International Doctoral School Algorithmic Decision Theory: MCDA and MOO

Lecture 4: Multiobjective Combinatorial Optimization

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Overview

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Mathematical Formulation

$$\begin{aligned} \min z(x) &= Cx \\ \text{subject to } Ax &= b \\ x &\in \{0,1\}^n \end{aligned}$$

$$\begin{aligned} x \in \{0,1\}^n &\longrightarrow n \text{ variables, } i = 1,\ldots,n \\ C \in \mathbb{Z}^{p\times n} &\longrightarrow p \text{ objective functions, } k = 1,\ldots,p \\ A \in \mathbb{Z}^{m\times n} &\longrightarrow m \text{ constraints, } j = 1,\ldots,m \end{aligned}$$

Combinatorial structure: paths, trees, flows, tours, etc.

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Feasible Sets

- $X = \{x \in \{0,1\}^n : Ax = b\}$
  feasible set in decision space
- $Y = z(X) = \{Cx : x \in X\}$
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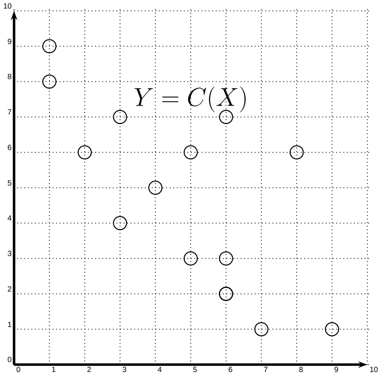

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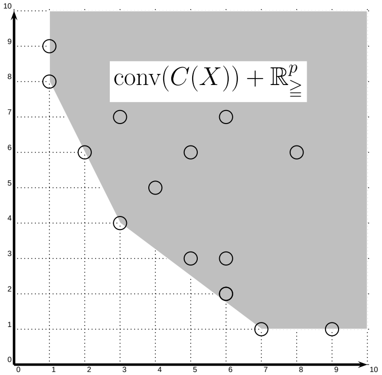
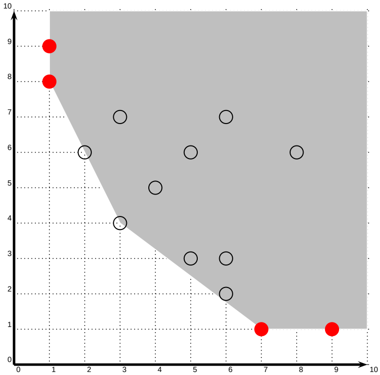


Lexicographic Optimality

- Individual minima
  $z_k(\hat{x}) \leqq z_k(x)$ for all $x \in X$
- Lexicographic optimality (1)
  $z(\hat{x}) \leqq_{lex} z(x)$ for all $x \in X$
- Lexicographic optimality (2)
  $z^\pi(\hat{x}) \leqq_{lex} z^\pi(x)$ for all $x \in X$ and some permutation z^π of $(z_1, \ldots, z_p)$



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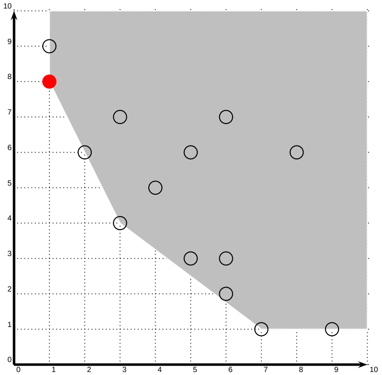

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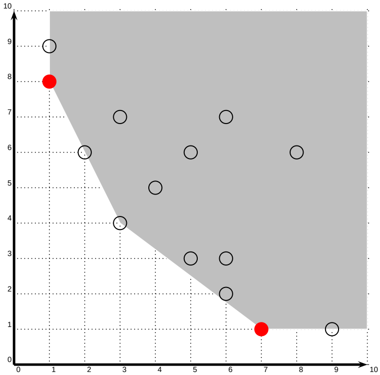

Efficient Solutions

- Weakly efficient solutions X_{wE}:
  there is no x with $z(x) < z(\hat{x})$
  $z(\hat{x})$ is weakly nondominated
  $Y_{wN} := z(X_{wN})$
- Efficient solutions X_E:
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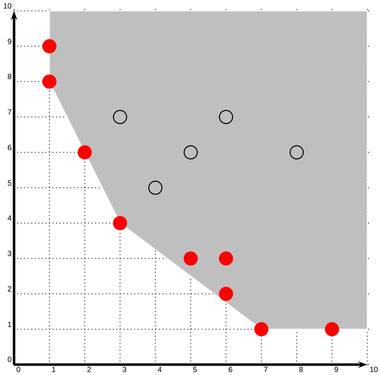

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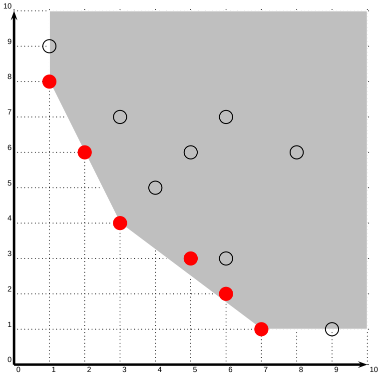

Efficient Solutions

- Supported efficient solutions X_{SE}: There is $\lambda > 0$ with $\lambda^T C\hat{x} \leqq \lambda^T Cx$ for all $x \in X$
  - $C\hat{x}$ is extreme point of $\text{conv}(Y) + \mathbb{R}^p_{\geqq} \rightarrow X_{SE1}$
  - $C\hat{x}$ is in relative interior of face of $\text{conv}(Y) + \mathbb{R}^p_{\geqq} \rightarrow X_{SE2}$
- Nonsupported efficient solutions X_{NE}: $C\hat{x}$ is in interior of $\text{conv}(Y) + \mathbb{R}^p_{\geqq}$

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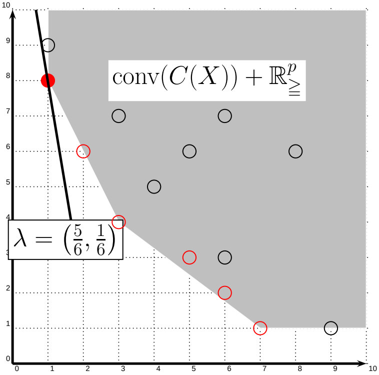

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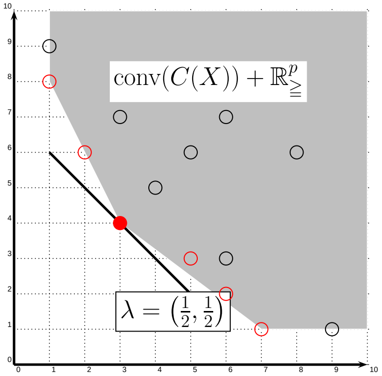

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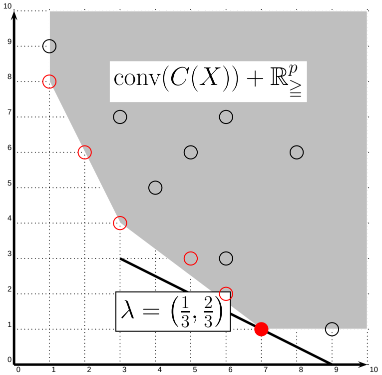

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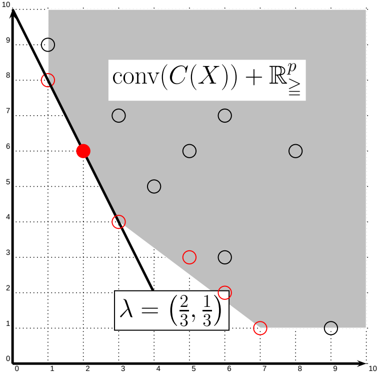

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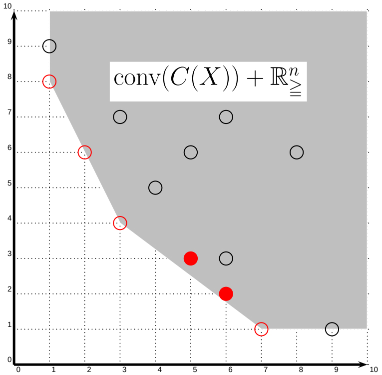
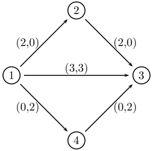
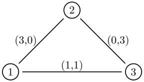


Nonsupported Effcient Solutions

- Shortest path: 

$Y_N = \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right\}$

- Spanning tree: 

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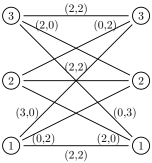
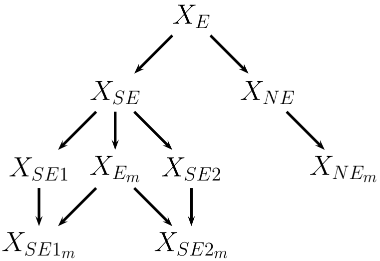
- Assignment: 

$Y_N = \left\{ \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \end{pmatrix} \right\}$

Classification of Efficient Sets

Hansen 1979:

- $x^1, x^2 \in X_E$ are equivalent if $Cx^1 = Cx^2$
- Complete set: $\hat{X} \subset X_E$ such that for all $y \in Y_N$ there is $x \in \hat{X}$ with $z(x) = y$
- Minimal complete set contains no equivalent solutions
- Maximal complete set contains all equivalent solutions



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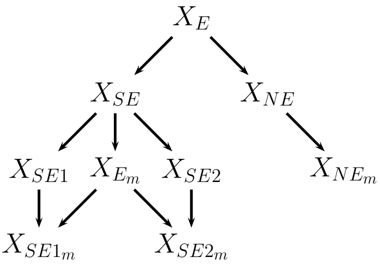

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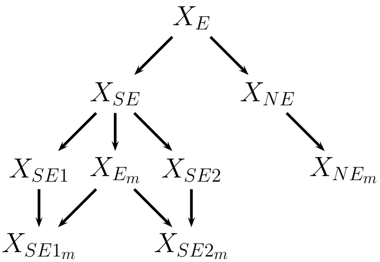

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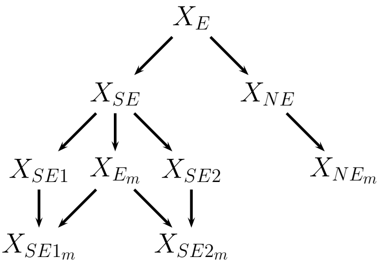

Overview

MOCO Problems Are Hard

- Decision problem: Given $b \in \mathbb{Z}^p$: Does there exist $x \in X$ such that $Cx \leqq b$?
- Counting problem: Given $b \in \mathbb{Z}^p$: How many $x \in X$ satisfy $Cx \leqq b$?
- How many efficient solutions (nondominated points) do exist?
- KNAPSACK: Given $a^1, a^2 \in \mathbb{Z}^n$ and $b_1, b_2 \in \mathbb{Z}$, does there exist $x \in \{0,1\}^n$ such that $(a^1)^T x \leqq b_1$ and $(a^2)^T x \geqq b_2$?
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The Unconstrained MOCO Problem

Observation

Multiobjective combinatorial optimization problems are NP-hard, #P-complete, and intractable.

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- Does there exist $x \in \{0,1\}^n$ such that $(c^1)^T x \leqq d_1$ and $(c^2)^T x \leqq d_2$?
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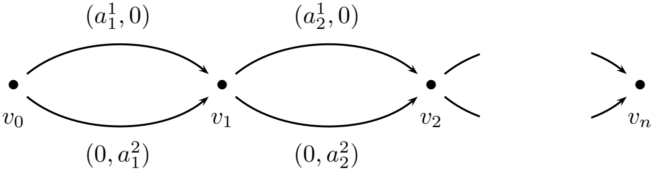
Multiobjective Shortest Path Problem

- NP-hard: Hansen 1979



- Exponentially many efficient paths

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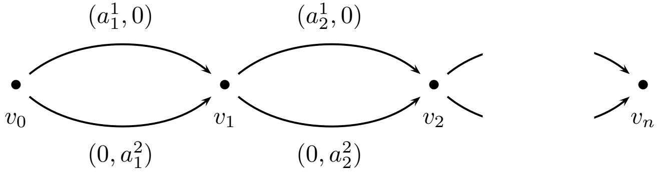

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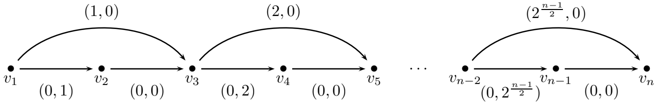

Other Examples

- Assignment (NP-hard: Serafini 1986, #P-hard: Neumayer 1994)
- Spanning tree (Intractable: Hamacher and Ruhe 1994)
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- Intractable: X_E, even Y_{SN}, can be exponential in the size of the instance
- Number of nondominated points for biobjective shortest path and assignment problems (Raith 2007, Przybylski 2006)

Shortest path			Assignment	
Nodes	Edges	$\lvert Y_N \rvert$	n	$\lvert Y_N \rvert$
4,902	19,596	6	10	13
4,902	19,596	1,594	20	82
3,000	33,224	15	40	243
14,000	153,742	17	60	470
330,386	1,202,458	21	80	671
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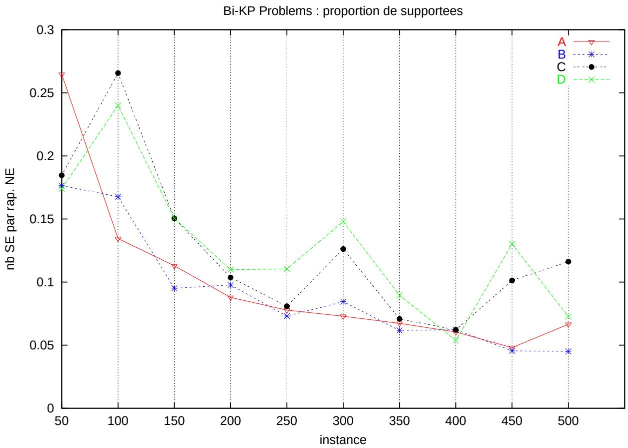

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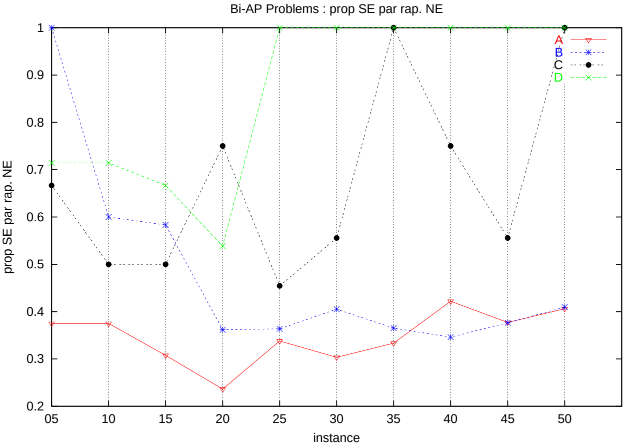

Overview

The Multiobjective Shortest Path Problem

- Digraph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with arc costs $c_{ij}^k, k = 1, \ldots, p, (i,j) \in \mathcal{A}$
- Given origin $s \in \mathcal{V}$, destination $t \in \mathcal{V}$ find efficient paths from s to t:

$$\min_{P \in \mathcal{P}} \sum_{(i,j) \in P} c_{ij}$$

  where $\mathcal{P}$ is set of all s-t paths
- Assume that all $c_{ij}^k \geqq 0$

Proposition

Let P_{st} be an efficient path from s to t. Then any subpath P_{uv} from u to v, where u and v are vertices on P_{st} is an efficient path from u to v.

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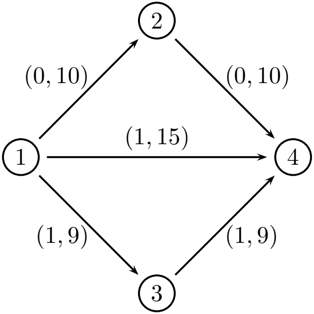
The Multiobjective Shortest Path Problem

Concatenations of efficient paths need not be efficient!



1-3 is efficient, 3-4 is efficient, 1-3-4 is not

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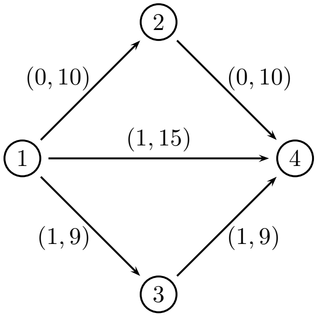

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The Multiobjective Shortest Path Problem

For a labelling algorithm we need

- Sets of nondominated labels at each node
- Store all costs, the predecessor, the label number at the predecessor, the number of the current label
- A list of permanent and temporary labels
- Make sure that a permanent label defines an efficient path: Choose the lexicographically smallest label from temporary list

Lemma

If P_1 and P_2 are two paths between nodes s and t and $c(P_1) \leq c(P_2)$ then $c(P_1) <_{lex} c(P_2)$.

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Algorithm (Multiobjective label setting algorithm)

Input: A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with p arc costs.

Initialization: Create label $L = (0, \ldots, 0, 0, 0, 1)$ at node s and let $\mathcal{TL} := \{L\}$.

While $\mathcal{TL} \neq \emptyset$ do

Let label $L = (c^1, \ldots, c^p, v_h, l, k)$ of node v_i be the lexicographically smallest label in $\mathcal{TL}$.

Remove L from $\mathcal{TL}$ and add it to $\mathcal{PL}$.

For all $v_j \in \mathcal{V}$ such that $(v_i, v_j) \in \mathcal{A}$ do

Create label $L' = (c^1 + c^1(v_i, v_j), \ldots, c^p + c^p(v_i, v_j), v_i, k, t)$ as the next label at node v_j and add it to $\mathcal{TL}$.

Delete all temporary labels of node v_j dominated by L', delete L' if it is dominated by another label of node v_j.

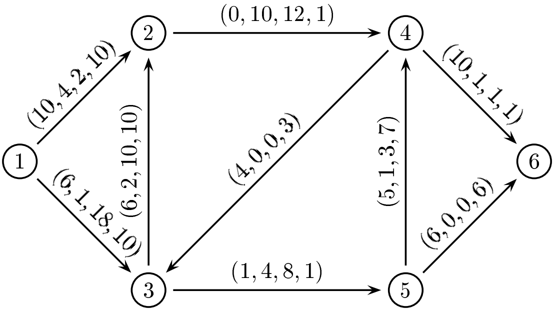
End for.

End while.

Use the predecessor labels in the permanent labels to recover all efficient paths from s to other nodes of $\mathcal{G}$.

Output: All efficient paths from node s to all other nodes of $\mathcal{G}$.

Multiobjective Label Setting Algorithm



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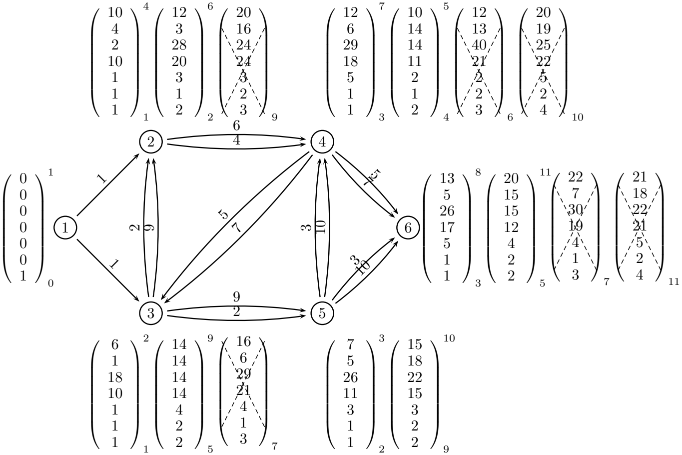

Multiobjective Label Correcting Algorithm

- Label setting fails if negative arc lengths are permitted
- Negative cycles C
  - Case 1: If $\sum_{a \in C} c_a^k < 0$ and $\sum_{a \in C} c_a^j > 0$ for $j \neq k$ there are infinitely many efficient paths
  - Case 2: If $\sum_{a \in C} c_a \leq 0$ there is no efficient path
- A label correcting algorithm is required
- Let $\mathcal{L}(i, k)$ be set of labels at node i in iteration k

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Algorithm (Multiobjective label correcting algorithm)

Input: A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with p arc costs.

Initialization: Set $d_{ii} := (0, \ldots, 0)$ for $i = 1, \ldots, n$.
Set $d_{ij} := (\infty, \ldots, \infty)$ if $v_i \neq v_j$ and $(v_i, v_j) \notin \mathcal{A}$.
Set $d_{ij} := (c^1(v_i, v_j), \ldots, c^p(v_i, v_j))$ otherwise.
Set $\mathcal{L}(i, 1) := \{d_{1i}\}, i = 1, \ldots, n$.

For $k := 1$ to $n - 1$ do

For $i := 1$ to n do

$$\mathcal{L}(i, k+1) := \min \bigcup_{j=1}^{n} \{d_{ji} + l_j^k : l_j^k \in \mathcal{L}(j, k)\}$$

End for.
If $\mathcal{L}(i, k+1) = \mathcal{L}(i, k)$ for all $i = 1, \ldots, n$ then

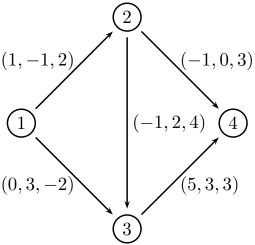
If $k = n - 1$ then STOP, a negative cycle exists.

STOP

End for.

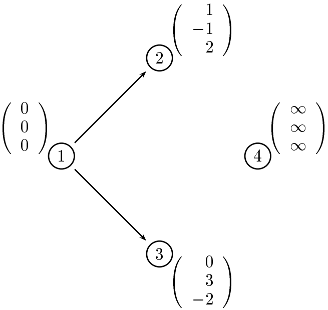
Output: All efficient paths from node v_1 to all other nodes.

Multiobjective Label Correcting Algorithm



$$d_{ij} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} & \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} \\ \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \\ \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \\ \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

Multiobjective Label Correcting Algorithm



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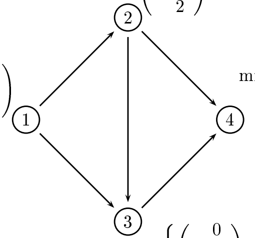

$$\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\min\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}+\begin{pmatrix}-1\\0\\3\end{pmatrix},\begin{pmatrix}0\\3\\-2\end{pmatrix}+\begin{pmatrix}5\\3\\3\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\-1\\5\end{pmatrix},\begin{pmatrix}5\\6\\1\end{pmatrix}\right\}$$

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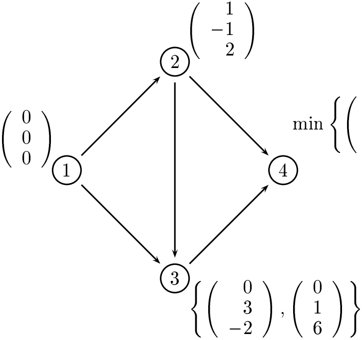
Multiobjective Label Correcting Algorithm



$$\min\left\{\begin{pmatrix}0\\-1\\5\end{pmatrix},\begin{pmatrix}5\\6\\1\end{pmatrix},\begin{pmatrix}0\\1\\6\end{pmatrix}+\begin{pmatrix}5\\3\\3\end{pmatrix}\right\}$$
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The Multiobjective Spanning Tree Problem

- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ with edge costs $c_{ij}^k, k = 1, \ldots, p; (i,j) \in \mathcal{E}$
- Find efficient spanning trees of $\mathcal{G}$:

$$\min_{T \in \mathcal{T}} \sum_{[i,j] \in T} c_{ij}$$

where $\mathcal{T}$ is set of all spanning trees of $\mathcal{G}$

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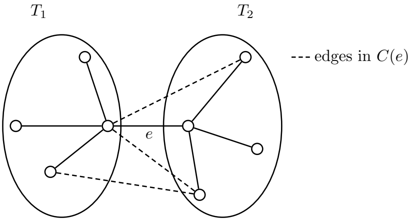
The Multiobjective Spanning Tree Problem

Theorem (Hamacher and Ruhe 1994)

T efficient spanning tree of $\mathcal{G}$

1. *Let $e \in \mathcal{E}(T)$ be an edge of T. Let $(\mathcal{V}(T_1), \mathcal{E}(T_1))$ and $(\mathcal{V}(T_2), \mathcal{E}(T_2))$ be the two connected components of $\mathcal{G} \setminus \{e\}$. Let $C(e) := \{f = (v_i, v_j) \in \mathcal{E} : v_i \in \mathcal{V}(T_1), v_j \in \mathcal{V}(T_2)\}$ be the cut defined by deleting e. Then $c(e) \in \min\{c(f) : f \in C(e)\}$.*

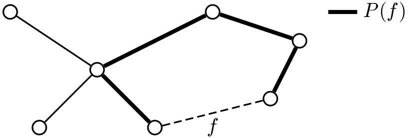


The Multiobjective Spanning Tree Problem

Theorem (Hamacher and Ruhe 1994)

T efficient spanning tree of $\mathcal{G}$

1. *Let $f \in \mathcal{E} \setminus \mathcal{E}(T)$ and let $P(f)$ be the unique path in T connecting the end nodes of f. Then $c(f) \leq c(e)$ does not hold for any $e \in P(f)$.*



The Multiobjective Spanning Tree Problem

Algorithm (Prim's spanning tree algorithm)

Input: *A graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *with* p *edge costs.*

$\mathcal{T}_0 := \{(\{1\}, \emptyset)\}$

For $k := 1$ *to* $n-1$ *do*

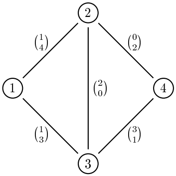
$\mathcal{T}_k := \{\mathcal{E}(T) \cup \{e_j\} : T \in \mathcal{T}_{k-1}, e_j \in argmin\{c(e) = c([v_i, v_j]) : v_i \in \mathcal{V}(T), v_j \in \mathcal{V} \setminus \mathcal{V}(T)\}\}$

End for

$\mathcal{T}_{n-1} := argmin\{c(T) : T \in \mathcal{T}_{n-1}\}$

Output: $\mathcal{T}_{n-1}$, *all efficient spanning trees of* $\mathcal{G}$.

The Multiobjective Spanning Tree Problem



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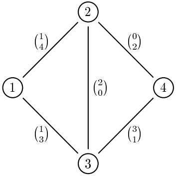

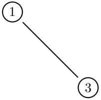

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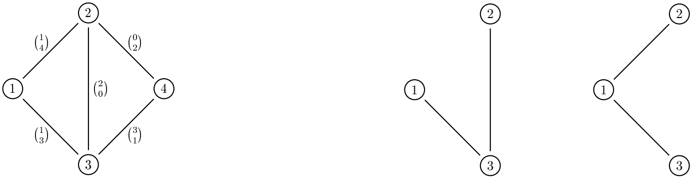

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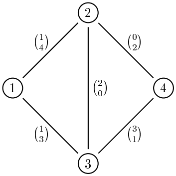

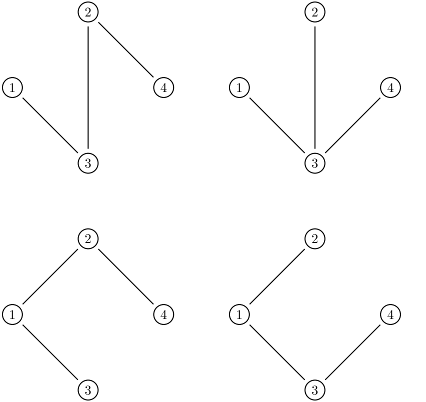

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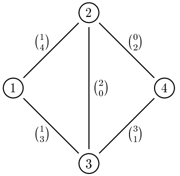

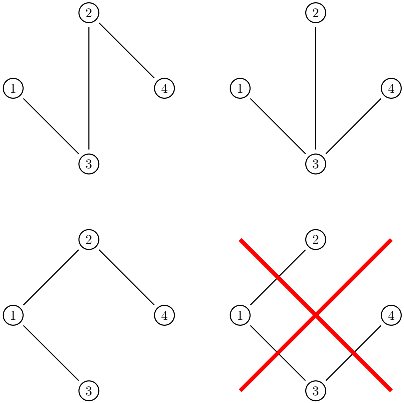

Overview

The Two Phase Method with 2 Objectives

- Phase 1: Compute $X_{SE(1)}$
  1. Find lexicographic solutions
  2. Recursively:
     Calculate λ
     Solve $\min_{x \in X} \lambda^T Cx$
- Phase 2: Compute X_{NE}
  1. Solve by triangle
  2. Use neighborhood (wrong)
  3. Use constraints (bad)
  4. Use variable fixing (possible)
  5. Use ranking (good)

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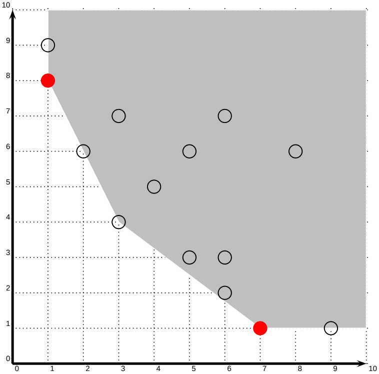

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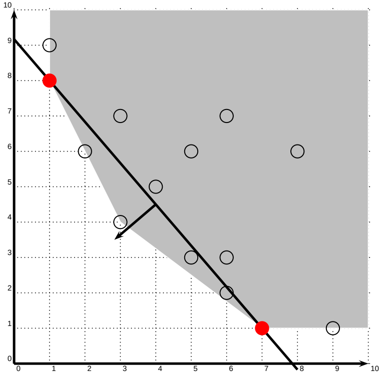

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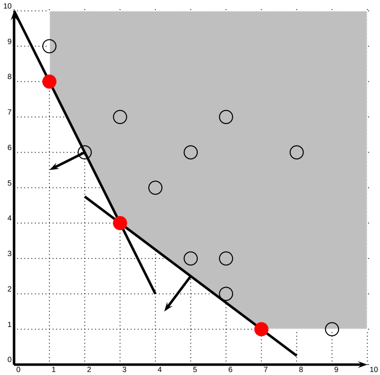

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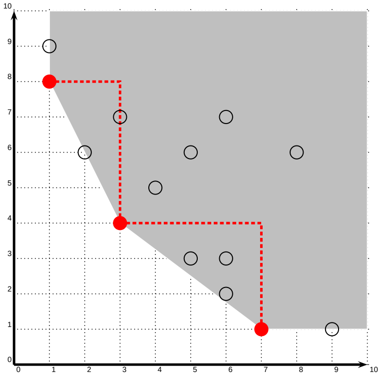

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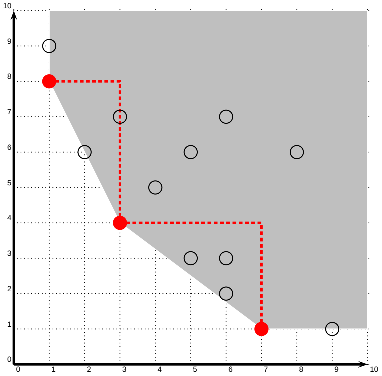

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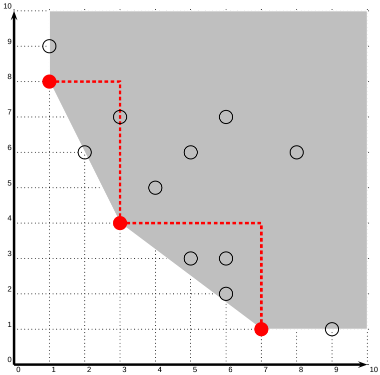
  5. Use ranking (good)



The Two Phase Method with 2 Objectives

- Phase 1: Compute $X_{SE(1)}$
  1. Find lexicographic solutions
  2. Recursively:
     Calculate λ
     Solve $\min_{x \in X} \lambda^T Cx$
- Phase 2: Compute X_{NE}
  1. Solve by triangle
  2. Use neighborhood (wrong)
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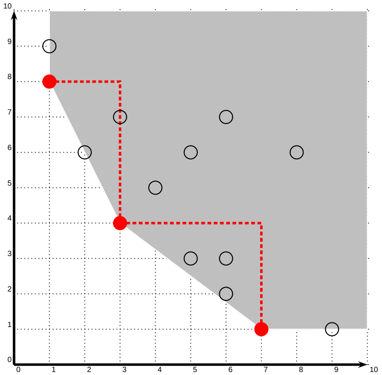

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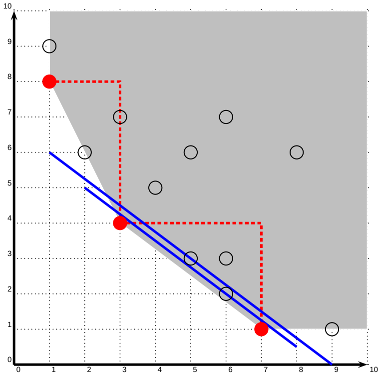
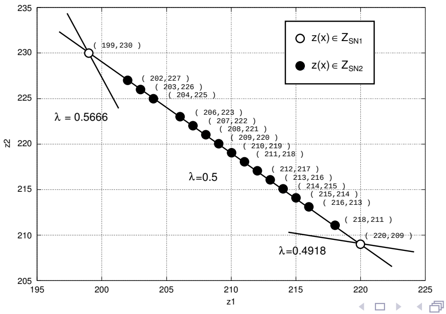


Enumeration Problems

- Finding maximal complete set:
  - Enumeration to find all optimal solutions of $\min_{x \in X} \lambda^T Cx$
  - Enumeration to find all $x \in X_{NE}$ with $Cx = y \in Y_{ND}$
- Finding minimal complete set:
  - Enumeration to find X_{SE2}



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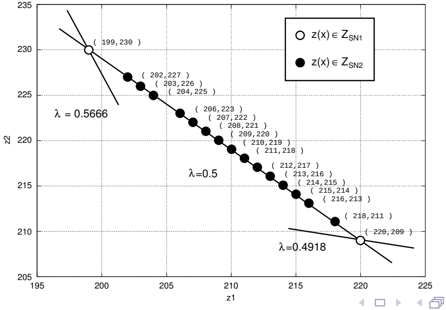

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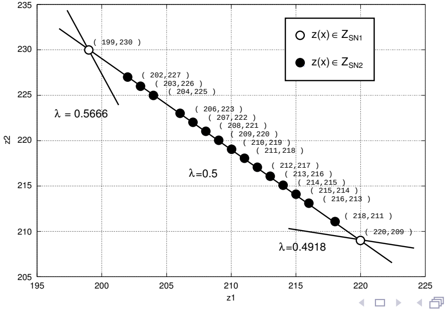

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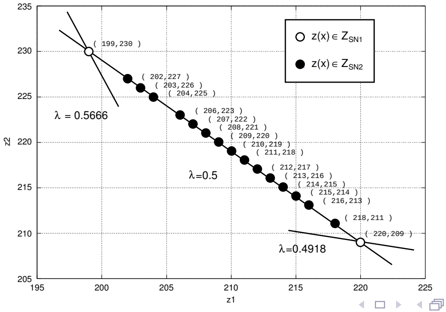

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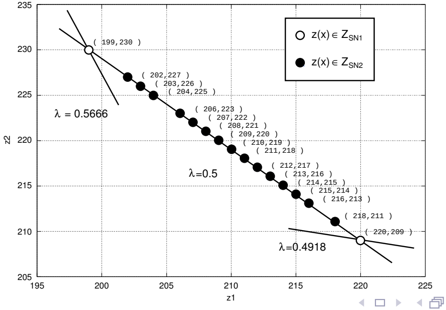
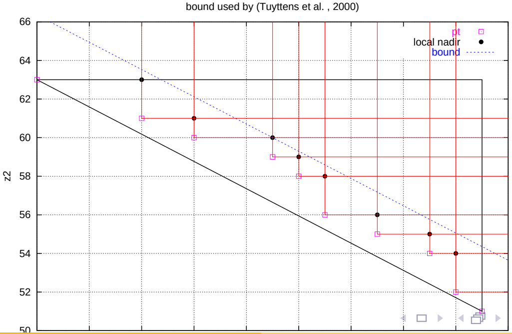


Bounds on $\lambda^T Cx$ in Phase Two

$\{x^i : 0 \leqq i \leqq q\}$ candidates for X_{NE} sorted by increasing z^1 in $\Delta(x^r, x^s)$

$$\gamma := \max_{i=0}^{q-1}\{\lambda_1 z_1(x^{i+1}) + \lambda_2 z_2(x^i)\}$$

$$\beta_0 := \max\left\{\gamma, \lambda^1 z^1(x^0) + \lambda^2 z^2(x^r), \lambda^1 z^1(x^s) + \lambda^2 z^2(x^q)\right\}$$



Bounds on $\lambda^T Cx$ in Phase Two

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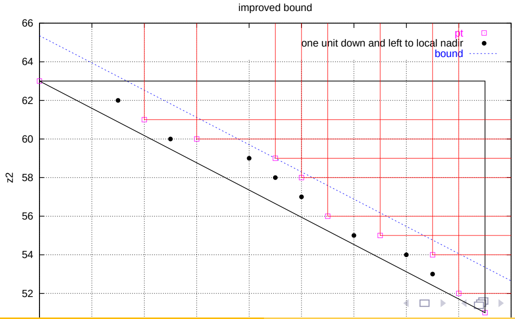
$$\delta_1 := \max_{i=0}^{q} \{\lambda^1 z^1(x^i) + \lambda^2 z^2(x^i)\}$$

$$\delta_2 := \max_{i=1}^{q} \{\lambda^1 (z^1(x^i) - 1) + \lambda^2 (z^2(x^{i-1}) - 1)\}$$

$$\beta_1 := \max \left\{ \delta_1, \delta_2, \lambda^1(z^1(x^0) - 1) + \lambda^2(z^2(x^r) - 1), \lambda^1(z^1(x^s) - 1) + \lambda^2(z^2(x^q) - 1) \right\}$$



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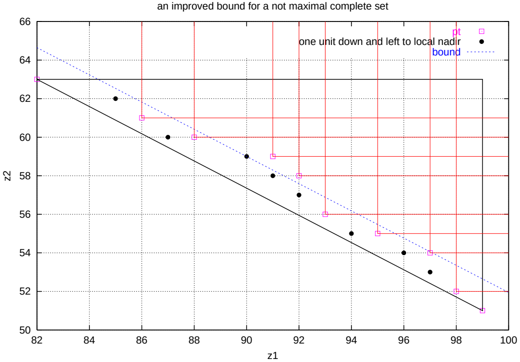

Two Phase Algorithm for Biobjective Assignment

Przybylski et al. 2006:

- Hungarian Method for $\min_{x \in X} \lambda^T Cx$
- Enumeration of all optimal solutions of $\min_{x \in X} \lambda^T Cx$ (Fukuda and Matsui 1992)
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Two Phase Algorithm for Biobjective Assignment

Results for 100×100 (1 GHz, 512 MB RAM):

Range of c_{ij}^k	Variable Fixing	Seek & Cut	Ranking
$[0, 20]$	14049.17	2251.72	228.26
$[0, 40]$	$\times$	17441.35	225.06
$[0, 60]$	$\times$	38553.18	399.65
$[0, 80]$	$\times$	53747.45	721.08
$[0, 100]$	$\times$	60227.31	711.97

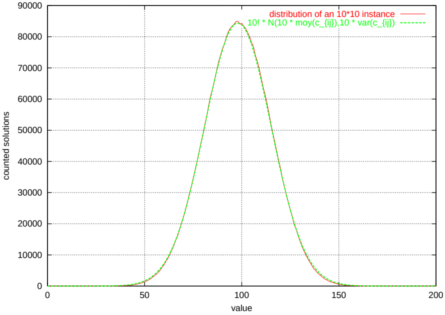
Two Phase Algorithm for Biobjective Assignment

Comparison with CPLEX 9.0 using constraints (3.4 GHz, 4 GB RAM)

Range of c_{ij}^k	CPLEX 9.0	Ranking
$[0, 20]$	200.63	85.58
$[0, 40]$	512.96	83.63
$[0, 60]$	1730.65	149.73
$[0, 80]$	3766.00	274.06
$[0, 100]$	4822.00	275.09

Two Phase Algorithm for Biobjective Assignment

- Objective values of an AP with $c_{ij} \in \{0, \dots, r-1\}$



- Proof by Przybylski and Bourdon 2006:
$\mu = \frac{n(r-1)}{2}, \sigma^2 = \frac{n(r^2-1)}{12}$

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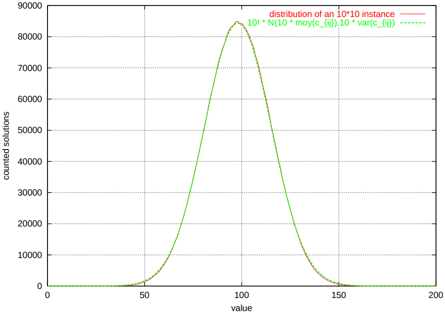

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The Two Phase Method with 3 Objectives

- Hyperplane defined by 3 points, possibly 6 lexicographically optimal points
- Normal vector defined by 3 nondominated points may not be positive

$$C^1 = \begin{pmatrix} 3 & 6 & 4 & 5 \\ 2 & 3 & 5 & 4 \\ 3 & 5 & 4 & 2 \\ 4 & 5 & 3 & 6 \end{pmatrix}, C^2 = \begin{pmatrix} 2 & 3 & 5 & 4 \\ 5 & 3 & 4 & 3 \\ 5 & 2 & 6 & 4 \\ 4 & 5 & 2 & 5 \end{pmatrix}, C^3 = \begin{pmatrix} 4 & 2 & 4 & 2 \\ 4 & 2 & 4 & 6 \\ 4 & 2 & 6 & 3 \\ 2 & 4 & 5 & 3 \end{pmatrix}$$

- 3 lexicographic points (11,11,14), (15,9,17) and (19,14,10)
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Weight Space Decomposition

$$W^0 := \left\{ \lambda : \lambda_1 > 0, \ldots, \lambda_p > 0, \lambda_p = 1 - \sum_{k=1}^{p-1} \lambda_k \right\}$$

$$W^0(y) := \{ \lambda \in W^0 : \lambda^T y \leqq \lambda^t Y' \text{ for all } y' \in Y \}$$

Proposition

1. *If y is supported but not extreme then $W^0(y) = \bigcap_{i=1}^{k} W^0(y^i)$ where $y^i, i \in \{1, \ldots, k\}$ are the extreme points of the face of conv(Y) that contains y in its relative interior.*
2. *Let y be a supported point, then $W^0(y) = \{\lambda \in W^0 : \lambda^T y \leqq \lambda^T y'$ for all supported extreme points y'.*
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Let $\{y^1, \ldots, y^n\}$ be the set of supported extreme points, then $W^0 = \bigcup_{i=1}^n W^0(y^i)$.

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Weight Space Decomposition

Proposition (Optimality Condition)

If S is a set of supported points then

$$Y_{SN1} \subseteq S \iff W^0 = \bigcup_{y \in S} W^0(y).$$

- Let S be a set of supported points
- Let $W_p^0(y) = \{\lambda \in W^0 : \langle \lambda, y \rangle \leq \langle \lambda, y^* \rangle \text{ for all } y^* \in S\}$
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Algorithm (Phase 1 with 3 Objectives)

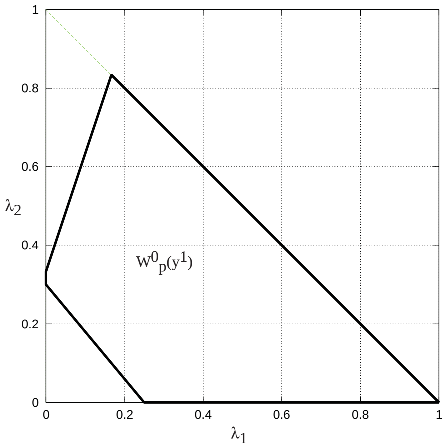
1. *Initialize S with the lexicographically optimal points*
2. *For all $\hat{y} \in S$*
   - *Compute $W_p^0(\hat{y})$*
   - *Investigate all facets F of $W_p^0(\hat{y})$ defined by $\lambda^T \hat{y} = \lambda^T y'$ for $y' \in S$ to determine whether F is also a facet of $W^0(\hat{y})$*
     - *If $\hat{y}$ minimizes $\lambda^T y$ for all $\lambda \in F$ then $\hat{y}$ and y' are adjacent and F is the common face of $W^0(\hat{y})$ and $W^0(y')$*
     - *If there are $y^* \in Y$ and $\lambda \in F$ such that $\lambda^T y^* < \lambda^T y$ then $W^0(\hat{y}) \subset W_p^0(\hat{y})$ and y^* is added to S and $W_p^0(\hat{y})$ is updated*

An Example

$$C^1 = \begin{pmatrix} 2 & 5 & 4 & 7 \\ 3 & 3 & 5 & 7 \\ 3 & 8 & 4 & 2 \\ 6 & 5 & 2 & 5 \end{pmatrix}, \; C^2 = \begin{pmatrix} 3 & 3 & 6 & 2 \\ 5 & 3 & 7 & 3 \\ 5 & 2 & 7 & 4 \\ 4 & 6 & 3 & 5 \end{pmatrix}, \; C^3 = \begin{pmatrix} 4 & 2 & 5 & 3 \\ 5 & 3 & 4 & 3 \\ 4 & 3 & 5 & 2 \\ 6 & 4 & 7 & 3 \end{pmatrix}$$

- Lexicographically optimal points: $y^1 = (9, 13, 16)$, $y^2 = (19, 11, 17)$, $y^3 = (18, 20, 13)$
- $S = \{y^1, y^2, y^3\}$

An Example



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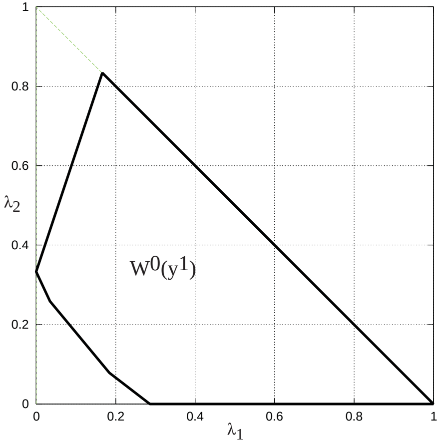

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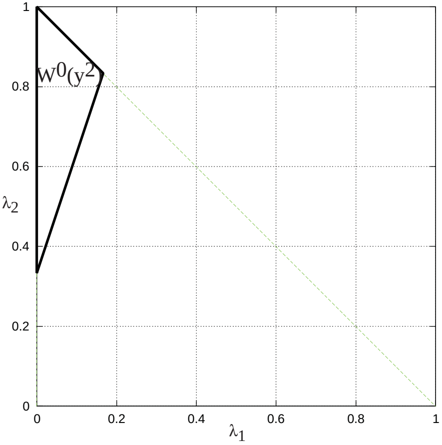

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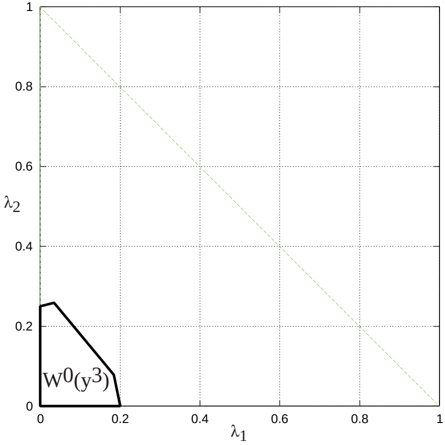

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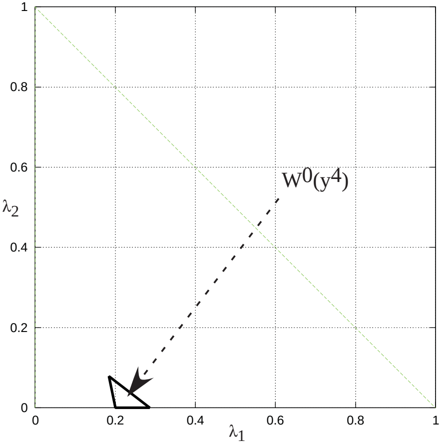

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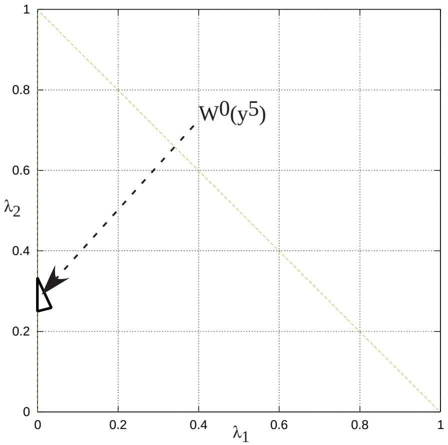

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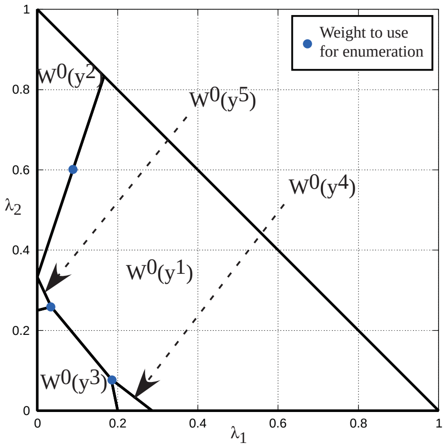
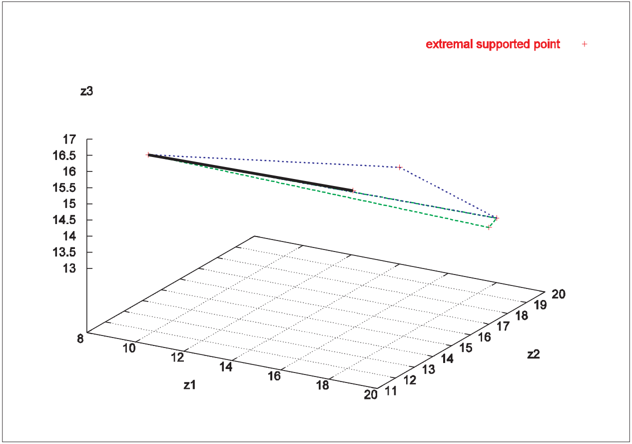

1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
λ_2
λ_1
Weight to use for enumeration
$W^0(y^2)$
$W^0(y^5)$
$W^0(y^4)$
$W^0(y^1)$
$W^0(y^3)$

An Example


extremal supported point
z3
17
16.5
16
15.5
15
14.5
14
13.5
13
8
10
12
14
16
18
20
z1
11
12
13
14
15
16
17
18
19
20
z2

Numerical Results

Size	$\lvert Y_N \rvert$	SC 2004	T-P 2003	LZT 2005	2 Phase
5	12	0.15	0.04	0.15	0.00
10	221	99865.00	97.30	41.70	0.08
15	483	×	544.53	172.29	0.36
20	1942	×	×	1607.92	4.51
25	3750	×	×	5218.00	30.13
30	5195	×	×	15579.00	55.87
35	10498	×	×	101751.00	109.96
40	14733	×	×	×	229.05
45	23941	×	×	×	471.60
50	29193	×	×	×	802.68

Comments

- Two phase method respects problem structure
- Successful for assignment, shortest path, network flow
- Requires ranking algorithm, which is available for most polynomially solvable problems

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Overview

Principle and Properties of Scalarization

Convert multiobjective problem to (parameterized) single objective problem and solve repeatedly with different parameter values

Desirable properties of scalarizations: (Wierzbicki 1984)

- Correctness: Optimal solutions are (weakly) efficient
- Completeness: All efficient solutions can be found
- Computability: Scalarization is not harder than single objective version of problem (theory and practice)
- Linearity: Scalarization has linear formulation

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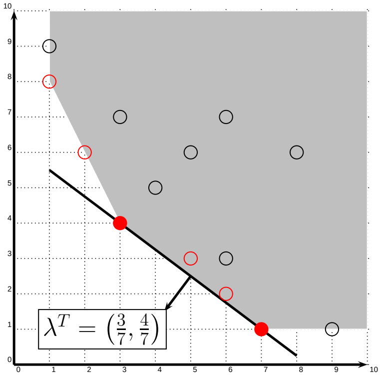
Scalarization Methods

- Weighted sum:
  $$\min_{x \in X} \left\{ \lambda^T z(x) \right\}$$
- ε-constraint:
  $$\min_{x \in X} \{ z_l(x) : z_k(x) \leq \varepsilon_k, k \neq l \}$$
- Weighted Chebychev:
  $$\min_{x \in X} \left\{ \max_{k=1,\ldots,p} \nu_k (z_k(x) - y_k^I) \right\}$$



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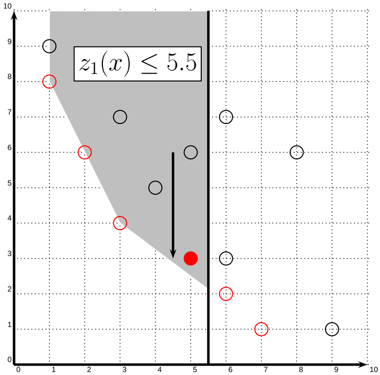

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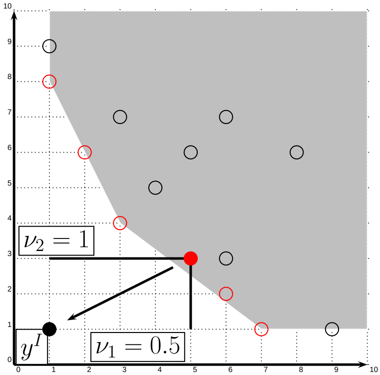

General Formulation

$$\min_{x \in X} \left\{ \max_{k=1}^{p} [\nu_k(c_k x - \rho_k)] + \sum_{k=1}^{p} [\lambda_k(c_k x - \rho_k)] \right\}$$

$$\text{subject to} \quad c_k x \leq \varepsilon_k \qquad k = 1, \ldots, p$$

Includes	Correct	Complete	Computable	Linear
Weighted sum	+	-	+	+
ε-constraint	+	+	-	+
Benson	+	+	-	+
Chebychev	+	(+)	(-)	+
Max-ordering	+	+	-	+
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General Formulation

Theorem (Ehrgott 2005)

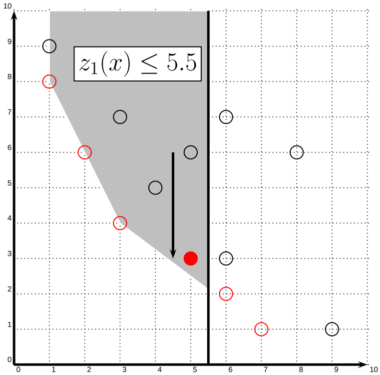
1. *The general scalarization is NP-hard.*
2. *An optimal solution of the Lagrangian dual of the linearized general scalarization is a supported efficient solution.*

Method of Elastic Constraints

$$\min_{x\in X} c_l x + \sum_{k\neq l} \mu_k w_k$$

$$\text{s.t.} \quad c_k x + v_k - w_k \leq \varepsilon_k \quad k \neq l$$

$$v_k, w_k \geq 0 \quad k \neq l$$



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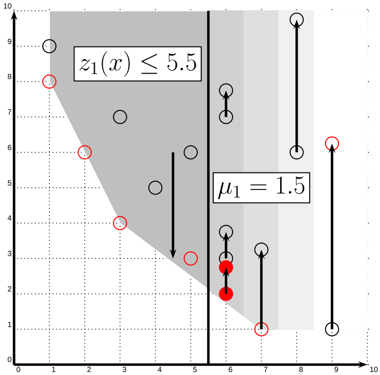

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Theorem (Ehrgott and Ryan 2002)

The method of elastic constraints

- *is correct and complete,*
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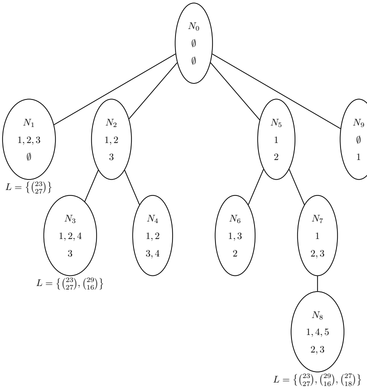
Multicriteria Branch and Bound

- Ulungu and Teghem 1997, Mavrotas and Diakoulaki 2002
- Branching: As in single objective case
- Bounding: Ideal point of problem at node is dominated by efficient solution
- Branching may be very ineffective
- Use lower and upper bound sets



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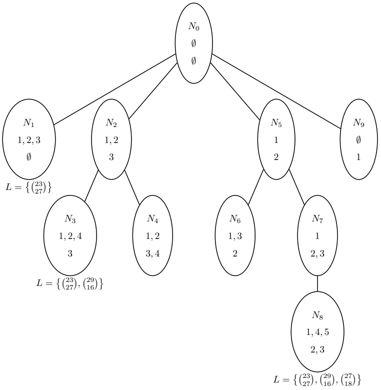

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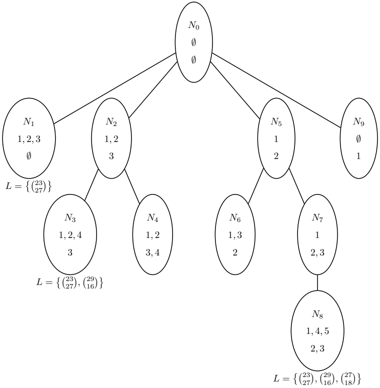

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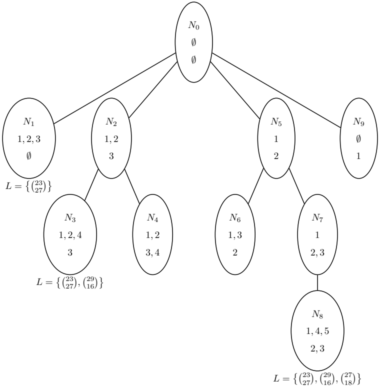

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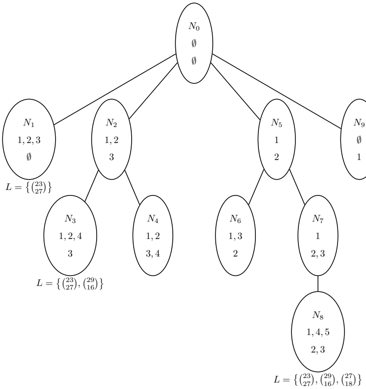

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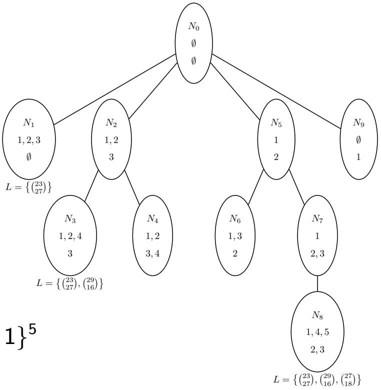

$$
\begin{array}{lrcl}
\max & 11x_1 + 5x_2 + 7x_3 + 13x_4 + 3x_5 & & \\
\max & 9x_1 + 2x_2 + 16x_3 + 5x_4 + 4x_5 & & \\
\text{s. t.} & 4x_1 + 2x_2 + 8x_3 + 7x_4 + 5x_5 & \leq & 16 \\
 & x & \in & \{0,1\}^5
\end{array}
$$

Bound Sets

Ehrgott and Gandibleux 2005:

1. Lower bound set L
   - is $\mathbb{R}^p_{\geqq}$-closed
   - is $\mathbb{R}^p_{\geqq}$-bounded
   - $Y_N \subset L + \mathbb{R}^p_{\geqq}$
   - $L \subset \left(L + \mathbb{R}^p_{\geqq}\right)_N$
2. Upper bound set U
   - is $\mathbb{R}^p_{\geqq}$-closed
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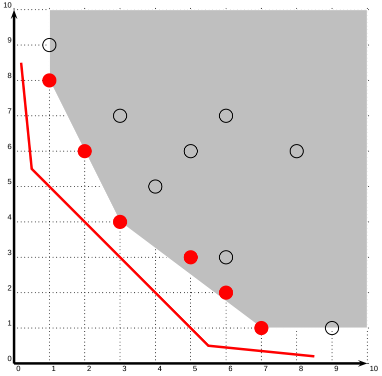

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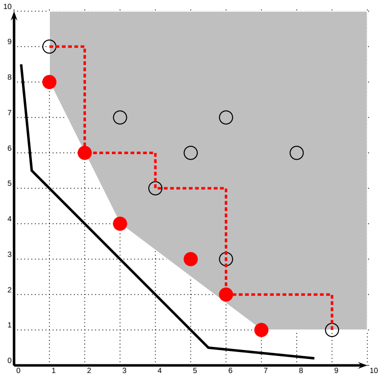

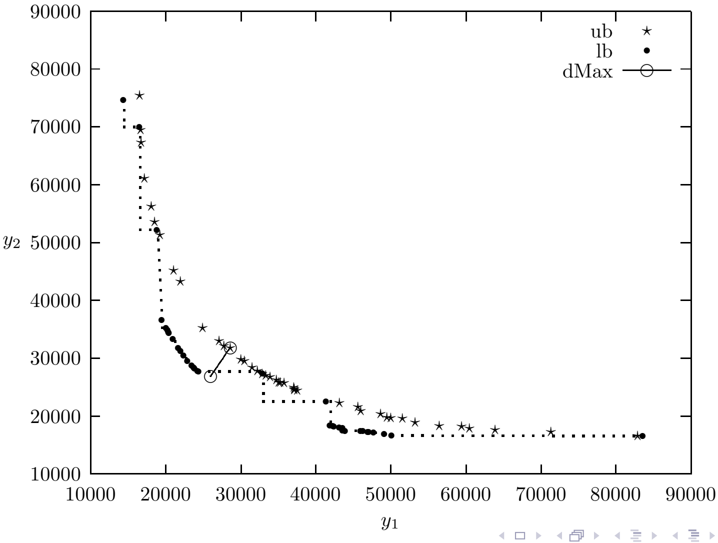
ub
lb
dMax
90000
80000
70000
60000
50000
40000
30000
20000
10000
y_2
10000
20000
30000
40000
50000
60000
70000
80000
90000
y_1

Overview

Conclusion

- MOCO problems as special multiobjective integer programmes
- Distinguish supported and non-supported efficient solutions
- MOCO problems are NP-hard, #P-hard, intractable
- MOCO problems can be solvable in practice
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- Spanning tree: Prim's (and Kruskal's) algorithm transferable, i.e. greedy algorithm works
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- MOCO problems can be solvable in practice
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- Spanning tree: Prim's (and Kruskal's) algorithm transferable, i.e. greedy algorithm works
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