with $Cx \leq Cx^0$
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A feasible solution $x^0 \in X$ is efficient if and only if the linear programme

$$\begin{array}{lrcl} \min & u^T b + w^T C x^0 & & \\ \textit{subject to} & u^T A + w^T C & \geqq & 0 \\ & w & \geqq & e \\ & u & \in & \mathbb{R}^m \end{array} \tag{3}$$

has an optimal solution $(\hat{u}, \hat{w})$ with $\hat{u}^T b + \hat{w}^T C x^0 = 0$.

Proof.

The LP (3) is the dual of the LP (2) □

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Theorem

A feasible solution $x^0 \in X$ is an efficient solution of the MOLP (1) if and only if there exists a $\lambda \in \mathbb{R}^p_>$ such that

$$\lambda^T C x^0 \leq \lambda^T C x \tag{4}$$

for all $x \in X$.

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Proof.

- Let $x^0 \in X_E$
- By Lemma 4 LP (3) has an optimal solution $(\hat{u}, \hat{w})$ such that

$$\hat{u}^T b = -\hat{w}^T C x^0 \tag{5}$$

- $\hat{u}$ is also an optimal solution of the LP

$$\min\left\{u^T b : u^T A \geqq -\hat{w}^T C\right\}, \tag{6}$$

which is (3) with $w = \hat{w}$ fixed

- $\Rightarrow$ There is an optimal solution of the dual of (6)

$$\max\left\{-\hat{w}^T C x : Ax = b,\ x \geqq 0\right\} \tag{7}$$

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Proof.

- By weak duality $u^T b \geq -\hat{w}^T Cx$ for all feasible solutions u of (6) and for all feasible solutions x of (7)
- We already know that $\hat{u}^T b = -\hat{w}^T Cx^0$ from (5)
- $\Rightarrow x^0$ is an optimal solution of (7)
- Note that (7) is equivalent to

$$\min \left\{ \hat{w}^T Cx : Ax = b, \ x \geqq 0 \right\}$$

with $\hat{w} \geqq e > 0$ from the constraints in (3)

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with $\hat{w} \geqq e > 0$ from the constraints in (3)

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- By weak duality $u^T b \geq -\hat{w}^T Cx$ for all feasible solutions u of (6) and for all feasible solutions x of (7)
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Overview

- Modification of the Simplex algorithm for LPs with two objectives

$$\begin{array}{lrcl} \min & ((c^1)^T x, (c^2)^T x) & & \\ \text{subject to} & Ax & = & b \\ & x & \geqq & 0 \end{array} \tag{8}$$

- We can find all efficient solutions by solving the parametric LP

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for all $\lambda = (\lambda_1, \lambda_2) > 0$

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- LPs with one parameter $0 \leqq \lambda \leqq 1$ and parametric objective
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- Let $\mathcal{B}$ be a feasible basis
- Recall reduced cost $\bar{c}_{\mathcal{N}} = c_{\mathcal{N}} - c_{\mathcal{B}}^T B^{-1} N$
- Reduced cost for the parametric LP

$$\bar{c}(\lambda) = \lambda \bar{c}^1 + (1 - \lambda)\bar{c}^2 \qquad (10)$$

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- $\mathcal{I} = \{i \in \mathcal{N} : \bar{c}_i^2 < 0, \bar{c}_i^1 \geqq 0\}$
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$$\lambda' := \max_{i \in \mathcal{I}} \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}. \tag{11}$$
- $\hat{\mathcal{B}}$ is optimal for all $\lambda \in [\lambda', \hat{\lambda}]$
- As soon as $\lambda < \lambda'$ new bases become optimal
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Algorithm (Parametric Simplex for biobjective LPs)

Input: Data A, b, C for a biobjective LP.

Phase I: Solve the auxiliary LP for Phase I using the Simplex algorithm. If the optimal value is positive, STOP, $X = \emptyset$. Otherwise let $\mathcal{B}$ be an optimal basis.

Phase II: Solve the LP (9) for $\lambda = 1$ starting from basis $\mathcal{B}$ found in Phase I yielding an optimal basis $\hat{\mathcal{B}}$. Compute $\tilde{A}$ and $\tilde{b}$.

Phase III: While $\mathcal{I} = \{i \in \mathcal{N} : \bar{c}_i^2 < 0, \bar{c}_i^1 \geq 0\} \neq \emptyset$.

$$\lambda := \max_{i \in \mathcal{I}} \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}.$$

$$s \in \operatorname{argmax} \left\{ i \in \mathcal{I} : \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2} \right\}.$$

$$r \in \operatorname{argmin} \left\{ j \in \mathcal{B} : \frac{\tilde{b}_j}{\tilde{A}_{js}}, \tilde{A}_{js} > 0 \right\}.$$

Let $\mathcal{B} := (\mathcal{B} \setminus \{r\}) \cup \{s\}$ and update $\tilde{A}$ and $\tilde{b}$.

End while.

Output: Sequence of λ-values and sequence of optimal BFSs.

Overview

Example

$$\begin{array}{ll} \min & \begin{pmatrix} 3x_1 + x_2 \\ -x_1 - 2x_2 \end{pmatrix} \\ \text{subject to} & \begin{array}{rcl} x_2 & \leqq & 3 \\ 3x_1 - x_2 & \leqq & 6 \\ x & \geqq & 0 \end{array} \end{array}$$

$LP(\lambda)$

$$\begin{array}{lrcrcrcrcl} \min & (4\lambda - 1)x_1 & + & (3\lambda - 2)x_2 & & & & & & \\ \text{subject to} & & & x_2 & + & x_3 & & & = & 3 \\ & 3x_1 & - & x_2 & & & + & x_4 & = & 6 \\ & & & & & & & x & \geqq & 0. \end{array}$$

- Use Simplex tableaus showing reduced cost vectors $\bar{c}^1$ and $\bar{c}^2$
- Optimal basis for $\lambda = 1$ is $\mathcal{B} = \{3, 4\}$, optimal basic feasible solution $x = (0, 0, 3, 6)$
- Start with Phase 3

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Iteration 1:

$\bar{c}^1$	3	1	0	0	0
$\bar{c}^2$	-1	-2	0	0	0
x_3	0	$\boxed{1}$	1	0	3
x_4	3	-1	0	1	6

$\lambda = 1, \bar{c}(\lambda) = (3, 1, 0, 0), \mathcal{B}^1 = \{3, 4\}, x^1 = (0, 0, 3, 6)$
$\mathcal{I} = \{1, 2\}, \lambda' = \max\left\{\frac{1}{3+1}, \frac{2}{1+2}\right\} = \frac{2}{3}$
$s = 2, r = 3$

Iteration 2

$\bar{c}^1$	3	0	-1	0	-3
$\bar{c}^2$	-1	0	2	0	6
x_2	0	1	1	0	3
x_4	$\boxed{3}$	0	1	1	9

$\lambda = 2/3, \bar{c}(\lambda) = (5/3, 0, 0, 0)$, $\mathcal{B}^2 = \{2, 4\}$, $x^2 = (0, 3, 0, 9)$
$\mathcal{I} = \{1\}$, $\lambda' = \max\left\{\frac{1}{3+1}\right\} = \frac{1}{4}$
$s = 1$, $r = 4$

Iteration 3

$\bar{c}^1$	0	0	-2	-1	-12
$\bar{c}^2$	0	0	7/3	1/3	9
x_2	0	1	1	0	3
x_1	1	0	1/3	1/3	3

$\lambda = 1/4, \bar{c}(\lambda) = (0, 0, 5/4, 0)$, $\mathcal{B}^3 = \{1, 2\}$, $x^3 = (3, 3, 0, 0)$
$\mathcal{I} = \emptyset$

- Weight values $\lambda^1 = 1, \lambda^2 = 2/3, \lambda^3 = 1/4, \lambda^4 = 0$
- Basic feasible solutions x^1, x^2, x^3
- In each iteration $\bar{c}(\lambda)$ can be calculated with the previous and current $\bar{c}^1$ and $\bar{c}^2$.
- Basis $\mathcal{B}^1 = (3,4)$ and BFS $x^1 = (0,0,3,6)$ are optimal for $\lambda \in [2/3, 1]$.
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- Values $\lambda = 2/3$ and $\lambda = 1/4$ correspond to weight vectors $(2/3, 1/3)$ and $(1/4, 3/4)$
- Contour lines for weighted sum objectives in decision are parallel to efficient edges

$$\frac{2}{3}(3x_1 + x_2) + \frac{1}{3}(-x_1 - 2x_2) = \frac{5}{3}x_1$$
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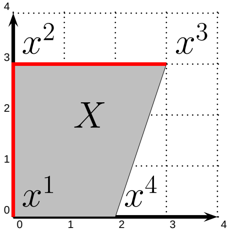

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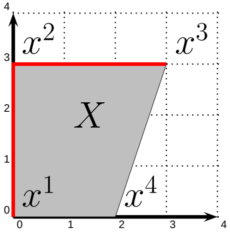

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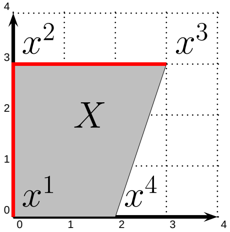
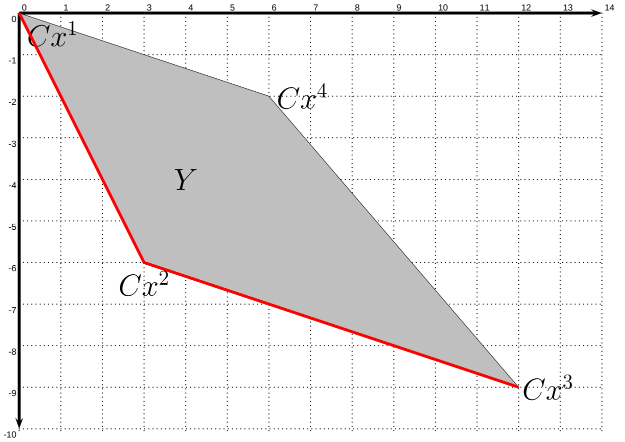


Feasible set in decision space and efficient set

- Weight vectors $(2/3, 1/3)$ and $(1/4, 3/4)$ are normal to nondominated edges



Objective space and nondominated set

- Algorithm finds all nondominated extreme points in objective space and one efficient bfs for each of those
- Algorithm does not find all efficient solutions just as Simplex algorithm does not find all optimal solutions of an LP

Example

$$\begin{array}{lll} \min & (x_1, x_2)^T & \\ \text{subject to} & 0 \leq x_i \leq 1 & i = 1, 2, 3 \end{array}$$

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Overview

- $\min\{Cx : Ax = b, x \geqq 0\}$
- Let $\mathcal{B}$ be a basis and $\bar{C} = C - C_{\mathcal{B}} A_{\mathcal{B}}^{-1} A$ and $R = \bar{C}_{\mathcal{N}}$
- How to calculate "critical" λ if $p > 2$?
- At $\mathcal{B}_1 : \bar{C}_{\mathcal{N}} = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}, \lambda' = 2/3, \lambda = (2/3, 1/3)^T$ and $\lambda^T \bar{C}_{\mathcal{N}} = (5/3, 0)^T$
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Lemma

If $\mathcal{X}_E \neq \emptyset$ then $\mathcal{X}$ has an efficient basic feasible solution.

Proof.

- There is some $\lambda > 0$ such that $\min_{x \in \mathcal{X}} \lambda^T Cx$ has an optimal solution
- Thus $LP(\lambda)$ has an optimal basic feasible solution solution, which is an efficient solution of the MOLP

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Definition

1. A feasible basis $\mathcal{B}$ is called efficient basis if $\mathcal{B}$ is an optimal basis of LP(λ) for some $\lambda \in \mathbb{R}^p_>$.
2. Two bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called adjacent if one can be obtained from the other by a single pivot step.
3. Let $\mathcal{B}$ be an efficient basis. Variable $x_j, j \in \mathcal{N}$ is called efficient nonbasic variable at $\mathcal{B}$ if there exists a $\lambda \in \mathbb{R}^p_>$ such that $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$, where r^j is the column of R corresponding to variable x_j.
4. Let $\mathcal{B}$ be an efficient basis and let x_j be an efficient nonbasic variable. Then a feasible pivot from $\mathcal{B}$ with x_j entering the basis (even with negative pivot element) is called an efficient pivot with respect to $\mathcal{B}$ and x_j

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- Therefore R always contains positive and negative entries

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Let $\mathcal{B}$ be an efficient basis. There exists an efficient nonbasic variable at $\mathcal{B}$.

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Let $\mathcal{B}$ be an efficient basis. There exists an efficient nonbasic variable at $\mathcal{B}$.

It is not possible to define efficient nonbasic variables by the existence of a column in R with positive and negative entries

Example

$$R = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

- $\lambda^T r^2 = 0$ requires $\lambda_2 = 2\lambda_1$
- $\lambda^T r^1 \geqq 0$ requires $-\lambda_1 \geqq 0$, an impossibility for $\lambda > 0$

Lemma

Let $\mathcal{B}$ be an efficient basis and x_j be an efficient nonbasic variable. Then any efficient pivot from $\mathcal{B}$ leads to an adjacent efficient basis $\hat{\mathcal{B}}$.

Proof.

- x_j efficient entering variable at basis $\mathcal{B}$
- $\Rightarrow$ there is $\lambda \in \mathbb{R}^p_>$ with $\lambda^T R \geqq 0$ and $\lambda^T r^j = 0$
- $\Rightarrow$ x_j is nonbasic variable with reduced cost 0 in LP(λ)
- Reduced costs of LP(λ) do not change after a pivot with x_j entering
- Let $\hat{\mathcal{B}}$ be the resulting basis with feasible pivot and x_j entering
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How to identify efficient nonbasic variables?

Theorem

Let $\mathcal{B}$ be an efficient basis and let x_j be a nonbasic variable. Variable x_j is an efficient nonbasic variable if and only if the LP

$$\begin{array}{lrcl} \max & e^t v & & \\ \text{subject to} & Rz - r^j\delta + Iv & = & 0 \\ & z, \delta, v & \geqq & 0 \end{array} \qquad (12)$$

has an optimal value of 0.

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Need to show: ALL efficient bases can be reached by efficient pivots

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Two efficient bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called connected if one can be obtained from the other by performing only efficient pivots.

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- Assume $\hat{B}$ is first basis (for $\Phi = 1$)
- After several pivots get an optimal basis $\tilde{\mathcal{B}}$ for LP(λ)
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- $\mathcal{B}$ and $\hat{\mathcal{B}}$ two efficient bases
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$$c(\Phi) = \Phi \hat{\lambda}^T C + (1 - \Phi)\lambda^T C \tag{16}$$

- Assume $\hat{B}$ is first basis (for $\Phi = 1$)
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- 3 cases
  - $\mathcal{X} = \emptyset$, infeasibility
  - $\mathcal{X} \neq \emptyset$ but $\mathcal{X}_E = \emptyset$, no efficient solutions
  - $\mathcal{X} \neq \emptyset, \mathcal{X}_E \neq \emptyset$

  result in three phase multiobjective Simplex algorithm
- Phase I: Solve $\min\{e^T z : Ax + Iz = b, x \geqq 0, z \geqq 0\}$
  If optimal value is nonzero, $X = \emptyset$
  Otherwise find bfs of $Ax = b, x \geqq 0$ from optimal solution
- Phase II: Find efficient bfs by solving appropriate $LP(\lambda)$
  Note: $LP(\lambda)$ can be unbounded even if $X_E \neq \emptyset$
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  If unbounded then $X_E = \emptyset$
  Otherwise find optimal $\hat{w}$ and solve
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  Optimal bfs x^1 exists and is efficient bfs for MOLP
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Algorithm (Multicriteria Simplex Algorithm.)

Input: *Data* A, b, C *of an MOLP.*

Initialization: *Set* $\mathcal{L}_1 := \emptyset$, $\mathcal{L}_2 := \emptyset$.

Phase I: *Solve the LP* $\min\{e^T z : Ax + Iz = b, x, z \geqq 0\}$.
If the optimal value of this LP is nonzero, STOP, $\mathcal{X} = \emptyset$.
Otherwise let x^0 *be a basic feasible solution of the MOLP.*

Phase II: *Solve the LP*
$\min\{u^T b + w^T C x^0 : u^T A + w^T C \geqq 0, w \geqq e\}$.
If the problem is infeasible, STOP, $\mathcal{X}_E = \emptyset$.
Otherwise let $(\hat{u}, \hat{w})$ *be an optimal solution.*
Find an optimal basis $\mathcal{B}$ *of the LP* $\min\{\hat{w}^T C x : Ax = b, x \geqq 0\}$.
$\mathcal{L}_1 := \{\mathcal{B}\}$, $\mathcal{L}_2 := \emptyset$.

Algorithm

Phase III:

While $\mathcal{L}_1 \neq \emptyset$

Choose $\mathcal{B}$ *in* $\mathcal{L}_1$, *set* $\mathcal{L}_1 := \mathcal{L}_1 \setminus \{\mathcal{B}\}$, $\mathcal{L}_2 := \mathcal{L}_2 \cup \{\mathcal{B}\}$.
Compute $\tilde{A}$, $\tilde{b}$, *and* R *according to* $\mathcal{B}$.
$\mathcal{EN} := \mathcal{N}$.
For all $j \in \mathcal{N}$.

Solve the LP $\max\{e^T v : Ry - r^j \delta + Iv = 0; y, \delta, v \geqq 0\}$.
If this LP is unbounded $\mathcal{EN} := \mathcal{EN} \setminus \{j\}$.

End for
For all $j \in \mathcal{EN}$.

For all $i \in \mathcal{B}$.

If $\mathcal{B}' = (\mathcal{B} \setminus \{i\}) \cup \{j\}$ *is feasible and* $\mathcal{B}' \notin \mathcal{L}_1 \cup \mathcal{L}_2$
then $\mathcal{L}_1 := \mathcal{L}_1 \cup \mathcal{B}'$.

End for.

End for.

End while.

Output: $\mathcal{L}_2$.

Example

- There can be exponentially many efficient bfs
- $$\begin{array}{llcll} \min & x_i & & & i = 1, \ldots, n \\ \min & -x_i & & & i = 1, \ldots, n \\ \text{subject to} & x_i & \leqq & 1 & i = 1, \ldots, n \\ & -x_i & \leqq & 1 & i = 1, \ldots, n. \end{array}$$
- n variables, $m = 2n$ constraints, $p = 2n$ objective functions
- all 2^n extreme points of the feasible set are efficient

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Overview

Example

$$\begin{array}{lrcl} \min & \begin{pmatrix} 3x_1 + x_2 \\ -x_1 - 2x_2 \end{pmatrix} & & \\ \text{subject to} & x_2 & \leqq & 3 \\ & 3x_1 - x_2 & \leqq & 6 \\ & x & \geqq & 0 \end{array}$$

$LP(\lambda)$

$$\begin{array}{lrcrcrcrcl} \min & (4\lambda - 1)x_1 & + & (3\lambda - 2)x_2 & & & & & & \\ \text{subject to} & & & x_2 & + & x_3 & & & = & 3 \\ & 3x_1 & - & x_2 & & & + & x_4 & = & 6 \\ & & & & & & & x & \geqq & 0. \end{array}$$

- Phase I: MOLP is feasible
  $x^0 = (0, 0)$
- Phase II: Optimal weight
  $\hat{w} = (1, 1)$
- Phase II: First efficient solution
  $x^2 = (0, 3)$
- Phase III: Efficient entering variables s^1, x^2
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  $x^1 = (0, 0), x^3 = (3, 3)$
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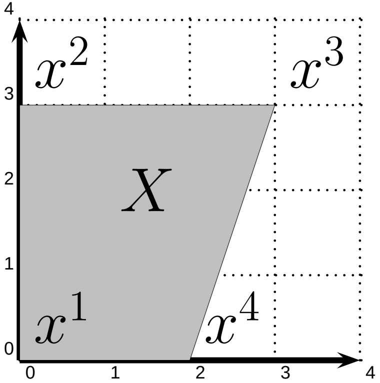

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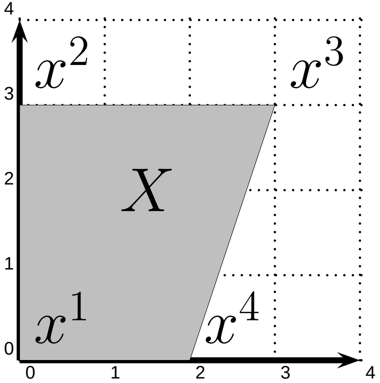

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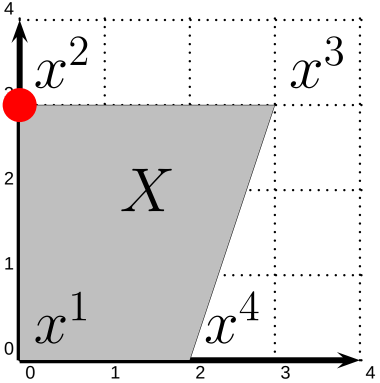

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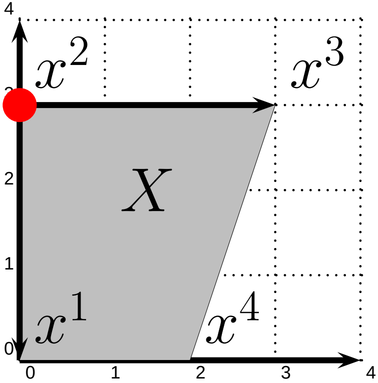

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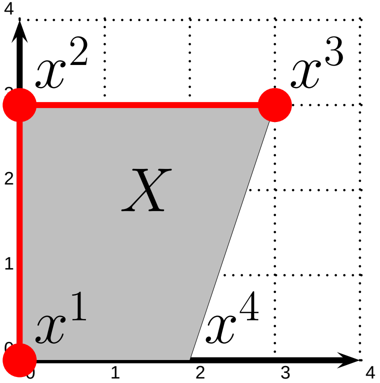

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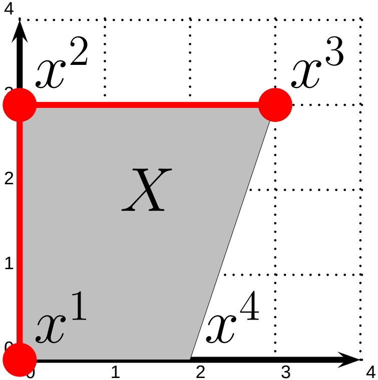

Example

$$\begin{array}{lrrrcl}
\min & -x_1 & -2x_2 & & & \\
\min & -x_1 & & +2x_3 & & \\
\min & x_1 & & -x_3 & & \\
\text{subject to} & x_1 & +x_2 & & \leq & 1 \\
 & & x_2 & & \leq & 2 \\
 & x_1 & -x_2 & +x_3 & \leq & 4.
\end{array}$$

Slack variables x_4, x_5, x_6 introduced to write the constraints in equality form $Ax = b$

- Phase I: $\mathcal{B} = \{4, 5, 6\}$ is a basis with bfs $x^0 = (0, 0, 0, 1, 2, 4)$
- Phase II:

$$\begin{array}{lll} \min & u_1 + 2u_2 + 4u_3 & \\ \text{subject to} & u^T \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} + w^T \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} & \geqq 0 \\ & w & \geqq e \end{array}$$

$\hat{w} = (1, 1, 1)$
$\min\{-x_1 - 2x_2 + x_3 : Ax = b, x \geqq 0\}$
$\mathcal{B}^1 = \{2, 5, 6\}$, $x^1 = (0, 1, 0, 0, 1, 3)$ is efficient bfs,
$\mathcal{L}_1 = \{\{2, 5, 6\}\}$

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Phase III

Iteration 1:

$\mathcal{B}^1 = \{2, 5, 6\}$, $\mathcal{L}_1 = \emptyset$, $\mathcal{L}_2 = \{\{2, 5, 6\}\}$

$\bar{c}^1$	1	0	0	2	0	0	2
$\bar{c}^2$	-1	0	2	0	0	0	0
$\bar{c}^3$	1	0	-1	0	0	0	0
x_2	1	1	0	1	0	0	1
x_5	-1	0	0	-1	1	0	1
x_6	2	0	1	1	0	1	5

$\mathcal{EN} := \{1, 3, 4\}$

- Check x_1

1	1	2	-1	0	0	0	0
1	0	2	-1	1	0	0	0
-1	2	0	1	0	1	0	0
1	-1	0	-1	0	0	1	0

LP has optimal solution, x_1 is efficient

- Check x_3

1	1	2	-1	0	0	0	0
1	0	2	0	1	0	0	0
-1	2	0	-2	0	1	0	0
1	-1	0	1	0	0	1	0

LP has optimal solution, x_3 is efficient

- Check x_1

1	1	2	-1	0	0	0	0
1	0	2	-1	1	0	0	0
-1	2	0	[1]	0	1	0	0
1	-1	0	-1	0	0	1	0

LP has optimal solution, x_1 is efficient

- Check x_3

1	1	2	-1	0	0	0	0
1	0	2	0	1	0	0	0
-1	2	0	-2	0	1	0	0
1	-1	0	[1]	0	0	1	0

LP has optimal solution, x_3 is efficient

- Check x_4

1	1	2	-2	0	0	0	0
1	0	2	-2	1	0	0	0
-1	2	0	0	0	1	0	0
1	-1	0	0	0	0	1	0

LP is unbounded, x_4 is not efficient

- $\mathcal{EN} = \{1, 3\}$
Feasible pivot x_1 enters and x_2 leaves: basis $\mathcal{B}^2 = \{1, 5, 6\}$
Feasible pivot x_3 enters and x_6 leaves: basis $\mathcal{B}^3 = \{2, 3, 5\}$
$\mathcal{L}_1 := \{\{1, 5, 6\}, \{2, 3, 5\}\}$

- Check x_4

1	1	2	-2	0	0	0	0
1	0	2	-2	1	0	0	0
-1	2	0	0	0	1	0	0
1	-1	0	0	0	0	1	0

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Iteration 2:
$\mathcal{B}^2 = \{1,5,6\}$ with BFS $x^2 = (1,0,0,0,2,3)$
$\mathcal{L}_1 = \{\{2,3,5\}\}$, $\mathcal{L}_2 = \{\{2,5,6\},\{2,3,5\}\}$

$\bar{c}^1$	0	-1	0	1	0	0	1
$\bar{c}^2$	0	1	2	1	0	0	1
$\bar{c}^3$	0	-1	-1	-1	0	0	-1
x_2	1	1	0	1	0	0	1
x_5	0	1	0	0	1	0	2
x_6	0	-2	1	-1	0	1	3

$\mathcal{EN} = \{2,3,4\}$

- Check x_2: Leads back to $\mathcal{B}^1 = (2, 5, 6)$
- Check x_3:

-1	1	1	-1	0	0	0	0
-1	0	1	0	1	0	0	0
1	2	1	-2	0	1	0	0
-1	-1	-1	1	0	0	1	0

  x_3 not efficient
- Check x_4

-1	1	1	-1	0	0	0	0
-1	0	1	-1	1	0	0	0
1	2	1	-1	0	1	0	0
-1	-1	-1	1	0	0	1	0

  x_4 not efficient
- $\mathcal{EN} = \emptyset$

- Check x_2: Leads back to $\mathcal{B}^1 = (2, 5, 6)$
- Check x_3:

-1	1	1	-1	0	0	0	0
-1	0	1	0	1	0	0	0
1	2	1	-2	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_3 not efficient

- Check x_4

-1	1	1	-1	0	0	0	0
-1	0	1	-1	1	0	0	0
1	2	1	-1	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_4 not efficient

- $\mathcal{EN} = \emptyset$

- Check x_2: Leads back to $\mathcal{B}^1 = (2, 5, 6)$
- Check x_3:

-1	1	1	-1	0	0	0	0
-1	0	1	0	1	0	0	0
1	2	1	-2	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_3 not efficient

- Check x_4

-1	1	1	-1	0	0	0	0
-1	0	1	-1	1	0	0	0
1	2	1	-1	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_4 not efficient

- $\mathcal{EN} = \emptyset$

- Check x_2: Leads back to $\mathcal{B}^1 = (2, 5, 6)$
- Check x_3:

-1	1	1	-1	0	0	0	0
-1	0	1	0	1	0	0	0
[1]	2	1	-2	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_3 not efficient

- Check x_4

-1	1	1	-1	0	0	0	0
-1	0	1	-1	1	0	0	0
[1]	2	1	-1	0	1	0	0
-1	-1	-1	1	0	0	1	0

x_4 not efficient

- $\mathcal{EN} = \emptyset$

Iteration 3

$\mathcal{B}^3 = \{2, 3, 5\}$ with bfs $x^3 = (0, 1, 5, 0, 1, 0)$

$\mathcal{L}_1 = \emptyset$, $\mathcal{L}_2 = \{\{2, 5, 6\}, \{1, 5, 6\}, \{2, 3, 5\}\}$

$\bar{c}^1$	1	0	0	2	0	0	2
$\bar{c}^2$	-5	0	0	-2	0	-2	-10
$\bar{c}^3$	3	0	0	1	0	1	5
x_2	1	1	0	1	0	0	1
x_5	-1	0	0	-1	1	0	1
x_3	2	0	1	1	0	1	5

$\mathcal{EN} = \{1, 4, 6\}$

- Check x_1

-1	1	-1	1	0	0	0	0
[1]	2	0	-1	1	0	0	0
-5	-2	-2	5	0	1	0	0
3	1	1	-3	0	0	1	0

x_4 is not efficient

- Check x_4

-1	1	-1	-1	0	0	0	0
1	2	0	-2	1	0	0	0
-5	-2	-2	2	0	1	0	0
3	1	[1]	-1	0	0	1	0

x_4 is not efficient

- Check x_6: Leads back to $\mathcal{B}^1$

- Check x_1

-1	1	-1	1	0	0	0	0
[1]	2	0	-1	1	0	0	0
-5	-2	-2	5	0	1	0	0
3	1	1	-3	0	0	1	0

x_4 is not efficient

- Check x_4

-1	1	-1	-1	0	0	0	0
1	2	0	-2	1	0	0	0
-5	-2	-2	2	0	1	0	0
3	1	[1]	-1	0	0	1	0

x_4 is not efficient

- Check x_6: Leads back to $\mathcal{B}^1$

- Check x_1

-1	1	-1	1	0	0	0	0
[1]	2	0	-1	1	0	0	0
-5	-2	-2	5	0	1	0	0
3	1	1	-3	0	0	1	0

x_4 is not efficient

- Check x_4

-1	1	-1	-1	0	0	0	0
1	2	0	-2	1	0	0	0
-5	-2	-2	2	0	1	0	0
3	1	[1]	-1	0	0	1	0

x_4 is not efficient

- Check x_6: Leads back to $\mathcal{B}^1$

Iteration 4: $\mathcal{L}_1 = \emptyset$, STOP
Output: List of efficient bases
$\mathcal{B}^1 = \{2,5,6\}, \mathcal{B}^2 = \{1,5,6\}, \mathcal{B}^3 = \{2,3,5\}$