International Doctoral School Algorithmic Decision Theory: MCDA and MOO

Lecture 3: MOLP Extensions

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Overview

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$$\min\{Cx : Ax = b, x \geqq 0\}$$

- $X = \{x \in \mathbb{R}^n : Ax \geqq b\}$
- $Y = \{Cx \in \mathbb{R}^p : x \in \mathcal{X}\}$
- $\hat{x} \in X$ is (weakly) efficient if there is no $x \in X$ with $Cx \leq C\hat{x}$ ($Cx < C\hat{x}$)
- If $\hat{x}$ is (weakly) efficient then $C\hat{x}$ is (weakly) non-dominated

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Theorem

$\hat{x} \in X$ is (weakly) efficient if and only if there exists $(\lambda \geq 0)$ $\lambda > 0$ such that $\hat{x}$ is an optimal solution of

$$\min\{\lambda^T x : Ax = b, x \geqq 0\}. \qquad P(\lambda)$$

Dual of weighted sum problem:

$$\max\{u^T b : u^T A \leqq \lambda^T C\} \qquad D(\lambda)$$

Theorem

$\hat{x} \in X$ is (weakly) efficient if and only if there exists $(\lambda \geq 0)$ $\lambda > 0$ and u with $u^T A \leqq \lambda^T C$ such that

$$(u^T A - \lambda^T C)x = 0.$$

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Feasibility of Dual

Lemma

- *$D(\lambda)$ is feasible for all $\lambda \geq 0$ if $\min\{c^T x : x \in X\}$ is bounded for all $c \in$ cone (C), the cone generated by the rows of C.*
- *Let $\bar{c}_k := \min\{c_k^i : i = 1, \ldots, p\}$. $D(\lambda)$ is feasible for all $\lambda \geqq 0$ if $\min\{\bar{c}^T x : Ax = b, x \geqq 0\}$ is bounded.*
- *$D(\lambda)$ is feasible for all $\lambda \geq 0$ if $c_{kj} \geqq 0$ for all k, j.*

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- *$D(\lambda)$ is feasible for all $\lambda \geq 0$ if $c_{kj} \geqq 0$ for all k, j.*

- Assume $u_{\bar{\Lambda}}$ feasible for $D(\lambda)$ for all $\lambda \in \bar{\Lambda} \subset \mathbb{R}^p_{\geqq}$
- Define $Q(\lambda) = \{j : u_{\bar{\Lambda}}^T a_j = c_j(\lambda)\}$
- $\hat{\Lambda} \subset \bar{\Lambda}$ is maximal with respect to $Q(\lambda)$ if for some $\hat{\lambda} \in \hat{\Lambda}$
  - $Q(\hat{\lambda}) = Q(\lambda)$ for all $\lambda \in \hat{\Lambda}$
  - $Q(\hat{\lambda}) \neq Q(\lambda)$ for all $\lambda \in \bar{\Lambda} \setminus \hat{\Lambda}$
- $\mathcal{Q}(\hat{\Lambda}) := Q(\hat{\lambda})$ for some $\lambda \in \hat{\Lambda}$
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Restricted primal for $\hat{\Lambda}$:

$$\min\{e^T y : Ax + y = b, x_i = 0 \text{ for } i \notin \mathcal{Q}(\hat{\Lambda}), x, y \geqq o\}$$

- If optimal value is 0 then optimal solution $\hat{x}$ is optimal for $P(\lambda)$ for all $\lambda \in \hat{\Lambda}$
- Otherwise improve dual solution

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Restricted dual for $\hat{\Lambda}$:

$$\max\{u^T b : w^T a_j \leqq 0 \text{ for } j \in \mathcal{Q}(\hat{\Lambda}), w \leqq e\}$$

- $\hat{w}(\hat{\Lambda})$ optimal solution
- If there is no $j \notin \mathcal{Q}(\hat{\Lambda})$ such that $\hat{w}(\hat{\Lambda}) > 0$ then $P(\lambda)$ infeasible for all $\lambda \in \hat{\Lambda}$, i.e. MOLP infeasible
- Otherwise

$$\hat{\varepsilon}(\lambda) = \min_j \left\{ \frac{c_j(\lambda) - (u_{\hat{\Lambda}}(\lambda))^T a_j}{\hat{w}(\hat{\Lambda})^T a_j} : \hat{w}(\hat{\Lambda})^T a_j > 0 \right\}$$

- $\Lambda^* \subset \Lambda$ maximal with repect to ε if the same for all $\lambda \in \Lambda^*$ and different for all other Λ: $\hat{\varepsilon}_{\Lambda^*}(\lambda)$
- $u_{\Lambda^*}(\lambda) = u_{\hat{\Lambda}}(\lambda) + \hat{\varepsilon}_{\Lambda^*}(\lambda)\hat{w}(\hat{\Lambda})$

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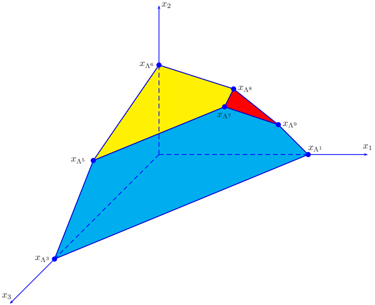
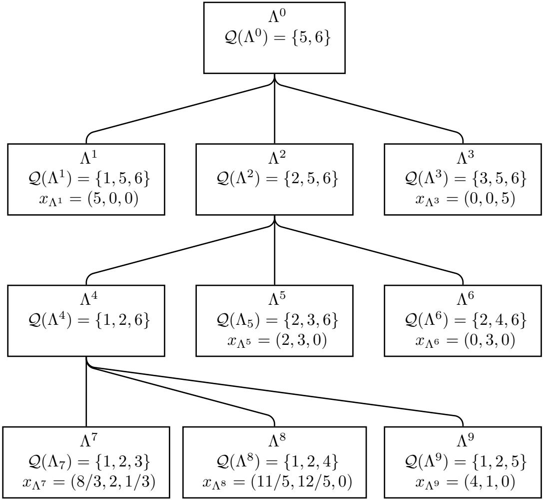
Algorithm

1. *Dual feasible* $u_{\bar{\Lambda}}$ *for all* $\lambda \in \bar{\Lambda}$*, partition* $\{\hat{\Lambda}_i \; : \; i \in I_0\}$ *of* $\bar{\Lambda}$*.*
2. *For* $i \in I_0$ *find* $\mathcal{Q}(\hat{\Lambda}_i)$, $u_{\hat{\Lambda}_i}(\lambda) := u_{\bar{\Lambda}}$, $\mathcal{L} = \mathcal{L} \cup \{(\hat{\Lambda}_i, u_{\hat{\Lambda}_i}(\lambda)\}$.
3. *While* $\mathcal{L} \neq \emptyset$, *choose* $(\hat{\Lambda}, u_{\hat{\Lambda}}(\lambda)) \in \mathcal{L}$ *and solve RP*($\hat{\Lambda}$)*.*
   - *If optimal value is 0: An optimal solution of P*(λ) *for all* $\lambda \in \hat{\Lambda}$ *is found.* $\mathcal{L} := \mathcal{L} \setminus \{(\hat{\Lambda}, u_{\hat{\Lambda}}(\lambda))\}$.
   - *Otherwise solve DRP*($\hat{\Lambda}$) *and let* $\hat{w}(\hat{\Lambda})$ *be an optimal solution.*
     - *If there is no* $j \notin \mathcal{Q}(\hat{\Lambda})$ *such that* $\hat{w}(\hat{\Lambda})^T a_j > 0$*: P*($\lambda$) *is infeasible for all* $\lambda \in \hat{\Lambda}$ *and MOLP is infeasible.*
     - *Otherwise compute the partition* $\{\Lambda_l^* \; : \; l \in I^*\}$ *of* $\hat{\Lambda}$ *where each* Λ_l^* *is maximal. For each* $l \in I^*$ *compute* $\hat{\varepsilon}_{\Lambda_l^*}(\lambda)$ *and update* $u_{\Lambda_l^*}(\lambda)$*. Compute* $\mathcal{Q}(\Lambda_l^*)$ *and set* $\mathcal{L} = \mathcal{L} \cup \{(\Lambda_l^*, u_{\Lambda_l^*}(\lambda))\}$*. Set* $\mathcal{L} = \mathcal{L} \setminus \{(\hat{\Lambda}, u_{\hat{\Lambda}}(\lambda))\}$*.*

$$
\begin{array}{lll}
\min & (-x_1, -x_2, -x_3) & \\
\text{s.t} & x_1 + x_2 + x_3 + s_1 & = & 5 \\
& x_1 + 3x_2 + x_3 + s_2 & = & 9 \\
& 3x_1 + 4x_2 + s_3 & = & 16 \\
& x_1, x_2, x_3, s_1, s_2 & \geq & 0
\end{array}
$$




Λ^0
$\mathcal{Q}(\Lambda^0) = \{5,6\}$
Λ^1
$\mathcal{Q}(\Lambda^1) = \{1,5,6\}$
$x_{\Lambda^1} = (5,0,0)$
Λ^2
$\mathcal{Q}(\Lambda^2) = \{2,5,6\}$
Λ^3
$\mathcal{Q}(\Lambda^3) = \{3,5,6\}$
$x_{\Lambda^3} = (0,0,5)$
Λ^4
$\mathcal{Q}(\Lambda^4) = \{1,2,6\}$
Λ^5
$\mathcal{Q}(\Lambda_5) = \{2,3,6\}$
$x_{\Lambda^5} = (2,3,0)$
Λ^6
$\mathcal{Q}(\Lambda^6) = \{2,4,6\}$
$x_{\Lambda^6} = (0,3,0)$
Λ^7
$\mathcal{Q}(\Lambda_7) = \{1,2,3\}$
$x_{\Lambda^7} = (8/3,2,1/3)$
Λ^8
$\mathcal{Q}(\Lambda^8) = \{1,2,4\}$
$x_{\Lambda^8} = (11/5,12/5,0)$
Λ^9
$\mathcal{Q}(\Lambda^9) = \{1,2,5\}$
$x_{\Lambda^9} = (4,1,0)$

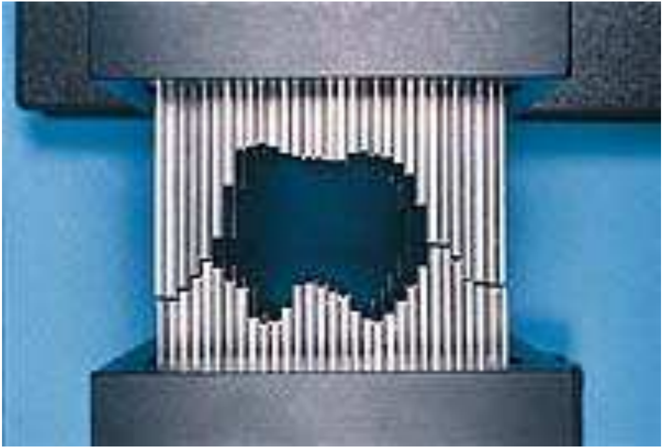
Theorem

Let the MOLP be nondegenerate. Then Algorithm 3.1 is finite and at termination the output gives an optimal solution of $P(\lambda)$ for each $\lambda \in \Lambda$.

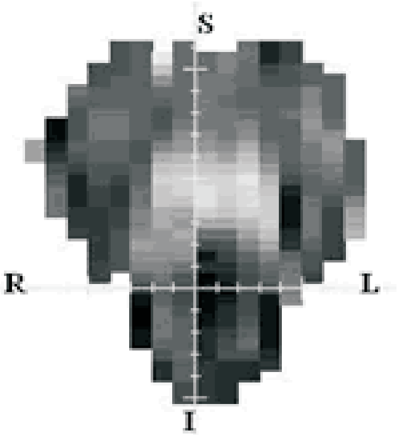
Overview

Delivery of Radiotherapy

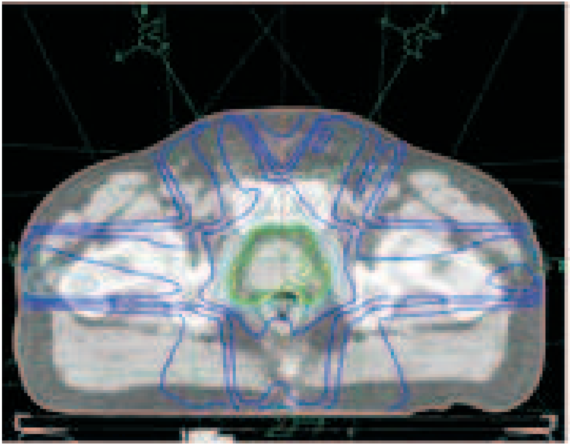
Intensity Modulation by Multileaf Collimator



Task: Find Intensity (Fluence) Map



that Produces Desired Dose Distribution



Modelling Intensity Optimization

- Many different (LP, NLP, MIP) models (Shao, 2005)
- Given: a beam directions, dose deposition matrix $A \in \mathbb{R}^{m \times n}$ with a_{ji} dose delivered to voxel j at unit intensity of bixel i
- Wanted: $x = (x_i : i = 1, \ldots, n)$ intensity profiles for all beams such that dose $d = Ax$ satisfies the treatment goals
- Goal 1: Destroy the tumour, physician prescribes lower and upper bound l_T and u_T for dose in tumour
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$$
\begin{array}{llcl}
\min & (y_T, y_C, y_N) & & \\
\text{s.t.} & A_T x + y_T e & \geqq & l_T \\
& A_T x & \leqq & u_T \\
& A_C x - y_C e & \leqq & u_C \\
& A_N x - y_N e & \leqq & u_N \\
& y_T & \leqq & \alpha \\
& y_C & \geqq & -u_C \\
& y_C & \leqq & \beta \\
& y_N & \leqq & \gamma \\
& x, y_T, y_N & \geqq & 0
\end{array} \tag{1}
$$

- Multiobjcetive version of elastic LP model of (Holder, 2003)
- Always feasible if α, β, γ are not too small

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 & y_T & \leqq & \alpha \\
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Overview

Multiobjective Linear Programming

$$\min\{Cx : Ax \geqq b, x \in \mathbb{R}^n\} \tag{2}$$

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Multiobjective Linear Programming

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- $\mathcal{Y} = \{Cx \in \mathbb{R}^p : x \in \mathcal{X}\}$

Benson's Algorithm

- (Benson, 1998): Solve MOLP in objective space
- $\mathcal{Y}' := \left(\mathcal{Y} + \mathbb{R}^p_{\geqq}\right) \cap \left(y' - \mathbb{R}^p_{\geqq}\right)$
- dim $\mathcal{Y}' = p$ and $\mathcal{Y}'_N = \mathcal{Y}_N$

$$P_1(y) \qquad \min\{z : Ax \geqq b, Cx - ez \leqq y\}$$

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Algorithm (Benson's Algorithm)

Init: *Compute* $\bar{p} \in int\ \mathcal{Y}'$
Construct p-dimensional simplex $\mathcal{S}^0 \supset \mathcal{Y}'$
Store vertex set $vert(\mathcal{S}^0)$
Set $k = 0$ *and go to* **It** $k1$

It $k1$**:** *If* $vert(\mathcal{S}^k) \subset Y'$ *go to* **It** $k5$: $\mathcal{Y}' = \mathcal{S}^k$
Otherwise choose $y^k \in vert(\mathcal{S}^k) \setminus \mathcal{Y}'$

It $k2$**:** *Find* $0 < \alpha_k < 1$ *such that* $\alpha_k y^k + (1 - \alpha_k)\bar{p} \in bd\ \mathcal{Y}'$
Set $q^k = \alpha_k y^k + (1 - \alpha_k)\bar{p}$

It $k3$**:** *Set* $\mathcal{S}^{k+1} = \mathcal{S}^k \cap \{y \in \mathbb{R}^p : \langle w^k, y\rangle \geqq \langle b, u^k\rangle\}$
(u^{k^T}, w^{k^T}) *is optimal solution to* $D_1(q^k)$

It $k4$**:** *Find* $vert(\mathcal{S}^{k+1})$, *set* $k = k + 1$ *and go to* **It** $k1$

It $k5$**:** $\mathcal{Y}_{NE} = \mathcal{Y}'_{NE} = \{y \in vert(\mathcal{S}^K) : y < y'\}$

$$C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ -3 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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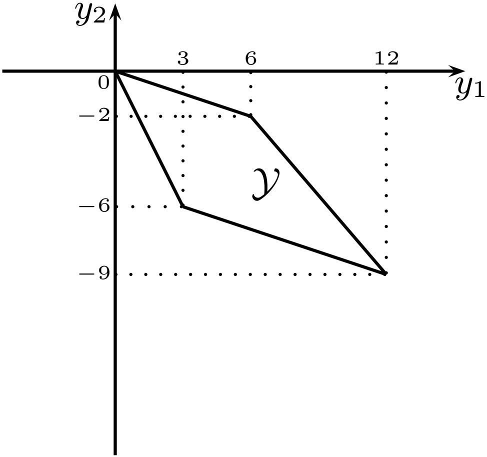
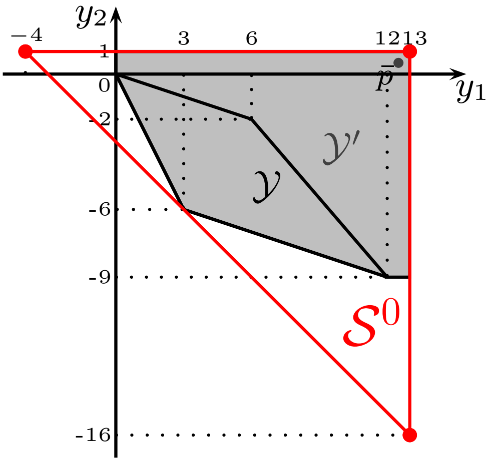


- Initial cover and interior point
- First cut
- Second cut
- Third cut
- Fourth cut



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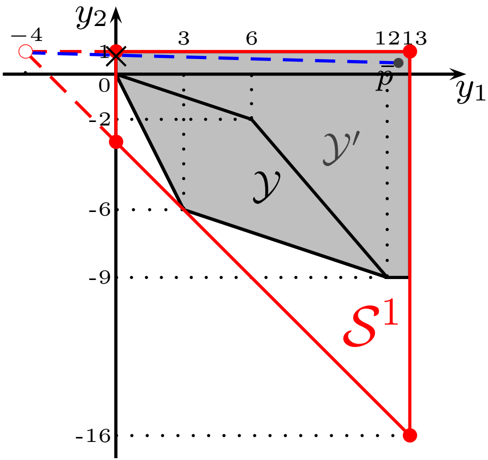

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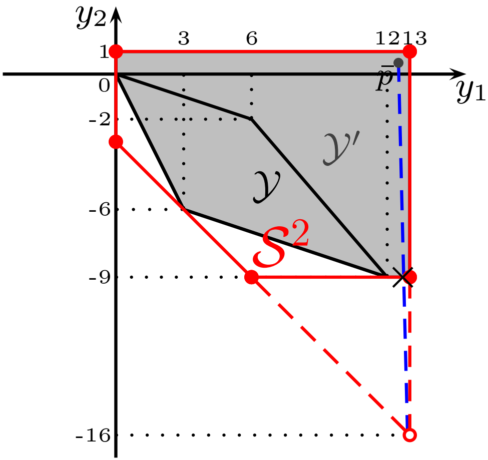

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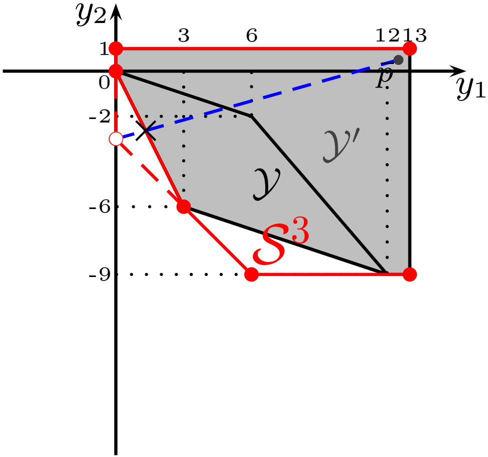

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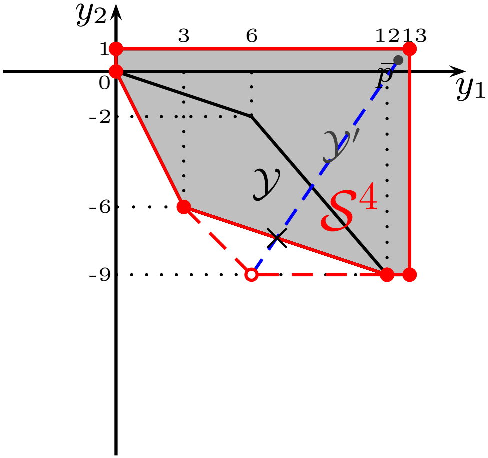

Approximating the Nondominated Set (Shao and Ehrgott, 2007a)

- If $d(y^k, q^k) < \epsilon$ do not construct hyperplane
- Keep $y^k \in \mathcal{O}$ and $q^k \in \mathcal{I}$ for outer and inner approximation

Algorithm (Approximation Algorithm)

It $k1$: *If* $vert(\mathcal{S}^k) \subset \mathcal{Y}' \cup \mathcal{O}$ *go to* **It** $k5$
Otherwise choose any $y^k \in vert(\mathcal{S}^k) \setminus (\mathcal{O} \cup \mathcal{Y}')$

It $k3$: *If* $d(y^k, q^k) \leqq \epsilon$ *add* y^k *to* $\mathcal{O}$, *add* q^k *to* $\mathcal{I}$, *go to* **It** $k1$
Otherwise $\mathcal{S}^{k+1} = \mathcal{S}^k \cap \{y \in \mathbb{R}^p : \langle w^k, y\rangle \geqq \langle b, u^k\rangle\}$
$({u^k}^T, {w^k}^T)$ *is optimal solution to* $D(q^k)$

It $k5$: $\mathcal{V}_o(\mathcal{S}^K) = vert(\mathcal{S}^K)$, $\mathcal{V}_i(\mathcal{S}^K) = (vert(\mathcal{S}^K) \setminus \mathcal{O}) \cup \mathcal{I}$
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- $\epsilon = 2.0$
- Two cuts as before
- $d(y^1, q^1) = 1.366$, $d(y^2, q^2) = 1.973$
- $\mathcal{V}_o(\mathcal{S}^2) = \{(13,1), (0,1), (0,-3), (6,-9), (13,-9)\}$
- $\mathcal{V}_i(\mathcal{S}^2) = \{(13,1), (0,1), (1.316,-2.632), (7.114,-7.371), (13,-9)\}$
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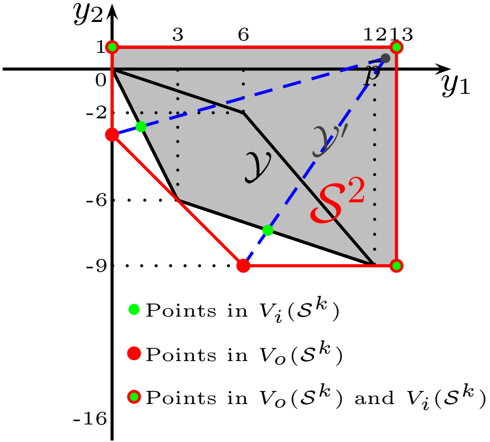

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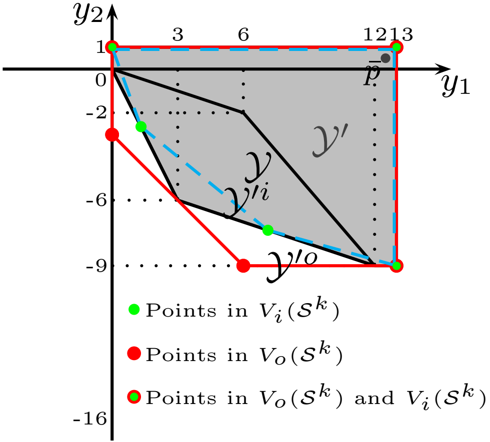

Proposition

1. $|\mathcal{V}_o(\mathcal{S}^K)| = |\mathcal{V}_i(\mathcal{S}^K)|$
2. $\mathcal{V}_i(\mathcal{S}^K) \subset bd(\mathcal{Y}')$
3. *For* $y \in \mathcal{V}_o(\mathcal{S}^K)$ *it holds* $y \notin bd(\mathcal{Y}')$ *if and only if* $y \notin \mathcal{V}_i(\mathcal{S}^K)$
4. *If* $y_{ov} \in \mathcal{V}_o(\mathcal{S}^K)$ *there exists* $y_{iv} \in \mathcal{V}_i(S^K)$ *with* $d(y_{ov}, y_{iv}) \leqq \epsilon$ *and vice versa*
5. $\mathcal{Y}_N'^i + \mathbb{R}^p_{\geqq} \subseteq \mathcal{Y}_N' + \mathbb{R}^p_{\geqq} \subseteq \mathcal{Y}_N'^o + \mathbb{R}^p_{\geqq}$

Proposition

If $y_o \in \mathcal{Y}'^{o}_{WN}$ there exists $y_i \in \mathcal{Y}'^{i}_{WN}$ such that $d(y_o, y_i) \leqq \epsilon$.

- $\hat{x} \in \mathcal{X}$ is (weakly) ε-efficient if there is no $x \in \mathcal{X}$ with $Cx \leq (<) C\hat{x} - \varepsilon$.
- $C\hat{x}$ is (weakly) ε-nondominated

Theorem

Let $\varepsilon = \epsilon e$, where $e = (1, \ldots, 1) \in \mathbb{R}^p$. Then $\mathcal{Y}'^{i}_{N}$ is a set of weakly ε-nondominated points for $\mathcal{Y}'$.

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Overview

The Geometric Dual Heyde and Löhne (2006)

- Primal MOLP:

$$\min\{Cx : x \in \mathbb{R}^n, Ax \geqq b\}$$

- $\mathcal{K} := \mathbb{R}_{\geqq} e^p = \{y \in \mathbb{R}^p : y_1 = \cdots = y_{p-1} = 0, y_p \geqq 0\}$
- Dual MOLP:

$$\max_{\mathcal{K}}\{D(u,\lambda) : (u,\lambda) \in \mathbb{R}^m \times \mathbb{R}^p, (u,\lambda) \geqq 0, A^T u = C^T \lambda, e^T \lambda = 1\}$$

$$D(u,\lambda) := (\lambda_1, \ldots, \lambda_{p-1}, b^T u)^T = \begin{pmatrix} 0 & I_{p-1} & 0 \\ b^T & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix}$$

- $\mathcal{P} := C(\mathcal{X}) + \mathbb{R}^p_{\geqq}$
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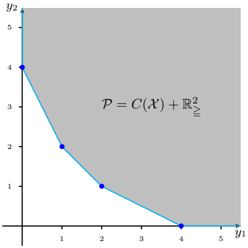

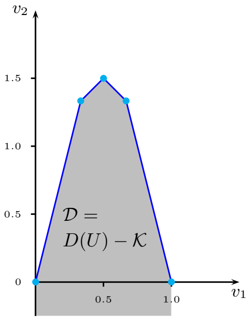

$$\varphi(y, v) := \sum_{i=1}^{p-1} y_i v_i + y_p \left(1 - \sum_{i=1}^{p-1} v_i\right) - v_p$$

For $x \in \mathcal{X}$ and $(u, \lambda) \in \mathcal{U}$: $\varphi(Cx, D(u, \lambda)) = \lambda^T Px - b^T u$

$$\begin{aligned}
\lambda(v) &:= \left(v_1, \ldots, v_{p-1}, 1 - \sum_{i=1}^{p-1} v_i\right)^T \\
\lambda^*(y) &:= (y_1 - y_p, \ldots, y_{p-1} - y_p, -1)^T \\
H(v) &:= \left\{y \in \mathbb{R}^p : \lambda(v)^T y = v_p\right\} \\
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For $\mathcal{F}^* \subset \mathbb{R}^p$ define $\Psi(\mathcal{F}^*) := \bigcap_{v \in \mathcal{F}^*} H(v) \cap \mathcal{P}$

Theorem (Heyde and Löhne (2006))

Ψ is an inclusion reversing one-to-one map between the set of all proper $\mathcal{K}$-maximal faces of $\mathcal{D}$ and the set of all proper weakly nondominated faces of $\mathcal{P}$ and the inverse map is given by

$$\Psi^{-1}(\mathcal{F}) = \bigcap_{y \in \mathcal{F}} H^*(y) \cap \mathcal{D}.$$

Moreover, for every proper $\mathcal{K}$-maximal face $\mathcal{F}^*$ of $\mathcal{D}$ it holds

$$\dim \mathcal{F}^* + \dim \Psi(\mathcal{F}^*) = p - 1$$

Overview

A Dual Algorithm (Ehrgott *et al.*, 2007)

$$P_2(v) \qquad \min\left\{\lambda(v)^T Cx : x \in \mathbb{R}^n, Ax \geqq b\right\}$$
$$D_2(v) \quad \max\left\{b^T u : u \in \mathbb{R}^m, u \geqq 0, A^T u = C^T \lambda(v)\right\}$$

Algorithm

Init: *For* $\hat{d} \in \text{int}\,\mathcal{D}$ *find optimal solution* x^0 *of* $P_2(\hat{d})$
Set $\mathcal{S}^0 := \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(Cx^0, v) \geqq 0\}$; $k := 1$

It $k1$: *If* $\text{vert}(\mathcal{S}^{k-1}) \subset \mathcal{D}$ *stop*
otherwise choose $s^k \in \text{vert}(\mathcal{S}^{k-1}) \setminus \mathcal{D}$

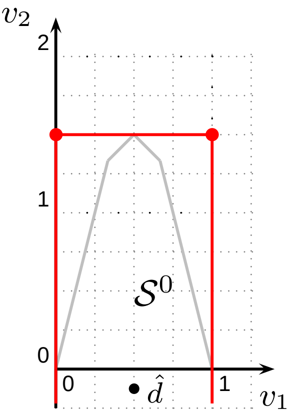
It $k2$: *Find* α^k *with* $v^k := \alpha^k s^k + (1-\alpha^k)\hat{d} \in \max_{\mathcal{K}} \mathcal{D}$

It $k3$: *Compute an optimal solution* x^k *of* $P_2(v^k)$

It $k4$: *Set* $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{v \in \mathbb{R}^p : \varphi(Cx^k, v) \geqq 0\}$

It $k5$: *Set* $k := k+1$ *and go to* **It** $k1$

- Initial cover and interior point
- First cut
- Second cut
- Third cut
- Fourth cut



- Initial cover and interior point
- First cut
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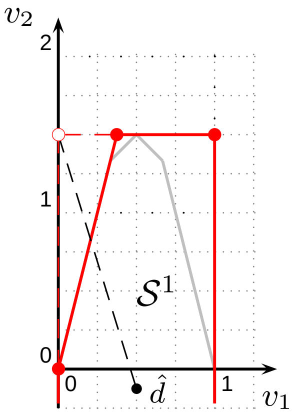
- Fourth cut



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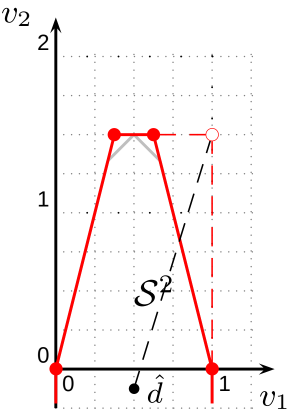
- First cut
- Second cut
- Third cut
- Fourth cut



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- First cut
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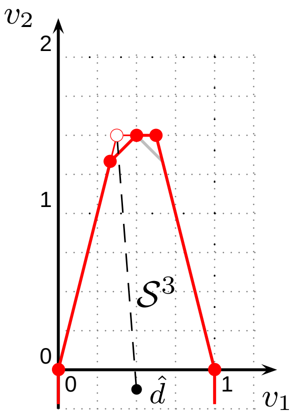
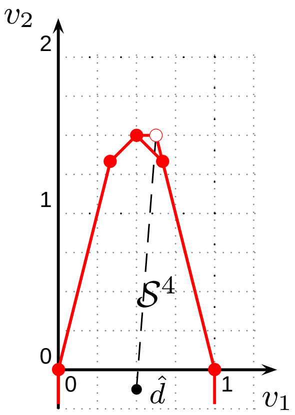
- Third cut
- Fourth cut



- Initial cover and interior point
- First cut
- Second cut
- Third cut
- Fourth cut



- Let $\mathcal{S} \subset \mathbb{R}^p$ be a polyhdron with $\mathcal{S} = \mathcal{S} - \mathcal{K}$ and $\text{proj}_{\mathbb{R}^{p-1}}(S) = \{t \in \mathbb{R}^{p-1} : t \geqq 0, \sum_{i=1}^{p-1} t_i \leqq 1\}$
- $\mathcal{D}(\mathcal{S}) = \{y \in \mathbb{R}^p : \varphi(y, v) \geqq 0, \text{ for all } v \in \text{vert}(\mathcal{S})\}$

Proposition

1. $\mathcal{D}(\mathcal{S}) = \mathcal{D}(\mathcal{S}) + \mathbb{R}^p_{\geqq}$
2. *Theorem 6 holds for* $\mathcal{D} = \mathcal{S}$ *and* $\mathcal{P} = \mathcal{D}(\mathcal{S})$
3. *If* $\mathcal{S}^1 \subset \mathcal{S}^0$ *then* $\mathcal{D}(\mathcal{S}^1) \supset \mathcal{D}(\mathcal{S}^0)$

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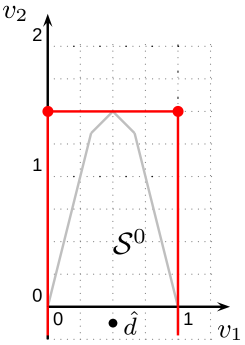
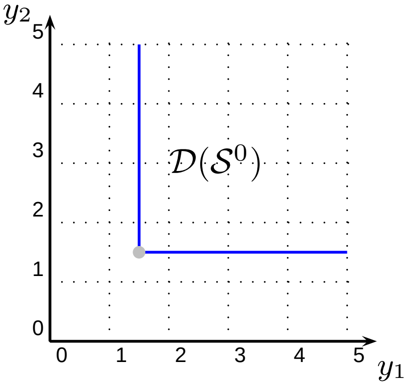
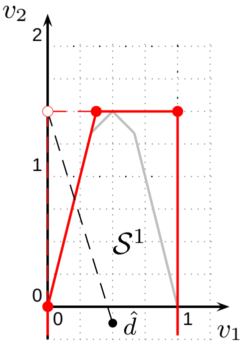
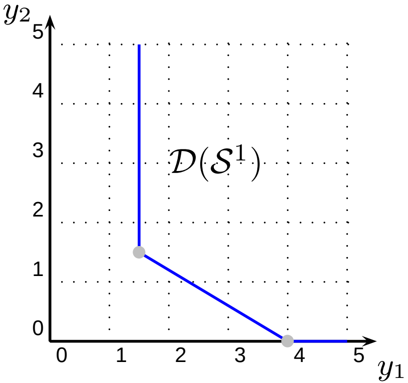
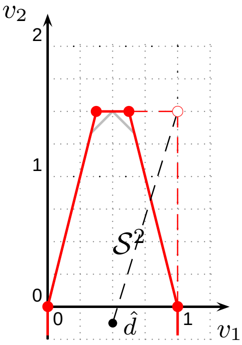
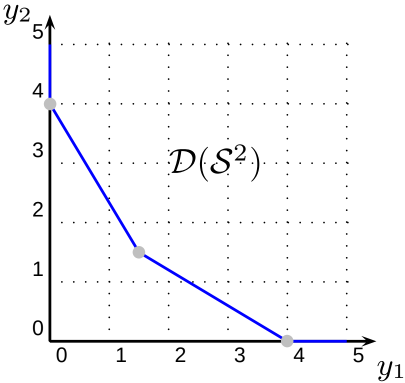

v_2
2
1
0
$\mathcal{S}^0$
0
$\hat{d}$
1
v_1

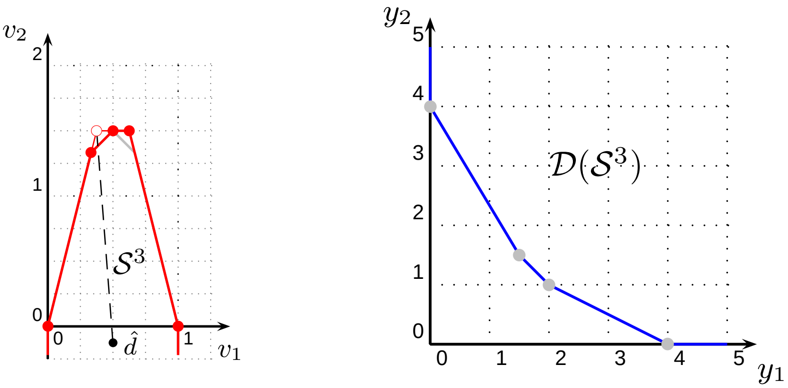

y_2
5
4
3
2
1
0
$\mathcal{D}(\mathcal{S}^0)$
0
1
2
3
4
5
y_1

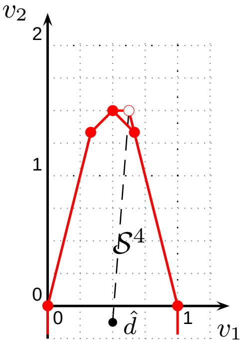

v_2
2
1
0
0
1
v_1
$\hat{d}$
$\mathcal{S}^1$

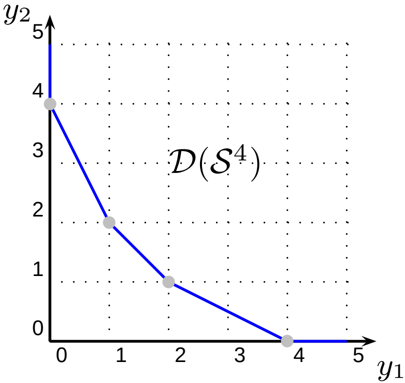

y_2
5
4
3
2
1
0
0
1
2
3
4
5
y_1
$\mathcal{D}(\mathcal{S}^1)$


v_2
2
1
0
0
1
v_1
$\mathcal{S}^2$
$\hat{d}$


y_2
5
4
3
2
1
0
0
1
2
3
4
5
y_1
$\mathcal{D}(\mathcal{S}^2)$


v_2
2
1
0
0
$\hat{d}$
1
v_1
$\mathcal{S}^3$
y_2
5
4
3
2
1
0
0
1
2
3
4
5
y_1
$\mathcal{D}(\mathcal{S}^3)$


v_2
2
1
0
0
1
v_1
$\hat{d}$
$\mathcal{S}^4$


y_2
5
4
3
2
1
0
0
1
2
3
4
5
y_1
$\mathcal{D}(\mathcal{S}^4)$

Dual Approximation (Shao and Ehrgott, 2007b)

- If $\text{vert}(\mathcal{S}^k) \subset \mathcal{D} + \epsilon e^p$ do not construct hyperplane
- If $v_p - f \leqq \epsilon$ then $v \in \mathcal{D} + \epsilon e^p$ where f is optimum of $D_2(v)$
- $\mathcal{D}^o := \mathcal{S}^{k-1} \supset \mathcal{D}$ is outer approximation of $\mathcal{D}$
- $$\mathcal{P}^i := \mathcal{D}(\mathcal{D}^o) \subset \mathcal{D}(\mathcal{D}) = \mathcal{P}$$

  is inner approximation of $\mathcal{P}$

Theorem

Let $\varepsilon = \epsilon e$, then the nondominated set of $\mathcal{P}^i$ is a set of ε-nondominated points of $\mathcal{P}$.

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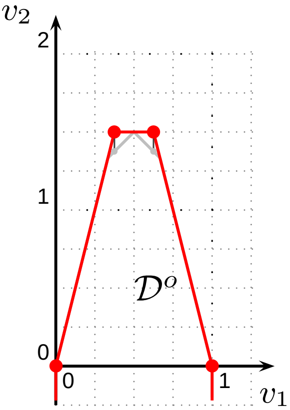
Let $\varepsilon = \epsilon e$, then the nondominated set of $\mathcal{P}^i$ is a set of ε-nondominated points of $\mathcal{P}$.

- $\epsilon = 3/20$
- Two cuts as before
- $d(v^1, bd^1) = 1/8$,
  $d(v^2, bd^2) = 1/8$
- $\mathcal{D}^o = \mathcal{S}^2$
- $\mathcal{P}^i = \mathcal{D}(\mathcal{D}^o)$



- $\epsilon = 3/20$
- Two cuts as before
- $d(v^1, bd^1) = 1/8$, $d(v^2, bd^2) = 1/8$
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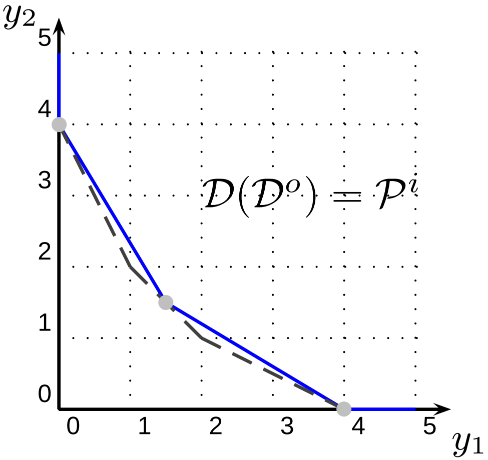
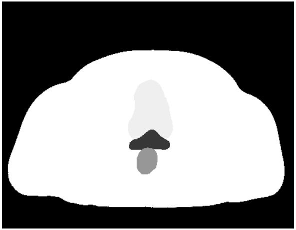


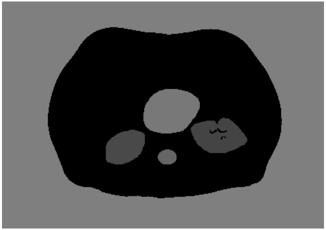
Overview

The Test Cases

Acoustic Neuroma



Prostate



Pancreatic Lesion

- Dose calculation inexact
- Inaccuracies during delivery
- Planning to small fraction of a Gy acceptable

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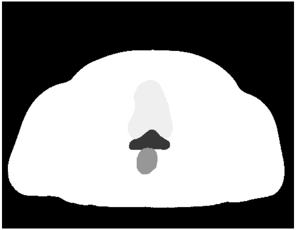
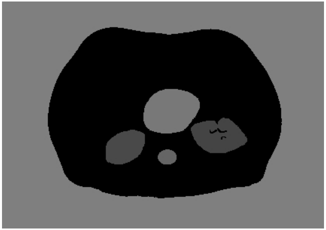
Acoustic Neuroma



Prostate



Pancreatic Lesion

- Dose calculation inexact
- Inaccuracies during delivery
- Planning to small fraction of a Gy acceptable

Case	AN	P	PL
Tumour voxels	9	22	67
Critical organ voxels	47	89	91
Normal tissue voxels	999	1182	986
Bixels	594	821	1140
u_T	87.55	90.64	90.64
l_T	82.45	85.36	85.36
u_C	60/45	60/45	60/45
u_N	0.00	0.00	0.00
α	16.49	42.68	17.07
β	12.00	30.00	12.00
γ	87.55	100.64	90.64

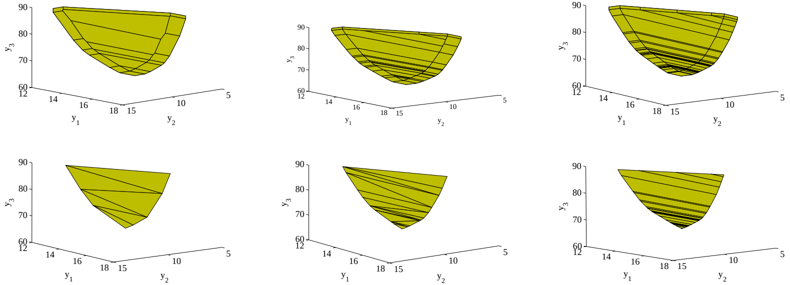
$$
\begin{array}{lrcl}
\min & (y_T, y_C, y_N) & & \\
\text{s.t.} & A_T x + y_T e & \geqq & l_T \\
& A_T x & \leqq & u_T \\
& A_C x - y_C e & \leqq & u_C \\
& A_N x - y_N e & \leqq & u_N \\
& y_T & \leqq & \alpha \\
& y_C & \geqq & -u_C \\
& y_C & \leqq & \beta \\
& y_N & \leqq & \gamma \\
& x, y_T, y_N & \geqq & 0
\end{array}
\qquad (3)
$$

	ϵ	Solving the dual			Solving the primal		
		Time	Vert.	Cuts	Time	Vert.	Cuts
AC	0.1	1.484	17	8	5.938	27	21
	0.01	3.078	33	18	8.703	47	44
	0	8.864	85	55	13.984	55	85
PR	0.1	4.422	39	19	14.781	56	42
	0.01	18.454	157	78	64.954	296	184
	0	792.390	3280	3165	995.050	3165	3280
PL	0.1	58.263	85	44	164.360	152	90
	0.01	401.934	582	298	1184.950	1097	586
	0.005	734.784	1058	539	2147.530	1989	1041

$\epsilon = 0.1$ | $\epsilon = 0.01$ | $\epsilon = 0$

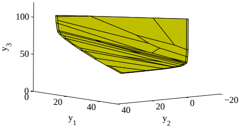
P:

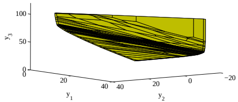
D:

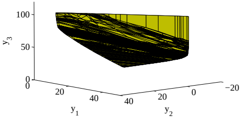


$\epsilon = 0.1$ | $\epsilon = 0.01$ | $\epsilon = 0$

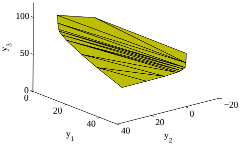
P:

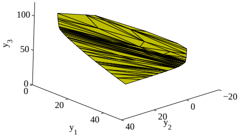


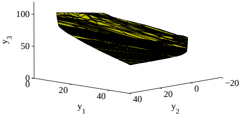




D:



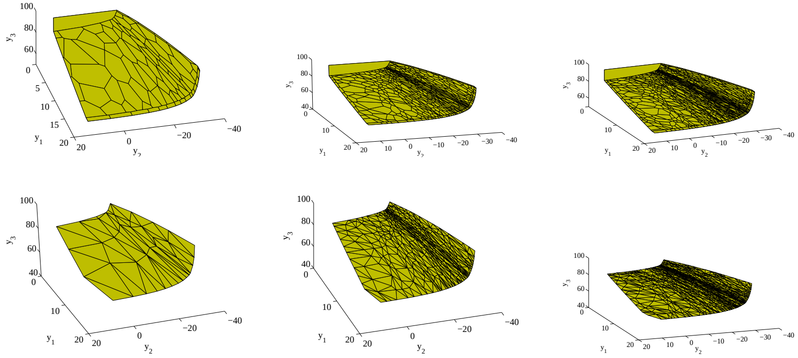




$\epsilon = 0.1$ $\epsilon = 0.01$ $\epsilon = 0.0005$

P:

D:



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