ELECTRE TRI-C: A Multiple Criteria Sorting Method Based on Central Reference Actions

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Contents

- Sorting problematic: the set of categories emerges naturally from the decision aiding context through an interaction process with the decision-makers.
- Nature of the categories: each category is defined in order to assign actions which will be subject to the same treatment or analysis.
- Absolute evaluation: the assignment of an action only takes into account the intrinsic evaluation of this action on all the criteria and does not depend on nor influence the category to which another action should be assigned.

Example (Credit analysis)

- Actions: Credit demand files
- Categories:
  - Accepted without additional information
  - Accepted with additional information
  - Sent to a particular department further analysis
  - Rejected under certain conditions
  - Rejected with no conditions at all

Example (Medical diagnosis)

- Actions: Patients waiting for treatment
- Categories: Set of pathologies studied

Profile limits

- Each category is bounded by a lower and an upper profile.
- The well-known method called up to now ELECTRE TRI based on profile limits, or boundary actions, will be designated here by ELECTRE TRI-B.

Central reference actions

- Each category is defined by a central reference action.
- ELECTRE TRI-C is, therefore, the designation of the procedures based on central reference actions.

Key concepts

Actions, criteria, and credibility index

- $a_1, a_2, \ldots$ are the potential actions. The set of such actions, A, can be partially known *a priori*.
- $F = \{g_1, \ldots, g_j, \ldots, g_n\}$ is a coherent family of criteria, with $n \geq 2$.
- $\Omega_j(a, a')$ is the advantage of a over a' on criterion $g_j \in F$,

$$\Omega_j(a, a') = \begin{cases} g_j(a) - g_j(a') & \text{if } g_j \text{ is to be maximized} \\ g_j(a') - g_j(a) & \text{if } g_j \text{ is to be minimized} \end{cases}$$

- Each criterion $g_j \in F$ will be considered as a pseudo-criterion.
  Definition
- $\sigma(a, a')$ is the credibility of the comprehensive outranking of a over a' when taking all the criteria from F into account. Definition

- $C = \{C_1, \ldots, C_h, \ldots, C_q\}$ is the set of pre-defined and ordered categories, where C_1 is the worst category, and C_q the best one, with $q \geq 2$.
- Each category C_h is defined by a central reference action b_h, $h = 1, \ldots, q$.
- $B = \{b_0, b_1, \ldots, b_h, \ldots, b_q, b_{q+1}\}$ is the set of $(q+2)$ reference actions.
- b_0 is a particular reference action with the worst possible evaluation $g_j(b_0)$ on criterion g_j, for all $g_j \in F$.
- b_{q+1} is a particular reference action with the best possible evaluation $g_j(b_{q+1})$ on criterion g_j, for all $g_j \in F$.

Definition (Dominance)

The set of reference actions, B, fulfills the (strict) dominance relation if and only if

$$\forall j,\ \Omega_j(b_{h+1}, b_h) \geq 0 \text{ and } \exists j,\ \Omega_j(b_{h+1}, b_h) > 0; h = 0, \ldots, q.$$

Definition (Strict separability condition)

The set of reference actions, B, fulfills the strict separability condition if and only if

$$\Omega_j(b_{h+1}, b_h) > p_j;\ j = 1, \ldots, n;\ \ h = 0, \ldots, q.$$

▸ Weak

Structural requirements

1. Conformity: Each central reference action, b_h, must be assigned to the category, C_h, $h = 1, \ldots, q$.
2. Monotonicity: If an action a strictly dominates a', then a is assigned to a category at least as good as the category a' is assigned to.
3. Homogeneity: Two actions must be assigned to the same category when they compare themselves in an identical manner with the reference actions.
4. Stability: After a modification of the set B by applying either a merging or a splitting procedure, the non-adjacent categories to the modified ones will remain with the same actions as before the modification.

Stability requirement

Definition (Basic modification procedures)

1. Merging procedure: The distinction between two consecutive categories, C_{h-1} and C_h, will be ignored by introducing a new central reference action, b'_h, such that:
   - $\Omega_j(b'_h, b_{h-1}) \geq 0$, for all $g_j \in F$
   - $\Omega_j(b_h, b'_h) \geq 0$, for all $g_j \in F$

2. Splitting procedure: The category C_h can be split into two new consecutive categories by introducing two new central reference actions, b'_h and b''_h, such that:
   - $b_{h+1} \; \Delta_F \; b''_h$, $b''_h \; \Delta_F \; b'_h$, and $b'_h \; \Delta_F \; b_{h-1}$
   - $\Omega_j(b''_h, b_h) \geq 0$, for all $g_j \in F$
   - $\Omega_j(b_h, b'_h) \geq 0$, for all $g_j \in F$

Slackness functions

Definition (Slackness functions)

Let $\lambda \in [0.5, 1]$ denote the chosen majority level:

1. Direct slackness function:
   $\xi_h^+(a, \lambda) = \sigma(a, b_h) - \lambda$, $h = (q+1), \ldots, 0$.
2. Reverse slackness function:
   $\xi_h^-(a, \lambda) = \sigma(b_h, a) - \lambda$, $h = 0, \ldots, (q+1)$.

Proposition

1. $\xi_h^+(\cdot)$ does not decrease when moving from a given category to a worst one.
2. $\xi_h^-(\cdot)$ does not decrease when moving from a given category to a best one.

Assignment rules of ELECTRE TRI-C

Definition (Descending assignment rule)

- Choose a majority level λ ($0.5 \leq \lambda \leq 1$).
- Decrease h from $(q+1)$ until the first value such that $\xi_h^+(a,\lambda) \geq 0$.
- If $\xi_h^+(a,\lambda) \leq |\xi_{h+1}^+(a,\lambda)|$, then assign action a to category C_h. Otherwise, assign a to C_{h+1}.

Definition (Ascending assignment rule)

- Choose a majority level λ ($0.5 \leq \lambda \leq 1$).
- Increase h from 0 until the first value such that $\xi_h^-(a,\lambda) \geq 0$.
- If $\xi_h^-(a,\lambda) \leq |\xi_{h-1}^-(a,\lambda)|$, then assign action a to category C_h. Otherwise, assign a to C_{h-1}.

▸ Example ▸ Comparing ELECTRE TRI-C rules ▸ ELECTRE TRI-B rules

Example

Actions	b_1	b_2	b_3	b_4	Descending	Ascending
a_1	$\succ$	$\succ$	R^λ	R^λ	C_2	C_4
a_2	R^λ	R^λ	R^λ	R^λ	C_1	C_4
a_3	$\succ$	$\succ$	$\prec$	$\prec$	C_3	C_3
a_4	$\succ$	$\succ$	$\prec$	$\prec$	C_2	C_3
a_5	I^λ	$\prec$	$\prec$	$\prec$	C_2	C_1
a_6	$\succ$	$\succ$	R^λ	R^λ	C_2	C_4
a_7	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_2
a_8	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_1
a_9	R^λ	R^λ	R^λ	R^λ	C_1	C_4
a_{10}	$\succ$	$\prec$	$\prec$	$\prec$	C_1	C_1

Note: $\lambda = 0.70$; Source: Data adapted from Merad *et al.*, 2004

▸ ELECTRE TRI-C rules ▸ Comparison ELECTRE TRI-C rules

Properties of the assignment rules I
ELECTRE TRI-C

Theorem

a) *The monotonicity, homogeneity, and stability requirements hold.*

b) *If the strict separability condition is fulfilled, then the conformity requirement holds.*

Remark

The properties of uniqueness and independence of the assignments are also fulfilled.

Definition (Dominance)

The set of reference actions, B, fulfills the (strict) dominance relation if and only if

$$\forall j,\ \Omega_j(b_{h+1}, b_h) \geq 0 \text{ and } \exists j,\ \Omega_j(b_{h+1}, b_h) > 0; h = 0, \ldots, q.$$

Definition (Weak separability condition)

The set of reference actions, B, fulfills the weak separability condition if and only if

$$\forall j,\ \Omega_j(b_{h+1}, b_h) \geq 0 \text{ and } \exists j,\ \Omega_j(b_{h+1}, b_h) > p_j; h = 0, \ldots, q.$$

Strict

Properties of the assignment rules II

ELECTRE TRI-C

Theorem

If the weak separability condition is fulfilled, then there exists a compatible majority level, λ^c, for which the conformity requirement holds, whenever the chosen majority level $\lambda \geq \lambda^c$, such that

$$\lambda^c = \frac{1}{2} + \frac{1}{2} \max_{h = 0, \ldots, q} \left\{ \sigma(b_h, b_{h+1}) \right\}$$

▸ Example

Numerical example
Compatible majority level

Example

	b_1	b_2	b_3	b_4
b_1	1.0000	0.3696	0.0000	0.0000
b_2	1.0000	1.0000	0.0217	0.0217
b_3	1.0000	1.0000	1.0000	0.0652
b_4	1.0000	1.0000	1.0000	1.0000

Compatible majority level:

$$\lambda^c = \frac{1}{2} + \frac{1}{2} \max_{h=0,\ldots,4} \{\sigma(b_h, b_{h+1})\} = 0.69$$

Source: Data adapted from Merad *et al.*, 2004

Theorem

An overview of ELECTRE TRI-B
Basic assumptions

- $\widehat{C} = \{\widehat{C}_1, \ldots, \widehat{C}_h, \ldots, \widehat{C}_q\}$ is the set of pre-defined and ordered categories, where $\widehat{C}_1$ is the worst category and $\widehat{C}_q$ the best one, with $q \geq 2$.
- Each category $\widehat{C}_h$ is defined by a lower profile limit, $\hat{b}_{h-1}$, and an upper profile limit, $\hat{b}_h$, such that $\hat{b}_h \Delta_F \hat{b}_{h-1}$, $h = 1, \ldots, q$.
- $\widehat{B} = \{\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_h, \ldots, \hat{b}_{q-1}, \hat{b}_q\}$ is the set of the $(q+1)$ profile limits.
- $\hat{b}_0$ and $\hat{b}_q$ play a similar role as b_0 and b_{q+1} when using central reference actions.

Definition (Pseudo-conjunctive assignment rule)

- Choose a majority level λ ($0.5 \leq \lambda \leq 1$).
- Decrease h from q until the first value such that $\xi^{+}_{h-1}(a, \lambda) \geq 0$.
- Assign action a to category $\widehat{C}_h$.

Definition (Pseudo-disjunctive assignment rule)

- Choose a majority level λ ($0.5 \leq \lambda \leq 1$).
- Increase h from 0 until the first value such that $\xi^{-}_{h}(a, \lambda) \geq 0$ and $\xi^{+}_{h}(a, \lambda) < 0$.
- Assign action a to category $\widehat{C}_h$.

Comparing ELECTRE TRI-B rules | ELECTRE TRI-C rules

Theorem

Consider $(q+2)$ *reference actions defined to apply* ELECTRE TRI-C *with q categories. When such reference actions are used as profile limits of the* $(q+1)$ *categories in* ELECTRE TRI-B*,*

a) *if an action* a *is assigned to* C_h *by the* ELECTRE TRI-C *descending rule, then* a *is assigned to* $\widehat{C}_h$ *or* $\widehat{C}_{h+1}$ *by the* ELECTRE TRI-B *pseudo-conjunctive rule.*

b) *if an action* a *is assigned to* C_t *by the* ELECTRE TRI-C *ascending rule, then* a *is assigned to* $\widehat{C}_k$*, with* $k \geq t$*, by the* ELECTRE TRI-B *pseudo-disjunctive rule.*

▸ Example ▸ Comparing results II

Numerical example
Comparing assignment results

Example

	ELECTRE TRI-C		ELECTRE TRI-B	
Actions	Descending	Ascending	Pseudo-conjunctive	Pseudo-disjunctive
a_1	C_2	C_4	$\widehat{C}_3$	$\widehat{C}_5$
a_2	C_1	C_4	$\widehat{C}_1$	$\widehat{C}_5$
a_3	C_3	C_3	$\widehat{C}_3$	$\widehat{C}_3$
a_4	C_2	C_3	$\widehat{C}_3$	$\widehat{C}_3$
a_5	C_2	C_1	$\widehat{C}_2$	$\widehat{C}_2$
a_6	C_2	C_4	$\widehat{C}_3$	$\widehat{C}_5$
a_7	C_1	C_2	$\widehat{C}_2$	$\widehat{C}_2$
a_8	C_1	C_2	$\widehat{C}_1$	$\widehat{C}_1$
a_9	C_1	C_4	$\widehat{C}_1$	$\widehat{C}_5$
a_{10}	C_1	C_1	$\widehat{C}_1$	$\widehat{C}_1$

Note: $\lambda = 0.70$; Source: Data adapted from Merad *et al.*, 2004

Conclusions

- In ELECTRE TRI-C the categories are defined through central reference actions instead of profile limits.
- Central reference actions and profile limits are two alternative ways of defining ordered categories.
- ELECTRE TRI-C fulfills the properties of uniqueness, independence, conformity, monotonicity, homogeneity, and stability.
- When the set of reference actions does not fulfill the strict separability condition, but only a weak separability condition, then a compatible majority level is required.
- A comparison with ELECTRE TRI-B shows the main similarities of the two methods.

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Appendix

7 Appendix: Additional results

Pseudo-criterion model

Definition (Pseudo-criterion)

A pseudo-criterion is a function g_j associated with two threshold functions, $q_j(\cdot)$ and $p_j(\cdot)$, satisfying the following condition: for all actions a in the sets of actions, $g_j(a) + p_j$ and $g_j(a) + q_j$ are non-decreasing monotone functions of $g_j(a)$. (Roy, 1996)

Key concepts

Remark

Consider an ordered pair of actions (a, a'), and the two thresholds associated to the pseudo-criterion model,

a) $a \; P_j \; a' \Leftrightarrow \Omega_j(a, a') > p_j(\cdot)$

b) $a \; Q_j \; a' \Leftrightarrow q_j(\cdot) < \Omega_j(a, a') \leq p_j(\cdot)$

c) $a \; I_j \; a' \Leftrightarrow -q_j(\cdot) \leq \Omega_j(a, a') \leq q_j(\cdot)$

Definition (Credibility index)

The credibility of the comprehensive outranking of an action a over a', which means that a may be judged at least as good as a' when taking all the criteria from F into account, is defined by aggregating the comprehensive concordance index, $c(a, a')$, and the partial discordance indices, $d_j(a, a')$, as follows:

$$\sigma(a, a') = c(a, a') \prod_{j=1}^{n} T_j(a, a')$$

where,

$$T_j(a, a') = \begin{cases} \frac{1 - d_j(a,a')}{1 - c(a,a')} & \text{if } d_j(a, a') > c(a, a') \\ 1 & \text{otherwise} \end{cases}$$

Binary relations

Definition (λ-binary relations)

1. λ-outranking: $aS^\lambda a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0$
2. λ-indifference: $aI^\lambda a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0 \ \wedge \ \sigma(a',a) - \lambda \geq 0$
3. λ-incomparability: $aR^\lambda a' \Leftrightarrow \sigma(a,a') - \lambda < 0 \ \wedge \ \sigma(a',a) - \lambda < 0$
4. λ-preference: $a \succ a' \Leftrightarrow \sigma(a,a') - \lambda \geq 0 \ \wedge \ \sigma(a',a) - \lambda < 0$

Theorem

a) *If an action a is λ-indifferent to at least one reference action, then a is assigned by the descending rule to a category at least as good as the one a is assigned to when using the ascending rule.*

b) *If an action a is λ-incomparable to at least one reference action, then a is assigned by the descending rule to a category at most as good as the one a is assigned to when using the ascending rule.*

c) *Otherwise, both rules assign the action a to the same category or to two different but consecutive categories.*

▸ Example ▸ ELECTRE TRI-C rules

Roy and Bouyssou, 1993, p. 395

- If an action a is assigned to category $\widehat{C}_k$ by the pseudo-conjunctive rule and to $\widehat{C}_h$ by the pseudo-disjunctive rule, then $k \leq h$.
- Furthermore, the two assignment rules provide the same results if and only if there is no t such that $\xi_t^+(a, \lambda) < 0$ and $\xi_t^-(a, \lambda) < 0$ or there is at most one t such that $\xi_t^+(a, \lambda) \geq 0$ and $\xi_t^-(a, \lambda) \geq 0$.

ELECTRE TRI-B rules

Theorem

Consider $(q+1)$ profile limits defined to apply ELECTRE TRI-B *with q categories. When such profile limits are used as reference actions of the $(q-1)$ categories in* ELECTRE TRI-C,

a) *if an action a is assigned to $\widehat{C}_h$ by the* ELECTRE TRI-B *pseudo-conjunctive rule, then a is assigned to C_h or C_{h-1} by the* ELECTRE TRI-C *descending rule.*

b) *if an action a is assigned to $\widehat{C}_t$ by the* ELECTRE TRI-B *pseudo-disjunctive rule, then a is assigned to C_k, with $k \leq t$ by the* ELECTRE TRI-C *ascending rule.*

▸ Comparing results I

ELECTRE TRI-C particular results

Proposition

a) If a λ-outranks b_h, then a is assigned at least to C_h by the descending rule.

b) If b_h λ-outranks a, then a is assigned at most to C_h by the ascending rule.

c) If a is λ-preferred to b_h, then a is assigned at least to C_h by both rules.

d) If b_h is λ-preferred to a, then a is assigned at most to C_h by both rules.

ELECTRE methods with interaction between criteria: An extension of the concordance index

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Contents

Introduction

- This presentation is devoted to an extension of the comprehensive concordance index of ELECTRE methods.

- Such an extension have been considered to take into account the interaction between criteria.

- Three types of interaction effects has been considered, mutual strengthening, mutual weakening, and antagonistic.

- In real-world decision-making situations is reasonable to consider the interaction between a small number of pairs of criteria.

- Various conditions, boundary, monotonicity, and continuity have been imposed.

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Introduction

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- Environmental problems
- Constructions of indices (in this case several pairs ... of interaction criteria)

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Illustrative Example 1

Choosing the site for constructing a new hotel

- Criteria:

  g_1: land purchasing and construction costs (investment costs) [min];

  g_2: annual operating costs (annual costs) [min];

  g_3: personnel recruitment possibilities (recruitment) [max];

  g_4: target client perceptions of the city district (image) [max];

  g_5: facility of access for the target clients (access) [max].

- Mutual strengthening between criteria: investment and annual costs.
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Illustrative Example 1

The comparison of site a with sites b, c, and d, in terms of the two financial criteria g_1 and g_2

	b	c	d
g_1	a is better than b	a is worse than c	a is better than d
g_2	a is worse than b	a is better than c	a is better than d

According to the classic definition of the concordance index, the role that g_1 and g_2 should have for supporting the answer to the assertion "a is at least as good as b (or c or d)" is characterized by the following weights,

k_1 in the comparison with b,

k_2 in the comparison with c,

$k_1 + k_2$ in the comparison with d.

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Illustrative Example 1

DMR considers the weights k_1 and k_2 appropriate, when only one criterion, supports a decision that one action is better than another one.

However, he/she judges that the sum $k_1 + k_2$ is not sufficient to characterize the role of this criteria pair when both supports the decision

Because in this case each criterion is strengthened by the other given the degree of complementarity between them.

If one action is better than another one with respect to criteria g_1 and g_2 conjointly, it would be interesting to be able to take this mutual strengthening effect into account.

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DMR considers the weights k_1 and k_2 appropriate, when only one criterion, supports a decision that one action is better than another one.

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The comparisons of site a with sites b, c, d' in terms of the two purely ordinal criteria, g_4 and g_5

	b	c	d'
g_4	a is better than b	a is worse than c	a is better than d'
g_5	a is worse than b	a is better than c	a is better than d'

The DMR judges that the sum $k_4 + k_5$ is too high to characterize the role of this criteria pair when both supports the decision that one action is better than another one, because in this case each criterion is weakened by the other due to the degree of redundancy between them.

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LAUNCHING A NEW DIGITAL CAMERA MODEL

- Criteria:
  - g_1: purchasing costs (cost) [min];
  - g_2: weaknesses (fragility) [min];
  - g_3: user friendliness of the controls (workability) [max];
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Comparisons of digital camera model a with models b, c, and d, according to criteria g_1 and g_2:

	b	c	d
g_1	a is better than b	a at least as good as c	a is better than d
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According to the classic definition of the concordance index the role these criteria should play in supporting the assertion "model a is at least as good as model b (or c or d)" is characterized by the following weights,

$k_1 + k_2$ in the comparison with b,

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However, he/she considers that the same is not true when comparing a with d

Based on a customer survey, it seems that when one model is less fragile than another, the benefit derived from the lower cost is partially masked by the fact the model is less fragile.

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- Notation
- Pseudo criterion
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Notation

- $F = \{g_1, g_2, \ldots, g_i, \ldots, g_n\}$ denote a coherent set or family of criteria; for the sake of simplicity we shall use also F as the set of criteria indices (the same will apply later on for subsets of F);
- $A = \{a, b, c, \ldots\}$ denote a finite set of actions with cardinality m;
- $g_i(a) \in E_i$ denote the evaluation of action a on criterion g_i, for all $a \in A$ and $i \in F$, where E_i is the scale associated to criterion g_i (no restriction is imposed to the scale type).
- k_i is the relative importance or weight of criterion g_i.
- $C(aTb)$ represents the coalition of criteria in favor of the assertion "aTb", where $T \in \{P, Q, S\}$ (introduced later).
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A pseudo criterion is a function g_i associated with the two threshold functions $q_i(g_i(a))$ and $p_i(g_i(a))$ satisfying the following condition, for all $a \in A$ (Roy, 1991, 1996): $g_i(a) + p_i(g_i(a))$ and $g_i(a) + q_i(g_i(a))$ are non-decreasing monotone functions of $g_i(a)$.

By definition, for all pairs $(a, b) \in A \times A$ with $g_i(a) \geq g_i(b)$ and $q_i(g_i(a)) \leq p_i(g_i(a))$,

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If, $q_i(g_i(a)) = p_i(g_i(a))$, for all $a \in A$, then g_i is called a quasi criterion. For a quasi criterion there is no ambiguity zone, that is, weak preference Q_i

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The criteria weights and the concordance index

The concordance index can be defined as follows,

$$c(a,b) = \sum_{i \in F} \frac{k_i}{K} c_i(a,b), \quad \text{with} \;\; K = \sum_{i \in F} k_i$$

where,

$$c_i(a,b) = \begin{cases} 1, & \text{if} \quad g_i(a) + q_i(g_i(a)) \geq g_i(b), \quad (aS_ib), \\ \dfrac{g_i(a) + p_i(g_i(a)) - g_i(b)}{p_i(g_i(a)) - q_i(g_i(a))}, & \text{if} \quad g_i(a) + q_i(g_i(a)) < g_i(b) \leq g_i(a) + p_i(g_i(a)), \quad (bQ_ia), \\ 0, & \text{if} \quad g_i(a) + p_i(g_i(a)) < g_i(b), \quad (bP_ia). \end{cases}$$

When F is composed of quasi-criteria,

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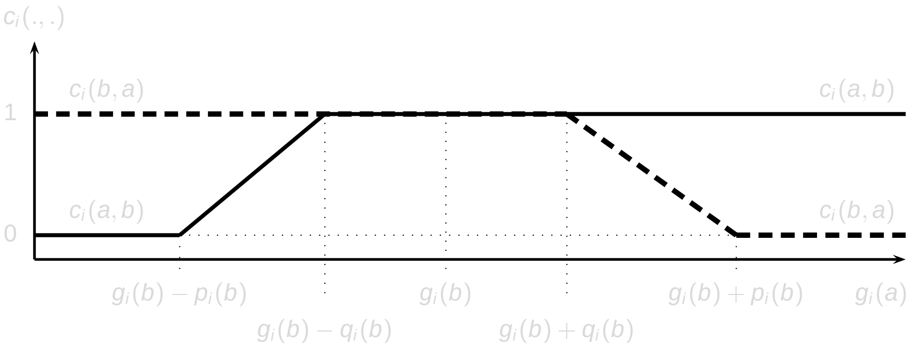
Partial concordance index, $c_i(a, b)$



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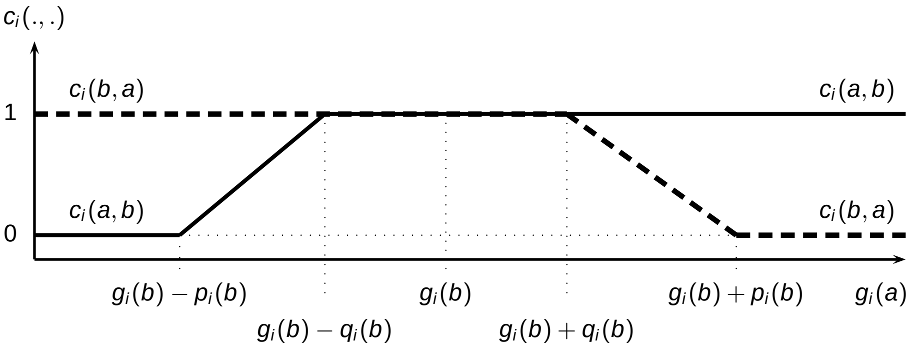

Properties of the comprehensive concordance index, $c(a, b)$

- Boundary conditions: $0 \leq c(a, b) \leq 1$.

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Types of interaction given two criteria g_i and g_j

- Mutual strengthening effect ($g_i, g_j \in \bar{C}(bPa)$)
  The passage from $k_i + k_j$ to $k_i + k_j + k_{ij}$, where $k_{ij} > 0$
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- Antagonistic effect ($g_i \in \bar{C}(bPa)$ and $g_h \in C(bPa)$)
  The passage from k_i to $k_i - k'_{ih}$, where $k'_{ih} > 0$

Positive net balance condition.

$$\forall i \in F, \quad \left(k_i\right) - \left(\sum_{\{i,j\}:k_{ij}<0} |k_{ij}| + \sum_h k'_{ih} \right) > 0$$

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  The passage from $k_i + k_j$ to $k_i + k_j + k_{ij}$, where $k_{ij} > 0$

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Positive net balance condition.

$$\forall i \in F, \quad \left(k_i\right) - \left(\sum_{\{i,j\}:k_{ij}<0} |k_{ij}| + \sum_h k'_{ih}\right) > 0$$

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On the antagonism effect: Remarks

- The presence of an antagonism coefficient $k'_{ih} > 0$ is compatible with both the absence of antagonism in the reverse direction ($k'_{hi} = 0$) and the presence of a reverse antagonism ($k'_{hi} > 0$).

- The antagonism effect does not double the influence of the veto effect; in fact, they are quite different. If criterion g_h has a veto power, it will always be considered, regardless of whether g_i belongs to the concordant coalition. The same is not true for the antagonism effect, which occurs only when the criterion g_i belongs to the concordant coalition.

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A procedure to assign numerical values to the coefficients:

1. Assign numerical values to the intrinsic weights. The SRF method can be used. (the “cards” should be ranked ignoring all the possible inter-criteria interactions.)

2. The analyst should ask the DRM about the possible interactions between criteria. Considering criterion g_1 and reviewing the remaining criteria $g_2, g_3, \ldots, g_n$, it should be easy (and relatively quick), given the very nature of the criteria, to recognize if there is or not interaction between the criteria and also identify the type of the interaction involved.

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3. A numerical value is assigned to the interaction coefficient associated with each pair identified in the previous step. Example of antagonism (cf. example, criteria g_1 and g_2):
   - Suppose that when using SRF the result is $k_1 = 6$ and $k_2 = 4$, and thus $k_1 + k_2 = 10$.
   - Since criterion g_2 is antagonistic with respect to g_1, the weight should be lower than 6, when comparing two digital camera models *a* and *d* (cf. Table).
   - The analyst can ask the DMR to set the value to be replaced to 6 in this comparison in order to adequately model the interaction that the DMR wants to take into account.
   - If the answer is 3.5, for example, the analyst should conclude that $k'_{12} = 2.5$.
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Definition of the new $c(a, b)$ (quasi criteria)

Let,

- $L(a, b)$ denote the set of all pairs $\{i, j\}$ such that $i, j \in \bar{C}(bPa)$;
- $O(a, b)$ denote the set of all ordered pairs (i, h) such that $i \in \bar{C}(bPa)$ and $h \in C(bPa)$.

$$c(a,b) = \frac{1}{K(a,b)}\left(\sum_{i \in \bar{C}(bPa)} k_i + \sum_{\{i,j\} \in L(a,b)} k_{ij} - \sum_{(i,h) \in O(a,b)} k'_{ih} \right)$$

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Function $Z(\cdot, \cdot)$

Function $Z(\cdot, \cdot)$ in the previous formula is used to capture the interaction effects in the ambiguity zone. It should be remarked that in the third summation $c_h(b, a)$ is always equal to 1.

Let $x = c_i(a, b)$ and $y = c_j(a, b)$ or $y = c_h(b, a)$. Consequently, $x, y \in [0, 1]$.

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Properties of $Z(\cdot,\cdot)$

Extreme value conditions: When leaving the ambiguity zones $c(a,b)$ should regain the form presented in formula. Thus, $Z(1,1) = 1$ and $Z(x,0) = Z(0,y) = 0$.

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Marginal impact condition: When the ambiguity diminishes we pass from $x + w$ to x, the relative marginal impact of the interactions is bounded from above,

$$\frac{1}{w}\Big(Z(x+w,y) - Z(x,y)\Big) \leq 1 \quad x,y,w,x+w \in [0,1]$$

Continuity: $Z(x,y)$ is a continuous function of each argument. This permits $c(a,b)$ to be a continuous function of $g_i(a)$ and $g_i(b)$ when $p_i(g_i(a)) > q_i(g_i(a))$, for all $a \in A$ and $g_i \in F$.

Boundary condition: For preserving the net balance condition, it is sufficient that $Z(x,y) \leq \min\{x,y\}$.

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Some possible forms for $Z(\cdot,\cdot)$

Among the multiple forms that can be chosen for $Z(x,y)$, we only present two of them which have an intuitive and meaningful interpretation.

$Z(x,y) = \min\{x,y\}$;

$Z(x,y) = xy$.

When x and y are both different from 1, i.e., when the two interacting criteria belong to the ambiguity zone, then the impact of the interaction is weaker with xy than with $\min\{x,y\}$.

Choosing the $\min\{x,y\}$ means that the reduction coefficient is not influenced by what happens it the other ambiguity zone. For these reasons formula xy seems preferable to $\min\{x,y\}$.

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Back to the examples: Digital camera model

	g_1[min]	g_2[min]	g_3[max]	g_4[max]	g_5[max]	g_6[min]	g_7[min]
a	220	Average	Average	Rather Good	Average	190 cm^3	155 g
b	300	Bad	Rather Good	Average	Rather Good	160 cm^3	145 g
c	160	Bad	Very Bad	Average	Rather Bad	140 cm^3	130 g
d	280	Very Good	Average	Very Good	Average	220 cm^3	170 g
q_i	25	1	1	1	1	10 cm^3	10 g
p_i	50	2	2	2	2	20 cm^3	20 g

Qualitative scale: very bad, bad, rather bad, average, rather good, good, very good.

Back to the examples: Digital camera model

- Consider the weights obtained using SRF: $k_1 = 6$, $k_2 = 4$, $k_3 = k_4 = k_5 = 1$, $k_6 = k_7 = 2$, where $K = 6 + 4 + 1 + 1 + 1 + 2 + 2 = 17$.
- The concordance index for (a, d) is $c(a, d) = \frac{(6+1+1+2+1)}{17} = \frac{11}{17} = 0.647$ (criterion g_7 is in the ambiguity zone, and it only counts for 50% of its overall weight).
- Now, consider the antagonistic effect, where $k'_{12} = 2.5$.

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- The new concordance index takes the value $c(a,d) = \frac{(6+1+1+2+2-2.5)}{17-2.5} = \frac{8.5}{14.5} = 0.586.$
- But, $c(d,a)$ remains the same (i.e., $c(d,a) = \frac{(4+3+1)}{17} = \frac{8}{17} = 0.471$).
- If s is defined at $s = 0.6$, when taking the antagonism effect into account, the actions become incomparable, although a was preferred to d before.
- This incomparability shows that this effect can imply significant changes.

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Fundamental results

Theorem (quasi criterion). *Monotonicity and boundary conditions hold for $c(a, b)$ as defined in slide 15.*

Theorem (pseudo criterion). *If function $Z(x, y)$ satisfies Extreme value conditions, Symmetry, Monotonicity, Marginal impact condition, and Continuity, then $c(a, b)$ satisfies Boundary conditions, Monotonicity and Continuity.*

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Fundamental results: monotonicity

Consider all the possible cases. The proof is based on the fact that if the difference $g_f(a) - g_f(b)$ decreases, either $c(a,b)$ remains constant or it decreases

1. Criterion f belongs to $C(bPa)$.
2. Criterion f belongs to $\bar{C}(bPa)$. Four subcases should be considered:
   a) Criterion f belongs to $C(aSb)$ and it continues in $C(aSb)$ after decreasing Δ_f.
   b) Criterion f moves from $C(aSb)$ to $C(bQa)$.
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Concordance index and Choquet integral

- Choquet integral is an aggregation operator permitting to model interactions between criteria.
- Choquet integral is used to build a value function giving a complete preorder, i.e. a transitive and strongly complete binary relation, rather than simply an outranking relation, being only reflexive and not transitive and complete, like in ELECTRE methods.
- Choquet integral is questionable especially with respect to two main points (Roy 2007): the evaluation of each criterion is supposed to be expressed, ...
- It is interesting to investigate more in detail the relationship between Choquet integral and the extension of the concordance index of ELECTRE methods. (Look at concordance index from the viewpoint of the Choquet integral.)

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Conclusions

- In this presentation we introduced three types of interaction that allow modeling a large number of dependence situations in real-world decision-making problems.
- We showed how to take into account these types of interaction in the concordance index used within the ELECTRE methods framework.
- Choquet integral and the min formula.
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