General Aspects of Social Choice Theory

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Overview

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  - ordinal vs. cardinal

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  - Sen's theorem - freedom aspects (1970)

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- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:
  - Arrow's theorem - general aspects (1951)
  - Sen's theorem - freedom aspects (1970)
  - Gibbard-Satterthwaite theorem - strategic aspects (1973/75)

Collective Decision Rule

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Example

Let $X = \{x, y, z\}$ and xPy,yIz and xIz. What properties does this relation satisfy?

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Usually in social choice theory we work with *linear orders*, i.e. strict rankings of the alternatives.

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What is a choice function?

Definition (Choice function)

Let $\mathcal{X}$ be the set of all non-empty subsets of X. A choice function is a function $C : \mathcal{X} \to \mathcal{X}$ s.t. $\forall S \in \mathcal{X}$, $C(S) \subseteq S$.

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Let $X = \{x, y, z\}$ and the choice function be s.t.
$C(\{x, y, z\} = C(\{x, y\}) = y$ and $C(\{x, z\}) = C(\{y, z\}) = z$.

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Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

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$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$.

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Also, let for all $B \in \mathcal{B}$, B^* denote its transitive closure, i.e. xB^*y if and only if there exists a sequence $z_1, z_2, ..., z_k \in X$ s.t. xBz_1, z_1Bz_2, ... ,z_kBy.

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The transitive closure rule assigns to all $p \in \mathcal{R}^n$ a choice function on X s.t. $\forall S \in \mathcal{X}$, $C(S) = M(S, B^*)$, where B^* is the transitive closure of the simple majority relation B for p.

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Let $X = \{x, y, z\}$, what does the transitive closure rule give for the Condorcet paradox?

Properties of social welfare functions

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Dictatorship

Arrow's impossibility theorem

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Definition (Nondictatorship)

$\nexists i \in N$ s.t. $\forall p \in \mathcal{R}^n$ and $x, y \in X$, xP_iy implies xPy.

Theorem (Arrow's theorem)

Let $|N| \geq 2$ and $|X| \geq 3$. There exists no SWF that satisfies UD, WP, IIA and ND.

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Example (Violation of IIA by Borda rule)

R_1	R_2	R_3	R'_1	R'_2	R'_3
a	d	d	a	d	d
c	c	c	b	c	c
b	a	a	d	a	a
d	b	b	c	b	b

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$G \subseteq N$ is decisive over the ordered pair $\{x, y\}$, $\bar{D}_G(x, y)$ iff xP_iy, $\forall i \in G$ implies xPy.

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Definition (Almost decisiveness)

$G \subseteq N$ is almost decisive over ordered pair $\{x, y\}$, $D_G(x, y)$ iff xP_iy, $\forall i \in G$ and yP_ix, $\forall i \in N \backslash G$ implies xPy.

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Lemma (Field expansion lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$*, if a group G is almost decisive over some ordered pair* $\{x, y\}$*, then it is decisive over every ordered pair, i.e.*
$[\exists x, y \in X : D_G(x, y)] \Rightarrow \left[\forall a, b \in X : \bar{D}_G(a, b)\right]$

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Lemma (Group contraction lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$*, if any group G with* $|G| > 1$ *is decisive, then so is some proper subgroup of G.*

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

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- aPx and yPb because of WP
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Field expansion lemma

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- Hence: $\bar{D}_G(a, b)$

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Partition G into G_1 and G_2

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- from field expansion lemma either G_1 or G_2 is decisive

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- WP and field expansion lemma implies that N is decisive
- by the group contraction lemma we can eliminate members of N until we are left with a dictator.

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- but many resolutions lead to other "dictator-like" results with veto rights or oligarchies

Sen's Liberal Paradox

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> *[Sen, 1970] If you prefer to have pink walls rather then white, the society should permit you to have this even if a majoritiy of the community would like to see your walls white.*

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There exist at least 2 individuals s.t. each of them is decisive over at least one pair of alternatives, i.e. if i is decisive over (x, y), then $xP_iy \Rightarrow xPy$.

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Theorem (Sen, 1970)

There exists no social decision function satisfying UD, WP and ML.

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- Leads to a cycle!

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Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem
  - it also works if we just consider the possibility of choices, i.e. acyclic social preferences
  - it does not use the rather criticized IIA condition
  - there is no satisfactory resolution via a broadening of the informational basis through interpersonal comparisons

Strategic aspects in voting

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> *My scheme is only intended for honest men! [Borda]*

> *Voters adopt a principle of voting which makes it more of a game of skill than a real test of the wishes of the electors. [Dodgson]*

> *Politicians are continually poking and pushing the world to get the results they want. The reason they do this is they believe (and rightly so) that they can change outcomes by their efforts. It is often the case that voting need not have turned out the way it did. [Riker]*

Manipulability

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Social choice rule $f : \mathcal{R}^n \to X$ is manipulable by i at profile p via R_i' if $f(p')P_i f(p)$.

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Definition (Manipulability)

Social choice rule $f : \mathcal{R}^n \to X$ is manipulable by i at profile p via R_i' if $f(p')P_i f(p)$.

Theorem (Gibbard-Satterthwaite)

Let $|N| \geq 2$ and $|X| \geq 3$. If f is non-manipulable and satisfies WP, it is a dictatorship.

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- There is an inconsistency between basic reasonable properties. [Arrow]
- There is an inconsistency between basic liberal aspects and the Pareto principle. [Sen]
- There is an inconsistency between basic strategic aspects and the Pareto principle. [Gibbard-Satterthwaite]

Literature

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