Aggregating Preferences with Positive and Negative Reasons

Meltem Öztürk and **Alexis Tsoukiàs** [1]

Abstract.
We discuss the problem of preference aggregation when the reasons supporting a preference for x wrt to y and the ones against are distinct and have to be considered independently. We show how it is possible to generalise the concordance/discordance principle in preference aggregation and we apply it to the problem of aggregating preferences expressed under intervals.

1 Introduction

Suppose you are comparing candidates for a Master course in AI. Candidate a has good notes in all classes, but not in Mathematics. Candidate b has average notes in all classes (therefore, worse than a, but definitely better in Mathematics). For the given Master course Mathematics is a very important class, but not that much to be able to decide alone. How do we compare a with b? If we make an average of the notes of the two candidates (giving an appropriate importance to Mathematics) we take the risk that a and b result equivalent (same global note), while they have totally different profiles. If we decide to reason in terms of majority of criteria then a will result better than b, but this cannot be accepted since a is really worse than b in Mathematics. If we use a "weighted" majority (so that Mathematics take the appropriate importance) the problem will not change unless the importance of Mathematics becomes dictatorial. Actually b cannot be considered better than a only because she has a better note in Math. Indeed these two candidates are "incomparable". The problem is how to handle such type of preference models.

The situation above described is the result of some intuitively sound rules such as:
- x is globally better than y iff it is the case for a majority of "criteria", (agents, dimensions, etc.);
- x cannot globally be better than y iff there is a "criterion" (agent, dimension) where it is "far worse" than y.

In the first case we have a "positive" information supporting the establishment of a preference between x and y. In the second case we have a negative information against establishing a preference between x and y. The problem arises from the fact that these positive and negative reasons are independent so that they could occur simultaneously.

Such issues are not really new in the literature. Rescher in [16] has been the first to introduce the concept of "bi-polarity" in scales measuring value. Roy (see [18]) has introduced the concept of concordance/discordance in Multiple Criteria Decision Analysis. Hwang and Yoon (see [9]) developed a similar idea in the frame of Multi-Objective Decision Making. Tsoukiàs and Vincke ([23] used specific logic formalisms (see also [21]) in order to extend preference models under the presence of positive and negative reasons (see [22] and [24]). More recently Dubois and Prade surveyed the different concepts of bi-polarity (see [4]), while Figueira and Greco ([5]) and Grabisch and Labreuche ([7] and [8]) introduced bi-capacities as a general tool under which such preferences could be considered.

In this paper we focus on the concept of positive and negative reasons in aggregating preferences expressed under different criteria, possibly in order to establish a final recommendation. The paper is organised as follows. In Section 2, we give further examples of the use of positive and negative reasons in decision making, mainly under a social choice perspective. In Section 3, we introduce our notation and set our problem. In Section 4, we show how this can be handled generalising the concepts of concordance and discordance. Section 5, is dedicated to an example of aggregating preferences expressed on intervals where the use of positive and negative reasons is essential. We conclude showing further research directions of this work.

2 Positive and negative reasons in social choice

Consider the Security Council of the United Nations. This is composed by 15 members (10 elected and 5 permanent). The decision rule for adopting a resolution requires that *at least 9 out of the 15 members agree* and that *no permanent member uses its veto*. It is easy to observe that in the above decision rule there exist agents having a "negative power". Such a "negative power" is not compensated by the "positive power" of each agent when forming a majority. It acts independently and only in a negative sense. Indeed we could associate to each agent both a positive power (1/15) and a negative one (0 or 1 depending on being a permanent member or not). These two powers cannot be combined between them, although they both influence the final decision.

Consider now a Parliament where a law on a very sensitive issue is introduced for discussion by the government. Consider now that the parliament decides to use the following rule in order to make a decision: "*A law proposal x is accepted (therefore preferred to its rejection) iff it meets the majority will and does not mobilise the minority aversion*". It should be observed that the minority is considered here as an independent decision power source. Such a "decision rule" is a regular practice in all mature democracies. Although the minority does not have the power to impose its political will, it has the possibility of expressing a "veto", at least occasionally. Such a "negative power" may not necessarily be codified somewhere, but is accepted. Actually, it is also a guarantee of the democratic game. When the present majority becomes a minority it will be able to use the same "negative power".

[1] LAMSADE - CNRS, Université Paris Dauphine, email: {ozturk,tsoukias}@lamsade.dauphine.fr

As it can be noted the use of independent positive and negative reasons within decision rules is common practice for electoral bodies, commissions, boards, etc., besides being an intuitive rule for comparing alternatives described under multiple attributes. It is therefore necessary to consider a specific model to handle them.

3 Notation and Problem

In the following A will represent a finite (countable) set of objects (candidates, alternatives, actions etc.) on which preferences are expressed and from which a choice or a ranking is expected to be established.

We are going to note with $\succeq$ (possibly subscribed $\succeq_i$) preference relations on the set A to be read as "x is at least as good as y" ($x \succeq y$ or $\succeq(x, y)$). We only impose reflexivity on such a relation. If necessary we may add other specific properties. $\succ$ will represent as usually the asymmetric part of $\succeq$. We will also use capital letters $P, Q, I \cdots$ in order to represent specific preference relations (characterised by their properties). As usual P^{-1} will represent the inverse relation of P ($P^{-1}(x, y) \equiv P(y, x)$).

We are going to use $\succeq^+$ in order to represent preference sentences of the type "there are positive reasons for considering x at least as good as y", while $\succeq^-$ will represent sentences of the type "there are negative reasons for which it should not be the case that x is at least as good as y". Both $\succeq^+$ and $\succeq^-$ are binary relations.

Given a set H of such preference relations (a set of criteria) and for each couple $(x, y) \in A$ we note as H^+_{xy} the subset of H for which $x \succeq^+ y$ holds (the coalition of criteria for which there are positive reasons for which x is at least as good as y: positive coalition). In the same way we are going to note as H^-_{xy} the subset of H for which $x \succeq^- y$ holds (the coalition of criteria for which there are negative reasons for which it should not be the case that x is at least as good as y: negative coalition).

Our problem can be summarised in two steps.

1. Establish for each couple $(x, y) \in A$ an overall preference relation ($\succeq$), possibly separating it in $\succeq^+$ and $\succeq^-$. This should correspond to a general rule to be applied recursively in the case there is a hierarchy of criteria to take into account. We call this the *preference aggregation step*.
2. Given such an overall preference relation, establish a final recommendation under form of a choice or a ranking on the set A, whenever this is required. We call this the *preference exploitation step*.

The reader can see an extensive discussion about the above two steps in classic Multiple Criteria Decision Analysis in [3].

4 Generalising Concordance and Discordance

4.1 Preference Aggregation

We introduce the general rule

$$x \succeq^+ y \iff \mathcal{P}^+(H^+_{xy}) \geq \gamma \quad (1)$$

$$x \succeq^- y \iff \mathcal{P}^-(H^-_{xy}) \geq \delta \quad (2)$$

where $\mathcal{P}^+$ ($\mathcal{P}^-$) represent a measure of the importance of the "positive" (respectively negative) coalition and γ and δ represent two thresholds.

We are not going to discuss in this paper how $\mathcal{P}^+$ ($\mathcal{P}^-$) is established, but without loss of generality we can assume that is a real valued function to the interval $[0, 1]$. Of couse the thresholds γ and δ are defined within the same interval.

The first rule should be read as: *when comparing x to y under all criteria, there are sufficient positive reasons to claim that x is at least as good as y iff the coalition of criteria where it is the case that x is at least as good as y is sufficiently strong*.

The second rule should be read as: *when comparing x to y under all criteria, there are sufficient negative reasons to claim that is not the case that x is at least as good as y iff the coalition of criteria where it is not the case that x is at least as good as y is sufficiently strong*.

In principle $\mathcal{P}^+$ and $\mathcal{P}^-$ are independently evaluated and therefore the strength of the positive and negative coalitions are not computed in the same way nor can be considered one the complement of the other. If we interpret the above rule within a social choice setting, we can consider $\mathcal{P}^+$ as the strength of the majority coalition, the γ threshold being the majority required to approve a bill, while $\mathcal{P}^-$ should be considered as the minority strength, the δ threshold representing the situation where a veto could be expressed. Consider again the United Nations Security Council example. The strength of the positive coalition is computed additively, the γ threshold being 3/5. The strength of the negative coalition is computed using the $\max$ operator, the δ threshold being 1.

The idea of using $\mathcal{P}^+$ and $\mathcal{P}^-$ has been already introduced in Multiple Criteria Decision Making methods. In the so called "outranking methods", the global preference relation S (to be read as "at least as good as") is generally established as

$$S(x, y) \iff C(x, y) \wedge \neg D(x, y) \quad (3)$$

where $C(x, y)$ is the concordance test (is there a weighted majority of criteria in favour of x wrt to y?) and $D(x, y)$ is the discordance test (is there a veto against x wrt to y?).

Example 1 *A typical application of the above rule can be seen in one of the oldest "outranking methods" (see [17]) where:*

$$C(x, y) \iff \frac{\sum_{j \in J_{xy}} w_j}{\sum_j w_j} \geq \gamma, \quad (4)$$

$$D(x, y) \iff \exists j : g_j(y) - g_j(x) > v_j \quad (5)$$

where:

- *g_j is a real valued function representing the evaluation of alternatives with respect to the criterion c_j (to be maximised);*
- *w_j is a non negative coefficient which represents the importance of the criterion c_j;*
- *J_{xy} represents the set of criteria for which x is at least as good as y; more precisely, $J_{xy} = \{j : g_j(y) - g_j(x) \leq q_j\}$ where q_j is the indifference threshold associated to criterion c_j; therefore, $J_{xy} = H^+_{xy}$;*
- *γ is a majority threshold;*
- *v_j is a veto threshold on criterion c_j;*
- *consequently H^-_{xy} will be the set of criteria where a veto is expressed against $S(x, y)$.*

In this case a sufficiently strong positive coalition is any subset of criteria for which the sum of the importance coefficients is at least γ. If such a coalition exists, it this means that we have a positive reason to consider that x is at least as good as y. On the other hand, we *consider that we have a negative reason to consider that x is at least as good as y when y is largely better than x on at least one criterion.*

However, this way to interpret the concordance/discordance principle presents a number of weak points. With the definition given in equation (3), both the concordance and the non-discordance have to be verified in order to establish the outranking relation. However, there is a big semantic difference between a situation where a majority of criteria supports that "x is at least as good as y", but there is a veto, a situation where there is neither majority nor veto and a situation where there is a minority of criteria in favour of the outranking relation. In other words, when comparing two alternatives x and y, the use of the concordance/discordance principle introduces four different epistemic situations but only two possible cases can occur (either the outranking relation holds or not).

Moreover, the principle does not work recursively. There is no way to consider the existence of positive and negative reasons for each single criterion which should be aggregated separately. This prevents the use of this method in a hierarchical structure of criteria and agents.

The two functions $\mathcal{P}^+$ and $\mathcal{P}^-$ are supposed to be measures of the strength of the positive and negative coalition of criteria respectively. It is reasonable to consider such functions as "fuzzy measures" or "valued binary relations" instead of using their "cuts" represented by the thresholds γ and δ. This is the approach adopted by several authors including [5], [7], [8] and [13]. The result will be a "bi-polar" (positive/negative) measure of the strength of preference for each pair of objects in A.

4.2 Preference Exploitation

Aggregating preferences will generally result in a binary relation which is neither necessarily complete nor transitive (see [2], [27]). The global relations $\succeq^+$ and $\succeq^-$ obtained after aggregating preferences are not necessarily orders. Thus, is difficult, if not impossible, to identify a best choice or a ranking of the set A, just using these relations. In order to obtain such a result (which we may call a final recommendation) it is necessary to further elaborate the information obtained from the aggregation step.

The literature offers a large variety of procedures for this purpose when conventional preference structures are considered (see for instance [28]). The interested reader can see more details in [3], chapter 7. However, very few, if any, procedures exist when positive and negative procedures are considered separately (see [19]). In this paper we present two procedures:
- the positive/negative net flow procedure;
- the positive/negative dominance ranking procedure.
Let us recall that the input of such procedures are the two binary relations $\succeq^+$ and $\succeq^-$ on the set A and the output is a ranking of the set A.

1. *The positive/negative net flow*. For each element $x \in A$ we compute a score

$$\sigma(x) = |\{y \in A : x \succeq^+ y\}| + |\{y \in A : y \succeq^- x\}| \qquad (6)$$
$$-|\{y \in A : y \succeq^+ x\}| - |\{y \in A : x \succeq^- y\}|$$

   We then rank the set A by decreasing values of σ. In other terms, for each element x we count the elements for which there are positive reasons such that x should be at least as good as them plus the elements for which there are negative reasons for which they should not be at least as good as x and we subtract the number of elements for which there are positive reasons for which they should be at least as good as x and the elements for which there are negative reasons for which x should not be at least as good as them. This procedure generalises the net flow procedure used in MCDM (see [1]).
2. *The positive/negative dominance ranking*. The procedure establishes two distinct rankings, one for the positive and one for the negative reasons and works as follows:
   - consider the graph associated to the relation $\succeq^+$;
   - identify the subset A_1^+ of A such that there are no entering arcs to any of its elements (the elements of A for which there are no other elements having positive reasons for which they should be at least as good as them);
   - establish A_1^+ as an equivalence class (the best), eliminate it from A and apply the same procedure to $A \setminus A_1^+$; this will identify the second best equivalence class A_2^+;
   - proceed until the set A is totally ranked from A_1^+ (the best) to A_n^+ (the least best);
   - consider the graph associated to the relation $\succeq^-$;
   - identify the subset A_1^- of A such that there are no entering arcs to any of its elements (the elements of A for which there are no other elements having negative reasons for which they should not be at least as good as them);
   - establish A_1^- as an equivalence class (the worst), eliminate it from A and apply the same procedure to $A \setminus A_1^-$; this will identify the second worst equivalence class A_2^-;
   - proceed until the set A is totally ranked from A_n^- (the least worst) to A_1^- (the worst);
   - the two rankings do not necessarily coincide. A partial ranking of A can be obtained from the intersection of these two rankings.

Several other procedures can be conceived. We limit ourselves in this paper to these two examples just in order to show how it is possible to obtain a final ranking after preferences have been aggregated using positive and negative reasons independently. Concluding this section we can make the following remarks.

Remark 1 *In the case $\mathcal{P}^+$ and $\mathcal{P}^-$ are considered as fuzzy measures the preference exploitation step will require different procedures. For an example the reader can see [13].*

Remark 2 *The idea of final recommendation adopted in this paper focusses on obtaining a choice or a ranking for the set A. However, in real decision support situations a final recommendation can be richer than that. For instance it can identify conflicts and incomparabilities to analyse before any further decision. A clear representation of the positive and negative reasons behind such critical issues is extremely beneficial in such situations.*

In the following we are going to show how the above ideas apply to a specific preference aggregation problem: the case where preferences are expressed through comparison of intervals.

5 Aggregating preferences on Intervals

Consider a set of four objects $A = \{a, b, c, d\}$ and a set of four criteria $H = \{h_1, h_2, h_3, h_4\}$. In order to simplify the presentation we consider that to all four criteria correspond attributes where the objects in A can take numerical values on a scale from 1 to 8. However, in this particular case we make the hypothesis that due to uncertainties on the real values of the objects these will take a value under form of an interval. Figure 1 shows how the objects in A are evaluated on the four attributes.

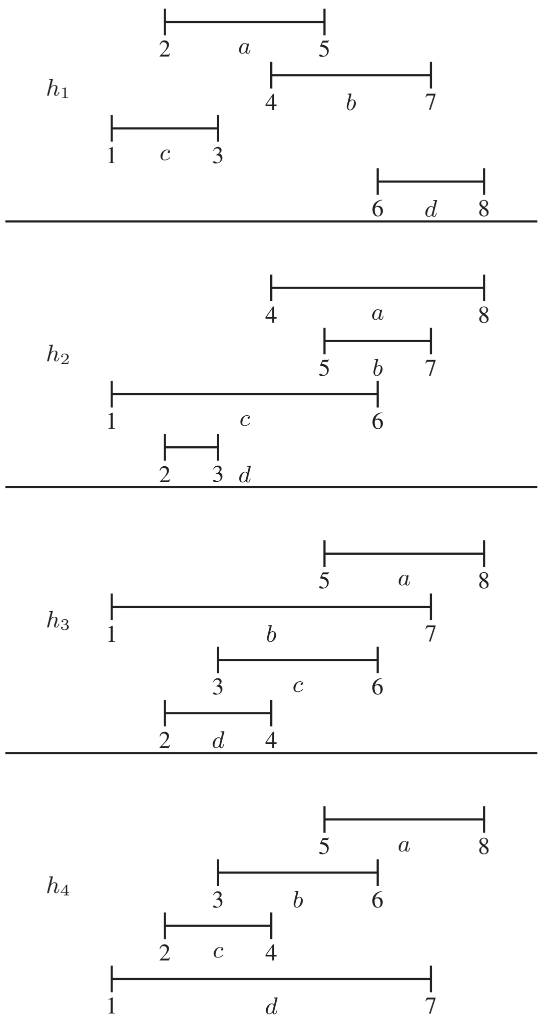

Figure 1. The values of A on the four attributes

Where is the problem? Conventionally, comparison of intervals is based on the hypothesis that "x is preferred to y" ($P(x, y)$ iff the lowest value of x is larger than the greatest value of y (the intervals thus being disjoint). In all other cases "x is indifferent to y" ($I(x, y)$) (for more on interval orders see [6], [15], [20]). If we interpret as usual the binary relation $\succeq$ as $P \cup I$ then the graphs representing this binary relation for the four criteria can be seen in table 1 (here represented under form of a 0-1 matrix).

Aggregating these preferences with a simple majority rule: "x is at least as good as y iff it is the case for at least three out of four criteria", will return all four objects indifferent. This is not surprising, since conventional interval orders only use positive information and are unable to differentiate between sure indifference and hesitation between indifference and preference.

In order to overcome this problem we are going to introduce positive and negative reasons both in comparing objects at each criterion level as well as at the aggregated one. For this purpose we are going to make use of a preference structure called PQI-interval order (see [26]). In this structure we consider three possibilities when comparing intervals:
- strict preference (P): when an interval is completely to the right of the other (exactly as in conventional interval orders);
- indifference (I): when one interval is completely included in the other;
- hesitation between preference and indifference or weak preference (Q): when an interval is to the right to the other, but they have a non empty intersection.
Applying this structure to the information previously presented we obtain the preference relations in table 2.

h_1	a	b	c	d
a	1	1	1	0
b	1	1	1	1
c	1	0	1	0
d	1	1	1	1

h_2	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	0	0	1	1

h_3	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	0	1	1	1

h_4	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	1	1	1
d	1	1	1	1

Table 1. The four Interval Orders

h_1	a	b	c	d
a	I	Q^{-1}	Q	P^{-1}
b	Q	I	P	Q^{-1}
c	Q^{-1}	P^{-1}	I	P^{-1}
d	P	Q	P	I

h_2	a	b	c	d
a	I	I	Q	P
b	I	I	Q	P
c	Q^{-1}	P^{-1}	I	I
d	Q^{-1}	P^{-1}	I	I

h_3	a	b	c	d
a	I	Q	Q	P
b	Q^{-1}	I	I	I
c	Q^{-1}	I	I	Q
d	P^{-1}	I	Q^{-1}	I

h_4	a	b	c	d
a	I	Q	P	Q
b	Q^{-1}	I	Q	I
c	P^{-1}	Q^{-1}	I	I
d	Q^{-1}	I	I	I

Table 2. The four Interval Orders

We are now going to interpret such preference relations in terms of positive and negative reasons. For this purpose we are going to use the $\succeq^+$ and $\succeq^-$ relations and define:

$$P(x,y) \iff \succeq^+ (x,y) \wedge \not\succeq^- (x,y) \wedge \not\succeq^+ (y,x) \wedge \succeq^- (y,x) \quad (7)$$

$$I(x,y) \iff \succeq^+ (x,y) \wedge \not\succeq^- (x,y) \wedge \succeq^+ (y,x) \wedge \not\succeq^- (y,x) \quad (8)$$

$$Q(x,y) \iff \succeq^+ (x,y) \wedge \not\succeq^- (x,y) \wedge \succeq^+ (y,x) \wedge \succeq^- (y,x) \quad (9)$$

In other words, while P and I represent "sure" situations of interval comparison, the relation Q represents an hesitation between them: indeed when comparing x to y we have positive reasons claiming that x is at least as good as y and no negative reasons claiming the opposite; when comparing y to x we have both positive and negative reasons (due to the fact that the larger value of y is larger than the smaller value of x, but smaller than the larger value of x). For further details on such models the reader can see [12], [22] and [25].

Applying this reasoning to the information concerning the set A we get the results in table 3 (for the positive reasons) and in table 4 (for the negative reasons).

$\succeq_1^+$	a	b	c	d
a	1	1	1	0
b	1	1	1	1
c	1	0	1	0
d	1	1	1	1

$\succeq_2^+$	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	0	0	1	1

$\succeq_3^+$	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	0	1	1	1

$\succeq_4^+$	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	1	1	1
d	1	1	1	1

Table 3. Positive Reasons

$\succeq_1^-$	a	b	c	d
a	0	1	0	1
b	0	0	0	1
c	1	1	0	1
d	0	0	0	0

$\succeq_2^-$	a	b	c	d
a	0	0	0	0
b	0	0	0	0
c	1	1	0	0
d	1	1	0	0

$\succeq_3^-$	a	b	c	d
a	0	0	0	0
b	1	0	0	0
c	1	0	0	0
d	1	0	1	0

$\succeq_4^-$	a	b	c	d
a	0	0	0	0
b	1	0	0	0
c	1	1	0	0
d	1	0	0	0

Table 4. Negative Reasons

In order to aggregate these positive and negative reasons let us apply now the principle introduced in equations 1-2. In this precise case we use the following specific rule:

$$x \succeq^+ y \iff \frac{|\{h_j : x \succeq_j^+ y\}|}{|H|} \geq \frac{3}{4} \tag{10}$$

$$x \succeq^- y \iff \frac{|\{h_j : x \succeq_j^- y\}|}{|H|} \geq \frac{1}{2} \tag{11}$$

Actually we use a very simple aggregation rule. Both $\mathcal{P}^+$ and $\mathcal{P}^-$ are additive and for both the positive and negative distribution of power we consider the criteria equivalent. The results of this aggregation can be seen in table 5.

$\succeq^+$	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	0	1	1	1

$\succeq_4^-$	a	b	c	d
a	0	0	0	0
b	1	0	0	0
c	1	1	0	0
d	1	0	0	0

Table 5. Positive and Negative Reasons after Aggregation

What do we get from these results?

First of all we are able to reconstruct a PQI preference structure at the aggregated level. We can establish the type of preference relation holding for any pair of objects in the set A. More precisely, applying equations 7-9 we get the results shown in table 6.

PQI	a	b	c	d
a	I	Q	Q	P
b	Q^{-1}	I	Q	I
c	Q^{-1}	Q^{-1}	I	I
d	P^{-1}	I	I	I

Table 6. The PQI preference structure after aggregation

We can now check whether such a PQI preference structure is also a PQI interval order and if it is the case we can try to reconstruct a numerical representation for each element of A under form of interval. Using results known in the literature ([10], [11]) we can prove that in this precise case this is indeed a PQI interval order, a numerical representation of which can be seen in figure 2.

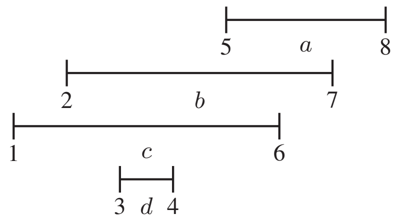

Figure 2. The global PQI interval order

In the case we definitely need a more operational result such as a ranking we can use any of the procedures introduced in section 4.2. More precisely adopting the positive/negative net flow procedure (see equation 7) we get $a > b > c, d$ ($>$ representing the ranking relation). The same result is obtained if we use the positive/negative dominance ranking. This is not surprising, if we consider the very simple nature of the aggregation procedure and the information available. However, this is not generally the case. Concluding this section we may make the following remarks.

Remark 3 *To our knowledge this is the only way through which is possible to aggregate preferences expressed on intervals without ending with only indifference, losing precious information. The identification of positive and negative reasons in intervals comparison allows to exploit information which in conventional preference modelling is usually neglected. The specific suggestions done in this paper should be considered as examples, since several other possibilities can be considered depending on the problem on hand.*

Remark 4 *The reader can check that modifying the parameters and rules in the aggregation and exploitation steps one can obtain significantly different results. This is not surprising since these are not preferential information obtained from the decision maker and are more or less arbitrary. Care should be taken to tune them robustly.*

Remark 5 *Fortunately the algorithmic part of the above procedures is "easy". Indeed as shown in [10] and [11], checking if a PQI preference structure is a PQI interval order and finding a numerical representation are all problems in* **P.**

6 Conclusions

In this paper we focus on the advantages of using independent positive and negative reasons in preference aggregation. More precisely:

- aggregating independent positive and negative reasons allows to clearly distinguish situations of sure preference from situations of hesitation as well as between incomparabilities due to conflicts (presence of both positive and negative reasons) and incomparabilities due to ignorance (absence of both positive and negative reasons);
- modelling independently positive and negative reasons allows to use the same principle for any level of preference modelling (single criterion, single agent, multiple criteria, multiple agents and their combinations), thus generalising the concordance/discordance principle;
- the use of positive and negative reasons when objects evaluated on intervals are compared allows to solve the problem of aggregating such preferences, a situation encountered not only in decision aiding, but in several other fields (see [14]).

Several research problems remain open in the paper. Among these we note the following:
- axiomatise preference aggregation and exploitation procedures based on the independent use of positive and negative reasons (of the type presented in this paper);
- study appropriate formalisms (multiple valued logics, argumentation theory etc.) enabling elegant and compact representations besides further extending the potentialities of this approach;
- further investigate the problem of aggregating PQI preference structures: under what conditions the aggregation of such preference structures will result in a PQI interval order or any other order representable by intervals?

REFERENCES

[1] D. Bouyssou, 'Ranking methods based on valued preference relations: A characterization of the net flow network', *European Journal of Operational Research*, **60**, 61–67, (1992).

[2] D. Bouyssou, 'Outranking relations: do they have special properties?', *Journal of Multi-Criteria Decision Analysis*, **5**, 99–111, (1996).

[3] D. Bouyssou, T. Marchant, M. Pirlot, A. Tsoukiàs, and Ph. Vincke, *Evaluation and decision models: stepping stones for the analyst*, Springer Verlag, Berlin, 2006.

[4] D. Dubois and H. Prade, 'Bipolarity in reasonning and decision: An introduction'. private communication.

[5] J. Figueira and S. Greco, 'Dealing with interactivity between bi-polar multiple criteria preferences in outranking methods', Research report 11-2003, INESC-Coimbra, Portugal, (2003). 73 pages.

[6] P.C. Fishburn, *Interval Orders and Interval Graphs*, J. Wiley, New York, 1985.

[7] M. Grabisch and C. Labreuche, 'Bi-capacities-i: definition, mobius transform and interaction', *Fuzzy Sets and Systems*, **151**(2), 211–236, (April 2005).

[8] M. Grabisch and C. Labreuche, 'Bi-capacities-ii: the choquet integral', *Fuzzy Sets and Systems*, **151**(2), 237–259, (April 2005).

[9] C.L. Hwang and K. Yoon, *Multiple attribut decision making : Methods and applications*, Springer-Verlag, 1981.

[10] A. Ngo The and A. Tsoukiàs, 'Numerical representation of PQI interval orders', *Discrete Applied Mathematics*, **147**, 125 – 146, (2005).

[11] A. Ngo The, A. Tsoukiàs, and Ph. Vincke, 'A polynomial time algorithm to detect PQI interval orders', *International Transactions in Operational Research*, **7**, 609–623, (2000).

[12] M. Öztürk, *Mathematical and Logical Structures for Intervals Comparison*, Ph.D. dissertation, Université Paris-Dauphine, Paris, 2005.

[13] M. Öztürk and A. Tsoukiàs, 'Modeling uncertain positive and negative reasons in decision aiding', *Decision Support Systems*, (to appear).

[14] M. Öztürk, A. Tsoukiàs, and Ph Vincke, 'Preference modelling', in *State of the Art in Multiple Criteria Decision Analysis*, eds., M. Ehrgott, S. Greco, and J. Figueira, 27 – 72, Springer Verlag, Berlin, (2005).

[15] M. Pirlot and Ph. Vincke, *Semi Orders*, Kluwer Academic, Dordrecht, 1997.

[16] N. Rescher, *Introduction to Value Theory*, Prentice Hall, Englewood Cliffs, 1969.

[17] B. Roy, 'Classement et choix en présence de points de vue multiples: La méthode ELECTRE', *Revue Francaise d'Informatique et de Recherche Opérationnelle*, **8**, 57–75, (1968).

[18] B. Roy, 'The outranking approach and the foundations of ELECTRE methods', *Theory and Decision*, **31**, 49–73, (1991).

[19] Greco S., Matarazzo B., Słowiński R., and Tsoukiàs A., 'Exploitation of a rough approximation of the outranking relation in multi-criteria choice and ranking', in *Trends in Multi-Criteria Decision Making*, eds., T.J. Stewart and R.C. van der Honert, 45 – 60, Springer Verlag, LNEMS 465, Berlin, (1998).

[20] W.T Trotter, *Combinatorics and partially ordered sets*, John Hopkins University Press, Baltimore, 1992.

[21] A. Tsoukiàs, 'A first-order, four valued, weakly paraconsistent logic and its relation to rough sets semantics', *Foundations of Computing and Decision Sciences*, **12**, 85–108, (2002).

[22] A. Tsoukiàs, P. Perny, and Ph. Vincke, 'From concordance/discordance to the modelling of positive and negative reasons in decision aiding', in *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, eds., D. Bouyssou, E. Jacquet-Lagrèze, P. Perny, R. Slowinski, D. Vanderpooten, and Ph. Vincke, 147–174, Kluwer Academic, Dordrecht, (2002).

[23] A. Tsoukiàs and Ph. Vincke, 'A new axiomatic foundation of partial comparability', *Theory and Decision*, **39**, 79–114, (1995).

[24] A. Tsoukiàs and Ph. Vincke, 'Extended preference structures in MCDA', in *Multicriteria Analysis*, ed., J. Climaco, 37–50, Springer Verlag, Berlin, (1997).

[25] A. Tsoukiàs and Ph. Vincke, 'Double threshold orders: A new axiomatization', *Journal of Multi-criteria Decision Analysis*, **7**, 285–301, (1998).

[26] A. Tsoukiàs and Ph. Vincke, 'A characterization of PQI interval orders', *Discrete Applied Mathematics*, **127**(2), 387–397, (2003).

[27] Ph. Vincke, 'Aggregation of preferences: a review', *European Journal of Operational Research*, **9**, 17–22, (1982).

[28] Ph. Vincke, 'Exploitation of a crisp relation in a ranking problem', *Theory and Decision*, **32**(3), 221–240, (1992).