A survey on non conventional Preference Modeling. [1]

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Introduction

Preferences are used in a lot of problem situations both in individual and organizational decision processes either in economy, psychology, sociology, planning, decision aid.... However the problem of preference modeling becomes to appear in the literature in a fragmentary and incomplete way. The first attempt to give an account about preference relations can be referred to Von Neumann and Morgenstern (1944) even if in a particular domain, economic theory. In this field it is Savage (1954) that first introduces a foundation of the subject. Models of preferences have been naturally associated to binary relations (preference relations are binary relations) where, by the same time, some important work had already been done on the subject of ordered sets and measurement (see: Dushnik and Miller, 1941; Tarski, 1954/55; Scott and Suppes, 1958).

However around the principal corpus of the theory (for a presentation see Roubens and Vincke, 1985) a lot of questions have been posed and discussed. Since Luce (1956) almost all assumptions done in the axiomatic foundation and operational uses of preference modeling have been challenged trying to verify the consistency, reliability and flexibility of the theory. The discussion moved from philosophical and logical foundations of the theory to specific problems about the properties of these particular binary relations (namely transitivity and complete comparability) and to particular operational problems as uncertainty, probability, risk and multidimensional preferences.

Notation and organization of the paper

In the following $\mathcal{R}$ and $\mathcal{I}$ will represent the set of real and integer numbers respectively and A will represent a set of elements on which preference relations are applied, with $a, b, c, ...$ being specific elements of A and $x, y, z, ...$ variables ranging on the set A. $A \times A$ represents the cartesian product of A with itself. We will use capital letters $P, Q, R, ...$ to represent the relations as subsets of $A \times A$ and small letters $p, q, r, ...$ to represent the relations as predicates holding among two elements of A ($p(a, b)$: the relation p holds among a and b) or among variables ($p(x, y)$). We will use classic logic with $\neg$, $\vee$, $\wedge$, $\rightarrow$, ... the usual logical connectives and set theory with $\cap$, $\cup$, $\subseteq$, ... the usual set operations. We will also write $P^{-1}, Q^{-1}, ...$ and $p^{-1}, q^{-1}, ...$ to represent the converse relation ($p^{-1}(x, y) \equiv p(y, x)$). We will also use the concepts of "decision process", "actor", "client", and "analyst" (see Moscarola, 1984, Roy, 1985) when we have to refer to preference modeling as a decision aid activity (see also Tsoukiàs, 1990).

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The paper will be organized as follows. In section 1 an account of the philosophical discussion around the existence of a logic of preference will be given. In section 2 the classic approach will be presented together with the different problems that in various versions have been posed. Section 3 will present the extensions of the classic approach (interval orders, partial orders etc.). In section 4 the valued preference relations will be examined and in section 5 a more qualitative approach will be presented. In the conclusions the principal fields of research will be accounted. An extended, but not complete bibliography is given. Interested readers can refer to surveys for more complete references.

1 The logic of preference

The increasing importance of preference modeling immediately interested people from other disciplines particularly logicians and philosophers. The strict relation with deontic logic (see Åqvist, 1986) posed some questions of the kind:
- does it exist a general logic where whatsoever preferences can be represented and used?
- if yes, what is the language and what are the axioms?
- is it possible, via this formalization, to give a definition of bad or good as absolute values?

It is clear that this attempt had a clear positivist and normative objective: that is to define the one well formed logic which people should follow while expressing preferences. The first work in the subject is the one made by Halldén (1957), but it is Von Wright's book (1963) that tries to give the first axiomatization of a logic of preference. Inspired by this work some important contributions have been done (Houthakker, 1965; Chisholm and Sosa, 1966a and 1966b; Rescher, 1967; Hansson, 1968a and 1968b). An influence of this idea can also be seen in Jeffrey (1965) and Rescher (1969), but on related fields (statistics and value theory respectively). The discussion apparently has been concluded by Von Wright (1972), but Huber (1974 and 1979) continued on and recently Halldin (1986) and Widmeyer (1988 and 1990) also treated the subject.

The general idea can be presented as follows. At least two questions should be clarified: preferences among what? how preferences have to be understood? Von Wright (1963) argues that preferences can be distinguished in extrinsic and intrinsic. The first ones are derived as *a reason from a specific purpose* while the second ones are *self-referential* to an actor expressing the preferences. In this sense intrinsic preferences are the expression of the system of values of the actor. Moreover preferences can be expressed among different things, the most general being (following Von Wright) *"states of affairs"*. That is, the expression "a is preferred to b" should be understood as the preference of a state (a world) where a occurs (whatever a represents: sentences, objects, relations etc.) over a state where b occurs. Von Wright, on this basis expressed a theory based on five axioms:

A^{W}1. $\forall x, y \;\; p(x,y) \;\rightarrow\; \neg p(y,x)$
A^{W}2. $\forall x, y, z \;\; p(x,y) \;\wedge\; p(y,z) \;\rightarrow\; p(x,z)$
A^{W}3. $p(a,b) \;\equiv\; p(a \wedge \neg b, \neg a \wedge b)$
A^{W}4. $p(a \vee b, c) \;\equiv\; p(a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c) \;\wedge\; p(a \wedge \neg b \wedge \neg c, \neg a \wedge \neg b \wedge c) \;\wedge\; p(\neg a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c)$
A^{W}5. $p(a,b) \;\equiv\; p(a \wedge c, b \wedge c) \;\wedge\; p(a \wedge \neg c, b \wedge \neg c)$

The first two axioms are asymmetry and transitivity of the preference relation while the following three axioms face the problem of combinations of states of affairs. The use of specific elements in place of the variables and quantifiers reflects the fact that Von Wright considered the axioms not as logical ones, but as "reasoning principles". This distinction has important consequences on the calculus level. In the first two axioms preference is considered as a binary relation (therefore the use of a predicate), in the three "principles" preference is a proposition. Von Wright does not make this distinction directly, considering the expression aPb ($p(a,b)$ in our notation) as a well formed formula in his logic. However this does not change the problem since the first two axioms are referred to the binary relation and the others not. The difference appears if one tries to introduce quantifications; in this case the three principles appear weak. The problem with this axiomatization is that empirical observation of human behaviour provides counterexamples of these axioms. Moreover from a philosophical point of view (following the normative objective that this approach assumed) a logic of intrinsic preferences about general states of affairs should allow to define what is good (the always preferred?) and what is bad (the always not preferred?). But this axiomatization fails to enable such a definition.

Chisholm and Sosa (1966a) rejected axioms A^W3 to A^W5 and built an alternative axiomatization based on the concepts of *"good"* and *"intrinsically better"*. Their idea is to postulate the concept of good and to axiomatize consequently preferences. So a *good* state of affairs is one that is always preferred to its negation ($p(a, \neg a)$); Chisholm and Sosa, 1966a, use this definition only for its operational potentialities as they argue that it does not capture the whole concept of "good"). In this case we have:

A^S1. $\forall x, y \;\; p(x,y) \; \rightarrow \; \neg p(y,x)$
A^S2. $\forall x, y, z \;\; \neg p(x,y) \; \wedge \; \neg p(y,z) \; \rightarrow \; \neg p(x,z)$
A^S3. $\forall x, y \;\; \neg p(x, \neg x) \; \wedge \; \neg p(\neg x, x) \; \wedge \; \neg p(y, \neg y) \; \wedge \; \neg p(\neg y, y) \; \rightarrow \; \neg p(y,x) \; \wedge \; \neg p(x,y)$
A^S4. $\forall x, y \;\; p(x,y) \; \wedge \; \neg p(y, \neg y) \; \wedge \; \neg p(\neg y, y) \; \rightarrow \; p(x, \neg x)$
A^S5. $\forall x, y \;\; p(y, \neg x) \; \wedge \; \neg p(y, \neg y) \; \wedge \; \neg p(\neg y, y) \; \rightarrow \; p(x, \neg x)$

However again in this axiomatization there exist counterexamples of the axioms. The assumption about the concept of good can be argued as it allows circularities in the definitions of preferences among combinations of states of affairs. On this criticism Hansson (1968a) considers only two foundamental, universally recognized, axioms:

A^H1. $\forall x, y, z \;\; s(x,y) \; \wedge \; s(y,z) \; \rightarrow \; s(x,z)$
A^H2. $\forall x, y \;\; s(x,y) \; \vee \; s(y,x)$

where s is a "large preference relation" and two specific preference relations are defined, p (strict preference) and i (indifference):

D^H1. $\forall x, y \;\; p(x,y) \; \equiv \; s(x,y) \; \wedge \; \neg s(y,x)$
D^H2. $\forall x, y \;\; i(x,y) \; \equiv \; s(x,y) \; \wedge \; s(y,x)$

He also introduces two more axioms, although he recognizes their controversial nature:

A^H3. $\forall x, y, z \;\; s(x,y) \; \wedge \; s(x,z) \; \rightarrow \; s(x, y \vee z)$
A^H4. $\forall x, y, z \;\; s(x,z) \; \wedge \; s(y,z) \; \rightarrow \; s(x \vee y, z)$

Von Wright in his replay (1972), trying to argue for his theory introduced a more general frame to define intrinsic *"holistic"* preferences or as he called them *"ceteris paribus"* preferences. In this approach he considers a set S of states where the elements are the ones of A (n elements) and all the 2^n combinations of these elements. Given two states s and t (elementary or combinations of m states of S) you have i ($i = 2^{n-m}$) combinations C_i of the other states. You call an s-world any state that holds when s holds. A combination C_i of states is also a state so you can define in the same way a C_i-world. Von Wright gives two definitions (a strong and a weak) of preference:

1. (strong): s is preferred to t under the circumstances C_i **iff** every C_i-world that is also s-world and not t-world is preferred to every C_i-world that is also t-world and not s-world.
2. (weak): s is preferred to t under the circumstances C_i **iff** some C_i-world that is s-world is preferred to a C_i-world that is t-world, but it does not exist a C_i-world that is t-world that is preferred to a C_i-world that is s-world.

Now s is *"ceteris paribus"* preferred to t **iff** it is preferred under all C_i. We leave the discussion to the interested readers, but we point out that with these definitions it is difficult to axiomatize both transitivity and complete comparability unless they are assumed as necessary truths for "coherence" and "rationality" (see Von Wright, 1972).

It can be concluded that the philosophical discussion about preferences failed the objective to give an unifying frame of generalized preference relations that could hold for any kind of states, based on a well defined axiomatization. It is still difficult (if not impossible) to give a definition of good or bad in absolute terms based on reasoning about preferences and the properties of these relations are not unanimously accepted as axioms of preference modeling.

It should be noticed here that this logical-philosophical discussion introduced some important observations about the relative truth of preferences depending on different factors. Von Wright (1972) insists that it is not possible to speak about preferences ignoring that these are preferences of a specific actor in a specific time. This specification is important because transitivity can be accepted as a rational principle only when it is referred to the same agent and the same temporal interval. However this discussion missed the point of the "motivations" of preferences and preference relations. The new question is:

- why do you need to build models of preferences?

The problem is not trivial because as we will see in the following, different answers to this question lead to different axiomatics for preference modeling.

The principal idea is that if preference modeling is to be viewed as an activity, as a process where actors and objects are involved, then the features of this process cannot be ignored. This changes the perspective and the frame of the assumptions you may use.

2 The classic approach

2.1 Introduction

The most common way to work with preferences is to study the underlying mathematical structure of preference relations. As already introduced, this structure has been identified in the study of binary relations (see Krantz et al., 1971, Roberts, 1979 and Barthélemy et al., 1982).

The questions that arise in this study can be summarized as follows:
- what are the basic preference relations?
- what are the properties of these relations?
- does it exist a way to build operational models of preferences for decision making purposes?

A thorough investigation of these subjects can be founded in Fishburn (1970). It is important to stress the last question. Preference modeling is viewed as one of the tools used to help a decision maker, facing a complex problem, to make a decision. Empirical evidence shows however that the kind of assumptions you do about the decision process, the behaviour of the decision maker, influence the way you build preference models (see Moscarola, 1984). The classic approach is coherent with the traditional assumptions about decision making:
- complete availability of information;
- existence of a rational decision maker (in the sense of optimization);
- perfect and sound statement of the problem.

In the other sections we will see that non conventional preference modeling has been developed denying sometimes axioms of the classic approach that are directly linked to the traditional assumptions about decision making. This led to introduce the concept of decision maker's "bounded rationality" (Simon, 1979), the analysis of the decision process and of the interaction client - analyst (see Checkland, 1980 and from another perspective Roy, 1985), the distinction among normative, prescriptive and descriptive approaches, the shift to the "decision aid" concept so as to represent more rich problem situations and more flexible operational attitudes towards the client's problem.

Coming back to the classic approach, we always have the same problem: *given a set of actions A and a system of values of the decision maker, expressed under sentences of preferences,* **choose** *the "best" element of A ("best" according to the values)*

2.2 Definitions

We first have to define the properties of the binary relations. For any binary relation V defined on a set A, the following properties can hold (quantifications hold for A):

- reflexivity	$\forall x$	$v(x,x)$.
- irreflexivity	$\forall x$	$\neg v(x,x)$.
- symmetry	$\forall x,y$	$v(x,y) \equiv v(y,x)$.
- asymmetry	$\forall x,y$	$v(x,y) \rightarrow \neg v(y,x)$.
- antisymmetry	$\forall x,y$	$v(x,y) \wedge v(y,x) \rightarrow x = y$.
- completeness	$\forall x,y$	$v(x,y) \vee v(y,x)$.
- transitivity	$\forall x,y,z$	$v(x,y) \wedge v(y,z) \rightarrow v(x,z)$.
- neg. transitivity	$\forall x,y,z$	$\neg v(x,y) \wedge \neg v(y,z) \rightarrow \neg v(x,z)$.
- semitransitivity	$\forall x,y,z,w$	$v(x,y) \wedge v(y,z) \rightarrow v(x,w) \vee v(w,z)$.
- Ferrers	$\forall x,y,z,w$	$v(x,y) \wedge v(z,w) \rightarrow v(x,w) \vee v(z,y)$.

Binary relations are sets of ordered couples, subsets of $A \times A$. It is therefore possible to define the usual set operations:

inclusion:	$V \subset T$ **iff** $\forall x,y \ v(x,y) \rightarrow t(x,y)$
union:	$V \cup T = \{(a,b) \mid v(a,b) \vee t(a,b)\}$
intersection:	$V \cap T = \{(a,b) \mid v(a,b) \wedge t(a,b)\}$
rel. product:	$V.T = \{(a,b) \mid \exists c: \ v(a,c) \wedge t(c,b)\}$

Properties of binary relations can also be written under set representations (for instance symmetry can be written as $V = V^{-1}$; transitivity as $V.V \subset V$ etc.).

Binary relations when applied on a set can generate an ordering of its elements. In this case the relations are called orders. Depending to their properties these orders can be (this is not an exhaustive list, see Roberts, 1979 and Roubens and Vincke, 1985):

- a total order	**iff**	is antisymmetric, complete and transitive;
- a weak order	**iff**	is complete and transitive (also called complete preorder);
- a strict weak order	**iff**	is asymmetric and neg. transitive;
- an equivalence	**iff**	is reflexive, symmetric and transitive;
- a partial order	**iff**	is reflexive, antisymmetric and transitive;
- a strict partial order	**iff**	is asymmetric and transitive;
- a quasi order	**iff**	is reflexive and transitive (also called partial preorder);
- a semi order	**iff**	is complete, Ferrers and semitransitive;
- an interval order	**iff**	is complete and Ferrers;

Preference relations are now binary relations. For the presentation of the classic approach we will use the outline of Fishburn (1970). In this case two relations are conceived:
strict preference (as a set P and as a predicate p);
indifference (as a set I and as a predicate i).
Preference is considered as a basic relation and indifference as the absence of strict preference:
$\forall x, y \;\; i(x, y) \;\; \equiv \;\; \neg p(x, y) \; \wedge \; \neg p(y, x)$.
P is asymmetric and negatively transitive (a strict weak order) and I is an equivalence relation. The relation $P \cup I$ is then a complete relation (a weak order). The most interesting result in this approach is the following theorem (there exist different versions, see Fishburn, 1970):

if P is a strict weak order and I an equivalence relation on A, A being a countable set, $P \cup I$ being complete, then there exists a real valued function $u : A \mapsto \mathcal{R}$ such that:

$$p(a, b) \Leftrightarrow u(a) > u(b)$$

$$i(a, b) \Leftrightarrow u(a) = u(b)$$

This is a very strong result because you are sure that there exists a function that preserves the relations and enables to give an exact numerical representation. This is the basis for the utility theory in decision making (for an extensive account see Fishburn, 1989) dominating, up to now, the field of decision making.

A very important extension of this approach has been the expected utility theory when stochastic considerations have to be introduced. In this case we assume that there exists a convex set $\mathcal{P}$ of probability distributions among the elements of A and the following axioms hold (quantifications hold among $\mathcal{P}$, $0 < \lambda < 1$, $\; 0 < \alpha < 1$, $\; 0 < \beta < 1$):

- there exists in $\mathcal{P}$ a preference relation P that is a strict weak order;

- $\forall \lambda, l, m, n \;\; p(m, n) \rightarrow p(\lambda m + (1 - \lambda) l, \; \lambda n + (1 - \lambda) l)$

- $\forall l, m, n \;\; \exists \alpha, \beta \;\; p(m, n) \;\; \wedge \;\; p(n, l) \rightarrow p(\alpha m + (1 - \alpha) l, \; n) \;\; \wedge \;\; p(n, \; \beta m + (1 - \beta) l)$

Then you can demonstrate (see Herstein and Milnor, 1953):

$$p(m, n) \;\; \Leftrightarrow \;\; \sum_A m(x) u(x) \;\; > \;\; \sum_A n(x) u(x)$$

where:
$m(x), n(x)$ are probability distributions and
$u(x)$ is an utility function on A.

2.3 Criticisms

In this section and in the following ones we will give two kinds of criticisms. One from a theoretical and the other from an operational point of view. We will not engage ourselves with the subject of expected utility theory and probability. The interested readers can look Allais, 1953 and 1979, Kahneman and Tversky, 1979 and Bell et al., 1988.

1. Theoretical problems. This theory has a perfect axiomatization but is applicable to a very limited number of cases, namely where perfect global rationality can be assumed.

First of all transitivity of both preference relations has been strongly contradicted. In two remarkable papers Luce (1956) and Tversky (1969) presented situations where transitivity cannot be allowed (see also Fishburn, 1991 for a survey). The first arguments against transitivity call for a "limited discrimination power" of the decision maker, and for the necessity to introduce thresholds (see also section 3). With a more deep investigation it can be realized that transitivity can hold only when the decision maker (the client) has a very stable and sure representation of his problem (remember the observations of von Wright, 1972) and this is not very frequent in decision aid situations.

The problem becomes more important when we enter the field of multidimensional preferences. Since Condorcet (1785) it is known that the aggregation of preference orders in a global order can generate cyclic preferences and therefore intransitivities. The result is that in the field of multicriteria decision aid the use of this theory imposes to the client some more assumptions that push him far from his real problem and his behaviour (for an introduction on aggregation procedures see Vincke, 1982). Again the question is: how do you do decision aid?

A more difficult problem is the assumption of completeness as a property of the relation $S = P \cup I$. This assumption is foundamental because the demonstration of the existence of the utility function is strongly embedded on it. On the other hand incomparabilities are very frequent in real decision situations because of lack of information, very conflicting evaluations, ambiguity of the client etc. (cfr also Roy, 1985). Moreover when aggregating preference orders (as in the multicriteria case), even if they are complete orders, the resulting global order is generally not a complete relation.

2. Operational problems. The problem in this case is to estimate the utility function representing the client's preferences. In the unidimensional case this is not very difficult. We will not discuss it anyway. The more interesting problems are in the multidimensional case (multiattribute utility theory: MAUT, see Vincke, 1985 for a first introduction).

Suppose you have already built for each attribute an utility function; you are then asked to build a global function representing the aggregation of the different attributes. The subject has been investigated in Keeney and Raiffa, 1976, where the basic techniques and the conditions under which their use is possible are introduced. Generally it is assumed that there exist two kinds of global utility functions:

$$\text{- the additive: } U(x) = \sum_{i=1}^{n} p_i u_i(x)$$

$$\text{- the multiplicative: } U(x) = \prod_{i=1}^{n} p_i u_i(x)$$

where $u_i(x)$ is the marginal utility function for the ith attribute and p_i is the "weight" of the attribute.

However there exist some problems.

1) The global function may not exist. The most important result in this field is Arrow's impossibility theorem (1951). The theorem states that in the social choice context it does not exist a global choice function fulfilling the following properties (generally assumed as foundamental ones):
- no dictatorship;
- pareto unanimity;
- independence from irrelevant alternatives.

More impossibility theorems have been established (see Kelly, 1978) and extensive studies have been dedicated to this argument. The important thing to notice is that the non existence of the social choice function does not preclude the possibility to aggregate different orders. You just have to find another way to do it (giving up for instance complete comparability).

2) The conditions under which the additive (or the multiplicative) functions can be established are again very strong and imply a fully rational client, a rather rare situation.

3) The weights are a very controversial concept. In the multiattribute utility theory the parameters p_i introduced as weights are effectively scaling factors representing the trade offs among the different attributes. Therefore this approach assumes implicitly a compensatory approach and again this can be a problem (you may have attributes that cannot be compensated). The existence of the utility function in the non compensatory case is not a trivial problem (see also section 3, Fishburn, 1974 and 1976, Vansnick, 1986, Bouyssou, 1986). In any case there exist "importance parameters" as a "veto threshold" on a criterion that cannot be introduced in an utility function.

How bad is all this? If you think that your client has to be forced in a rational scheme and you have the resources to do it then this theory is strong enough to provide you sure and well defined results. But what about the real problem (if there was one) your client had? Are you sure that your solution is not just a mathematical model with no operational validity?

3 The extensions of the classic approach

3.1 Introduction

As we have seen in the previous section the classic approach suffers some limitations both from theoretical and operational points of view. Before changing completely the approach it is interesting to explore if any extensions are possible in order to cover, at least partially, the space of possible decision situations.

In other words we will see what happens if we retract some of the conditions of the theorem that demonstrates the existence of the utility function. It exists also another possibility which is the idea to work without the utility function. Specially in the multidimensional case it could be interesting to aggregate directly the preference relations without a numerical coding (as the utility function) or with a numerical coding derived in other ways. In the first case however we have to leave the classic approach because working directly with the information, expressed in terms of binary comparisons, leads us to accept "inconsistencies" and "irrationalities" that could be present there and are not eliminated by the numerical coding. Therefore this idea will be discussed more extensively in section 5. In the second case there exists some very interesting work as the "minimal representation of semi orders" (see Pirlot, 1990 and 1991a), but for the moment there exist no other results.

3.2 Definitions

In the following we will make some changes in our presentation of preference relations. First we will introduce the concept of a preference structure. Generally it should be viewed as a n-uple $< P^1, ..., P^n >$ of relations that are exhaustive and disjoint when applied to a set A. To this structure we will associate a characteristic relation S. The relations $P^1, ..., P^n$ should be then defined as combinations of different states of the relation S. It is easy to show that assuming S to be a "large preference" relation of the kind "it is at least preferred as" then there exist four possible combinations:

$$\begin{array}{rcrcll} s(a,b) & \wedge & \neg s(b,a) & \equiv & p(a,b) & \text{strict preference} \\ \neg s(a,b) & \wedge & s(b,a) & \equiv & p(b,a) & \text{strict preference} \\ s(a,b) & \wedge & s(b,a) & \equiv & i(a,b) & \text{indifference} \\ \neg s(a,b) & \wedge & \neg s(b,a) & \equiv & r(a,b) & \text{incomparability} \end{array}$$

Therefore our preference structures are always triplets $< P, I, R >$ with S as characteristic relation; P is asymmetric, I is reflexive and symmetric and R is irreflexive and symmetric. According to particular properties of S it is possible to give other properties to P, I and R or even to have some of them empty. For instance if S is a complete relation then R is an empty relation. For an extensive study of the subject see Roubens and Vincke (1985). For the moment we are interested to notice that if S is a weak order then P and I are both transitive and R is empty. Then the theorem of the existence of the utility function can be restated as follows:

if S is a weak order on A with $S = P \cup I$ then there exists a real valued function $u : A \mapsto \mathcal{R}$ such that:

$$p(a,b) \Leftrightarrow u(a) > u(b)$$

$$i(a,b) \Leftrightarrow u(a) = u(b)$$

Now we can begin to retract some of the assumptions. First of all we can assume that I is not transitive. S will be no more a weak order. We have two possibilities (see also Fishburn, 1985).

1. We can impose $P.I.P \subset P$. Then S is a Ferrers relation called interval order and the following theorem is demonstrated (see Fishburn, 1970):

   if S is an interval order on A, with $S = P \cup I$, then there exist two real valued functions $u : A \mapsto \mathcal{R}$ and $q : A \mapsto \mathcal{R}^+$ such that:

$$p(a,b) \Leftrightarrow u(a) > u(b) + q(b)$$

$$i(a,b) \Leftrightarrow \mid u(a) - u(b) \mid \leq q(b)$$

2. We can impose $P.I.P \subset P$ and $P.P \cap I.I = \emptyset$. Then S is a Ferrers semitransitive relation called semi order and the following theorem can be demonstrated (see Fishburn, 1970):

   if S is a semi order on A, with $S = P \cup I$, then there exist a real valued function $u : A \mapsto \mathcal{R}$ and a constant $\delta \in \mathcal{R}^+$ such that:

$$p(a, b) \Leftrightarrow u(a) > u(b) + \delta$$

$$i(a, b) \Leftrightarrow \mid u(a) - u(b) \mid \leq \delta$$

Another step we can do is to negate transitivity also for the strict preference relation. Even if the question is open since Tversky (1969), only recently it has become subject of extensive studies (for a survey see Fishburn, 1991). The general idea in this case is a representation as follows:

$$p(a, b) \Leftrightarrow \phi(a, b) > 0$$

where $\phi(x, y)$ is a two variables real valued function and in some cases (as for the weak order) can be decomposed: $\phi(x, y) = u(x) - u(y)$. We will not present here more about the subject, but we want to point out that in the multidimensional case the noncompensatory approach has to be used so as to define the global function. (see Bouyssou, 1986).

Last but not least we can give up completeness of the characteristic relation S. What happens is that the theorems we have presented work only in one direction. So:

If S is a quasi order, then:

$$p(a, b) \Rightarrow u(a) > u(b)$$

$$i(a, b) \Rightarrow u(a) = u(b)$$

If S is a partial semi order, then:

$$p(a, b) \Rightarrow u(a) > u(b) + \delta$$

$$i(a, b) \Rightarrow \mid u(a) - u(b) \mid \leq \delta$$

This result has a limited operational interest because it does not allows you to infer univoquely a relation from the evaluation of the function. In Table 1 we give a synthesis of the principle preference structures that are used.

However any kind of partial order is always contained in at least one complete order of the same type (see Barthélemy et al., 1982). The analysis of the "dimension" of a partial order can offer very interesting information for an operational use of these structures.

Table 1: Synthesis of the principal preference structures (Vincke, 1989)

COMPLETE	CHARACTERISTICS	PARTIAL
total order or complete order	no indifference	partial order
weak order or complete preorder	indifference holds	quasi order or partial preorder
semi order	constant threshold	partial semiorder
interval order	variable threshold	partial int. order

3.3 Criticisms

Much of the criticisms done in the classic approach section both in the theoretical and operational aspects can be repeated here.

From a theoretical point of view these extensions try to overcome the counterexamples that come from empirical evidence, but always in a direction that imposes a scheme of perfect rationality to the client (the utility function). Moreover this is not always possible. For the moment only the intransitivity of indifference has been successfully treated. The operational implementation however of utility based methods using thresholds is yet to be investigated (how should these thresholds be elicited?).

But other problems still remain. Actually the client can be in such a situation where intransitivity or probability are not sufficient to represent his uncertainty, ambiguity and moreover his feeling in the decision process for which he asked you some aid.

Therefore for such situations new directions should be explored. In these cases a direct representation of the complexity should be pursued and other operational tools to work with preference relations should be defined.

4 Valued preference relations

4.1 Introduction

A possible answer to the problem of uncertainty and ambiguity arising in the preference modeling process consists in associating an index of credibility to the proposition "a is preferred to b" or to the corresponding predicate. In other words preference is not more a clear and precise statement of comparison, but a fuzzy operator more or less of the type "a is nice", "b is narrow" etc.. If the assumption is that uncertainty is a typical feature of preferences then it is necessary to define a calculus so as to handle operationally these situations. Fuzzy sets theory could then be the tool (see Zadeh et al., 1975 and Zimmermann, 1985 for a general perspective). The idea of a fuzzy preference relation appears in literature with two papers, the one of Roy (1977) and the other of Orlovsky (1978). In the first one the fuzzy outranking relation is defined as a theoretical background for the multicriteria method ELECTRE III where the credibility index is explicitly used. In the second a more foundational work is presented.

After these papers there has been enough research on the subject so as to have some important results while a lot of problems are still open. The interested reader can refer to Ovchinnikov 1981, Barrett et al., 1986 and 1990, Roubens, 1989, Ovchinnikov and Roubens, 1990 and 1991, Billot, 1991; a large part of the book of Kacprzyk and Roubens, 1988, is also dedicated to this approach. Before presenting this approach it should be noticed that no confusion has to be done with probability calculus (see Slowinski and Teghem, 1990). That is, the credibility of "a is preferred to b" is not the probability of "a is preferred to b" as the former has to do with characteristics of the preference modeling itself, while the latter depends on external factors. In other words the fuzzy approach aims to be a general framework for preference modeling where *sure* preferences are just a particular case of the theory. Probability on the other hand is just a tool to use when effectively stochastic situations are encountered.

4.2 Definitions

Let us recall first some basic notions about fuzzy sets. Given a set A, a fuzzy subset is a set F_A^c of ordered pairs $(x, \mu^c(x))$ where $x \in A$ and $\mu^c(x) : A \mapsto [0, 1]$ is a function (called also membership function) mapping each element of A to the interval $[0, 1]$. Intuitively $\mu^c(a)$ can be viewed as a number indicating the credibility of the proposition "a belongs to F_A^c" where c can be viewed as a particular property that elements of A can have (for instance the "nice" elements of A).

Now the application of this approach to preference relations is straightforward. Given A we can imagine a fuzzy binary relation V as a set F_A^v of ordered pairs $((x, y), \mu^v(x, y))$ where $(x, y) \in A \times A$ and $\mu^v(x, y) : A \times A \mapsto [0, 1]$ is again a membership function referred to the relation V. Intuitively $\mu^v(a, b)$ indicates the credibility of holding the relation v among the two elements a and b of A. The problem we have now is to apply this idea to the case of preference relations. Technically what we have to do is to translate the properties of binary relations in the fuzzy case and study the preference structures that can be defined. In the following, from the pair $((x, y), \mu^v(x, y)$ we will omit the first part and use only $\mu^v(x, y)$ to represent the fuzzy set or even more simple directly $v(x, y)$ if that does not generate any confusion.

The properties of a binary relation S can now be defined as follows:

reflexivity:	$\mu^s(x, x) = 1 \;\; \forall x \in A$
irreflexivity:	$\mu^s(x, x) = 0 \;\; \forall x \in A$
symmetry:	$\mu^s(x, y) = \mu^{s^{-1}}(x, y) \;\; \forall x, y \in A$
antisymmetry:	$\mu^s(x, y) > 0 \Rightarrow \mu^{s^{-1}}(x, y) = 0 \;\; \forall x, y, x \neq y \in A$
transitivity:	$\mu^s(x, y) \geq \sup_{z \in A}(\min(\mu^s(x, z), \mu^s(z, y))) \;\; \forall x, y, z \in A$
completeness:	$\max(\mu^s(x, y), \mu^{s^{-1}}(x, y)) = 1 \;\; \forall x, y, x \neq y \in A$

As usual different kinds of orders can be defined as in the classic approach: a fuzzy partial order is an antisymmetric transitive fuzzy binary relation, a fuzzy linear order is a complete fuzzy partial order etc..

Coming back to preference relations our objective is: given a characteristic relation S of a preference structure $< P, I, R >$, how is it possible to define them using fuzzy binary relations. As usual S is a "large preference relation", but in this case is a fuzzy relation also called valued relation, of the type "a is at least as good as b". The problems are:
- how should be defined the three basic preference relations P, I, R and what are their properties?
- how can we exploit valued preference relations?

The first problem that arises is the one to define suitable logical connectives for the fuzzy case so as to reproduce, if possible, the definitions we have already given in the classic approach. For this purpose some ideas have been borrowed from probabilistic calculus (see Schweizer and Sclar, 1983). Let us define now:

- a negation N as a strictly decreasing function $N : [0,1] \mapsto [0,1]$ satisfying

  $$N(N(x)) = x \;\; \forall x \in [0,1]$$

  N is a negation function **iff** it exists an automorphism ϕ of the unit interval such that

  $$N(x) = \phi^{-1}(1 - \phi(x))$$

  A classical negation function is $N(x) = 1 - x$.

- a t-norm T is defined as a function $T : [0,1] \times [0,1] \mapsto [0,1]$ such that $\forall x, y, z, u \in [0,1]$:
  i) $T(x,1) = x$
  ii) $T(x,y) \leq T(z,u)$ if $x \leq z$ and $y \leq u$
  iii) $T(x,y) = T(y,x)$
  iv) $T(x, T(y,z)) = T(T(x,y), z)$

  A t-norm is "archimedean" if $T(x,x) < x \;\; \forall x \in [0,1]$
  An archimedean t-norm is continuous if $T(x,y) = f(g(x) + g(y))$ with

  g a continuous and strictly decreasing function $g : [0,1] \mapsto R^+$ and $g(1) = 0$

  f a continuous function $f : R^+ \mapsto [0,1]$ such that:
  $f(x) = g^{-1}(x) \;\; \forall x \in [0, g(0)]$
  $f(x) = 0 \;\; \forall x \geq g(0)$

  In this way we can write

  $$T(x,y) = g^{-1}(\min(g(x) + g(y), g(0)))$$

- a t-conorm will be defined as a function $V : [0,1] \times [0,1] \mapsto [0,1]$ such that

  $$V(x,y) = N(T(N(x), N(y)))$$

  Obviously $V(x,y)$ fulfills conditions ii), iii), iv) of a t-norm and also the condition
  v) $V(x,0) = x \;\; \forall x \in [0,1]$

All this mathematical machinery has a very simple objective. We will associate to the classical logical connectives $\neg, \vee, \wedge$ the functions $N(x), V(x,y), T(x,y)$ respectively. So recalling the definitions about preference relations given in section 2 we have: given S as a characteristic relation of a preference structure its fuzzy equivalent will be μ^s. So for an ordered couple (a,b)

we will have the number $\mu^s(a,b)$ and therefore we can deduce the credibility of strict preference $\mu^p(a,b)$, indifference $\mu^i(a,b)$ and incomparability $\mu^r(a,b)$:

$$\mu^p(a,b) = T(\mu^s(a,b), N(\mu^{s^{-1}}(a,b)))$$

$$\mu^i(a,b) = T(\mu^s(a,b), \mu^{s^{-1}}(a,b))$$

$$\mu^r(a,b) = T(N(\mu^s(a,b)), N(\mu^{s^{-1}}(a,b)))$$

Given these definitions one has to choose the appropriate functions for $N(x)$, $T(x,y)$ etc. What is requested is that the three relations thus defined have to fulfill the same properties already defined in section 2. That is strict preference has to be irreflexive and antisymmetric, indifference has to be symmetric and reflexive and incomparability has to be irreflexive and symmetric. There exist some problems at this point that will be discussed later on.

The second problem that appears now is how to exploit these valued or fuzzy binary relations. Actually what we have are some numbers representing the credibility of the relation S for each couple (a,b) and eventually we have deduced some numbers about the credibility of the relations P, I, R. But what we need is to aggregate this information to a final ranking or choice. We have to remember also that in this case we may have both intransitivities and incomparabilities. It makes no sense to try to identify a real valued function representing the global preference.

There exist methods based on valued preference relations (see Roy, 1978, Brans et al., 1984, Yu, 1991, Massaglia and Ostanello, 1991). All these methods have been developed before any axiomatization has been presented in the field. They have been originated in the MCDA area to support decision procedures where a ranking or a sorting is requested. In all these methods the credibility index is calculated on the basis of the information arising when the comparisons done on each criterion have to be aggregated (for example the sum of the weights favouring a against b or the percentage of voters declaring that "a is at least as good as b" etc.). This is to mean that it is the method which associates a credibility index to its evaluations and not the client (remember the distinction between credibility and probability to this purpose).

We will not present here how these methods build the credibility index and the final prescription. We want to make two observations.

1. The lack of a stable axiomatization of the fuzzy approach did not allow until now to give a well defined partition of the $A \times A$ space in the foundamental preference relations when the input is the valued relation S. A definition of a "fuzzy partition" however has been recently introduced by Perny and Roy, (1991).

2. It does not exist until now a common characterization of these methods. Some research has been done (see Bouyssou, 1990, Bouyssou and Perny, 1990, Pirlot, 1991b) and some results are available at least for the flow methods. However the principal problem is the definition of a non-redundant set of independent properties that can characterize each method using valued preference relations. For the moment (Bouyssou, 1991) the properties that have been introduced are ($\sqsupseteq^S$ represents the ranking associated to S):

   (a) neutrality: $\sqsupseteq^S$ is neutral **iff**

$$\forall \text{ permutations } \sigma \text{ on } A, \; ; \; \forall x,y \in A \sqsupseteq^S (x,y) \;\; = \;\; \sqsupseteq^{\sigma S} (\sigma(x), \sigma(y))$$

Neutrality means that changing the labels of the alternatives does not modify the ranking.

(b) continuity: $\sqsupseteq$ is continuous **iff**

for all infinite sequences of relations S^i $(i = 1, 2, ...)$

$$S^i, i = 1, 2, ... \leadsto S \text{ and } \sqsupseteq^S (a, b) \text{ then } \neg \sqsupseteq^{S^i} \quad \forall S^i \text{ in the sequence}$$

where the infinite sequence $S^i, i = 1, 2, .. \leadsto S$ **iff**
$\forall \epsilon \in \mathcal{R} \;\; \exists k \in \mathcal{I} : \;\; \forall \;\; j \geq k$ and $\forall x, y \in A \quad | S^j(x, y) - S(x, y) |< \epsilon$

(c) ordinality: $\sqsupseteq^S (x, y)$ is ordinal **iff**

$$\sqsupseteq^S (x, y) = \sqsupseteq^{\phi(S)} (x, y) \;\; \forall x, y \in A$$

where $\phi(S)$ is an automorphism of the unit interval.

(d) monotonicity: $\sqsupseteq^S (x, y)$ is monotonic **iff**

$$\sqsupseteq^S (x, y) \quad \leq \; \sqsupseteq^{S*} (x, y) \;\; \forall x, y \in A$$

where $\sqsupseteq^{S*} (x, y)$ is identical to $\sqsupseteq^S (x, y)$ except that:
$\exists z \in A \setminus \{x\}$: $\mu^S(x, z) < \mu^{S*}(x, z)$ or $\mu^S(z, x) > \mu^{S*}(z, x)$ or
$\exists w \in A \setminus \{y\}$: $\mu^S(y, w) > \mu^{S*}(y, w)$ or $\mu^S(w, y) < \mu^{S*}(w, y)$.
A strong monotonicity can also be defined implying the antisymmetric part of $\sqsupseteq^{S*}$

(e) independence from circuits: $\sqsupseteq^S (x, y)$ is independent from circuits **iff**

$$\sqsupseteq^S (x, y) \;\; = \sqsupseteq^{S*} (x, y) \;\; \forall x, y \in A \text{ and } \forall S, S*$$

with $S - S* = C_\epsilon$ where S and $S*$ are the matrix of $\mu^S(x, y)$ and $\mu^{S*}(x, y)$ $\forall x, y \in A$ and C_ϵ is a matrix with everywhere 0 except a subset of pairs, defining an elementary oriented cycle (circuit) ω with value $\epsilon \in [-1, 1]$

4.3 Criticisms

As every theory the fuzzy approach presents some weak points. We will try to summarize distinguishing between problems with the formalism and operational problems.

1. **Problems with the formalism**. We can identify two particular problems: a specific one, having to do with the properties of the preference relations and a general one about the logic underlying the fuzzy approach.

- We recall that the three foundamental preference relations have some specific properties that characterize them. P is antisymmetric and irreflexive, I is symmetric and reflexive, R is symmetric and irreflexive. Now go back to the definitions of the fuzzy equivalents. The open question was to choose the appropriate function for N, T etc.. The problem here is that it is demonstrated (see Ovchinnikov and Roubens, 1991) that it is impossible to use the same function for T for the three relations and fulfill the properties of the relations. Indeed you have to define a $T^{\sqsupset}$ for the asymmetric part of S and another function $T^{\approx}$ for the symmetric part (see also Fodor and Roubens, 1991, for some results).

How bad is this? One could argue that the choice of the functions $T^{\sqsupset}$ and $T^{\approx}$ is arbitrary; therefore it is difficult to accept the theory from a formal point of view. The antisymmetric and symmetric part of S are thus not well defined. So how to be sure about the membership of a couple to a relation or to another? This implies a nested reasoning on fuzziness. We will not commit ourselves in such a discussion. We notice that this problem is strongly linked to some general problems of fuzzy logic.

- Fuzzy logic, the logic underlying the fuzzy preference relations theory, is not always a truth functional logic. This means that in fuzzy logic it is not always possible to define all the logical connectives in a truth functional way. Actually the logical implication has an almost arbitrary definition (see Urquhar, 1986). The reason for this is that if we use the intermediate values of truth also for the implication it is difficult to determine the necessary logical tautologies. The solutions proposed then suffer when sound inference rules and entailment have to be defined. What is worst with fuzzy logic is that the infinite possible truth values between the absolute truth and the false have an intuitive meaning that does not correspond with its formal semantics. In other words, high or low credibility is always (intuitively) a measure of truthness (or falsiness). But the formal semantics give space at least to "undefined" propositions. Again, how bad is all this? It is not here that you can find the answer. In the conclusions however we will try to give some directions about the importance of these problems.

2. **Operational Problems**. There exists a long list of real world applications (for a list see Jacuet-Lagreze and Siskos, 1983, Fandel and Spronk, 1985, Vincke, 1989) of methods using valued preference relations, the most of them successful. This accounts for a positive operational validation of these methods. However we have to point out the questions that can arise. Essentially there exists one problem and this is the construction of the credibility index. If it is built by the method itself it is difficult to control its consequences on the final prescription because of the theoretical problems we have already presented; of course it is also difficult to explain it to your client (not when everything works, but when you have the shortcomings). If you try to elicit this index directly from your client then the mathematical machinery could be tedious and intractable.

Last but not least: why should we use valued preference relations? Unlike the aims of the theory of fuzzy sets there is no specific reason to follow this approach, while it should be recognized that credibility indexes can help you in the mess of your problem. In other words: let your client fill as well as possible; with or without credibility.

5 The outranking relation

5.1 Introduction

As we have already seen intransitivity of preference and indifference can be argued on the basis of different counterexamples (see Luce, 1956 and Tversky, 1969). The same question can be posed about completeness of preference relations. Specially in the case of multicriteria evaluations, the aggregation of the preferences expressed on each criterion in a global preference relation leads to an intransitive and incomplete relation. Faced with this situation there exist two alternatives:
- either to explore all the possibilities that the classic approach allows without rejecting its foundamental assumptions and to force the real situations to the more suitable model;
- or to reject some of the classic assumptions and try to identify more rich preference relations introducing new preference relations.

The concept of outranking relation introduced by Roy (1968) has an empirical origin due to the will to represent more complex preference situations than the those admitted by the classic approach. The general idea is: "...an outranking relation is conceived so as to take in account, in the aggregation model, the particular case where two actions are incomparable.... Then, when accepting incomparability situations, willingness to clarify situations of preference or indifference, an outranking relation refers to preferences modelled only for these cases where the analyst is able to establish them with an objectivity and security that he judges satisfactory" (quoted from Ostanello, 1985). One of the intuitions that appears from such a definition is the will to keep in account not only the reasons for which a preference can be established but also the reasons that reject a preference even if this does not automatically establish another preference relation.

Such a flexibility is necessary only if we accept that the problem we are trying to solve is not always a "choice" but that other problem statements can be also accepted as "ranking", "sorting" etc. and also their combinations (see Roy, 1985 and Ostanello 1987). This is a major innovation from the decision aid point of view because it changes the perspective of the operational modelization of a problem. The problem now is how to represent adequately these ideas in a formalized way.

5.2 Definitions

From a formal point of view the idea is to introduce the outranking relation as the characteristic relation of preference structures based on four foundamental relations $< P, Q, I, R >$ ("strict preference", "weak preference", "indifference", "incomparability", respectively). In his methodology Roy (1985) defines the structure $< P, Q, I, R >$ a *f.s.p.r.* (foundamental system of preference relations) which means that the following properties are fulfilled (Roy and Vincke, 1984):
- the relations can be used to modellize an actor's preferences towards a set of actions A;
- the relations are exhaustive: for any pair of actions, at least one is satisfied;
- the relations are mutually exclusive: for any pair of actions, at most one is satisfied.

The four relations are thus defined, in a discursive way as follows:

Table 2: *Foundamental Preference Relations* (quoted from Roy and Vincke, 1984)

Relation	Definition	Properties
Indifference	Two actions are indifferent in the sense that there exist clear and positive reasons to choose equivalence.	reflexive symmetric
Strict preference	There exist clear and positive reasons to justify that one (well specified) of the two actions is significantly preferred to the other.	asymmetric
Weak preference	One (well specified) of the two actions is not strictly preferred to the other but it is impossible to say if the other is strictly preferred or indifferent to the first one.	asymmetric
Incomparability	Two actions are not comparable in the sense that none of the above three situations predominates	irreflexive symmetric

Operationally the outranking relation has been conceived to represent the global preference order that may arise while aggregating different complete orders corresponding to the different criteria. A strong intuition has been the introduction of the concepts of concordance (a "measure" of "positive" preference) and of discordance (a "measure" of "negative" preference) as a double evaluation in order to establish if an action outranks another one.

Different methods have been developed on this basis (the ELECTRE family, PROMETHEE, ORESTE, but most of them in versions with valued relations) with a big variety of real world applications (for a list see Jacuet-Lagreze and Siskos, 1983, Fandel and Spronk, 1985, Vincke, 1989)

5.3 Criticisms

Until now no complete axiomatization is available for the outranking relation. In other words it is not possible to give a complete characterization of this relation and of the four foundamental relations P, Q, I, R. The most problematic one is the weak preference relation that cannot be distinguished from the strict preference on a pure mathematical basis.

The only tentative done is the definition of the concept of "pseudo-order" (see Roy and Vincke, 1984) based on different kinds of thresholds. In this case, as in the classic approach (incomparability is now empty), it is possible to demonstrate that (see Vincke, 1980):

Given the relations P, Q, I, under specific conditions and when their union defines a complete relation, there exist three real valued functions: $g(x) : A \mapsto \mathcal{R}$, $q(x) : A \mapsto \mathcal{R}$, $s(x) : A \mapsto \mathcal{R}$ such that:

$$P(a, b) \Leftrightarrow g(a) - g(b) > s(b)$$

$$Q(a, b) \Leftrightarrow s(b) \geq g(a) - g(b) > q(b)$$

$$I(a, b) \Leftrightarrow q(b) \geq g(a) - g(b) \text{ and } q(a) \geq g(b) - g(a)$$

However this does not solve the axiomatization problem and does not introduce the incomparability as the existence of the real valued functions imposes the completeness of the associated order.

How bad is all this? The lack of an axiomatization gives of course a lot of damage to the "scientific" credibility of this approach, because a lot of space is left for arbitrary and confused interpretations. Moreover the approach claims to represent better situations where ambiguity, uncertainty, imprecision are present allowing qualitative evaluations. The problem is that without a stable axiom theory either you are obliged to use numerical representations, thus contradicting the qualitative nature of the approach or you risk to have not well understandable and defined results.

On the other hand experimental and operational validation of the use of the methods developed by this approach gives a strong support at least to the intuitions that are behind it. Let us make a summary of these intuitions:
- the reference for the analyst should be the decision process and not the decision itself;
- the problem statement (or the problem formulation) can change the aggregation procedure and therefore the results;
- intransitivity and incomparability are not expressions of "irrationality" or "inconsistency", but possible situations not always reducable to simpler ones;
- the establishment of the existence of a preference is always the result of a double evaluation (a positive and a negative one).

Recall now the question we posed in the first section: why do you need to build models of preferences? If your answer is: so as to help the client in a decision process, where she/he is involved, in order to improve her/his behaviour, then you should take these propositions as a new set of conceptual assumptions about decision aid (and preference modeling). We then have more flexibility in decision aid activities. The problem is to find now the best possible formalization for these assumptions even if we have to go outside the traditional mathematical structures. What we definitely claim is that the guideline is the decision aid and not the mathematics that support it. It is the former one that has to define the latter and not viceversa. In the following section we will give an outline of a new possible formalism that can also give the axiomatic basis of the approach.

5.4 A new formalism

Suppose that in a committee a part of it suggests than the candidate a is "better" that the candidate b and another part argues that the candidate a is "not better" than b. You have a conflictual situation where the couple (a, b) is both "inside" the relation "better" and "not inside" the same relation. If you want to represent these situations in a formal structure then you may introduce a distinction between elements "inside" a set, "not inside" a set, "outside" a set and "not outside" a set represented as follows (read LP as "I believe P" where L is an unary logical operator):
- "inside" the set P: $\in P$ or LP;
- "not inside" the set P: $\not\in P$ or $L\neg P$;
- "outside" the set P: $-\!\subset P$ or $\neg LP$;
- "not outside" the set P: $\not-\!\subset P$ or $\neg L\neg P$;
The following relations hold:
$LP \wedge L\neg P \equiv \mathbf{K}P$
$L\neg P \wedge \neg LP \equiv \mathbf{F}P$
$\neg LP \wedge \neg L\neg P \equiv \mathbf{U}P$
$LP \wedge \neg L\neg P \equiv \mathbf{T}P$
where $\mathbf{T}P, \mathbf{F}P, \mathbf{K}P, \mathbf{U}P$ represent the extensions of the predicate P with truth values t, f, u, k respectively: true, false, unknown, contradictory (for details see: Belnap, 1976 and 1977, Sandewall, 1985, Dunn, 1986, Ginsberg, 1988; for the particular case see also Tsoukiàs, 1991). To this structure it is possible to associate a well defined logic for both the propositional and first order predicate calculus. We will use the notation P^t, P^f, P^u, P^k to represent the subsets of P with the specific truth values.

With this structure and the relative logic we can build now the following theory:

- Properties of the binary relations (V being any kind of binary relation):

- non contradiction:	$\mathbf{K}V = \mathbf{U}V = \emptyset$
- reflexivity:	$\mathbf{T}V(x, x)\ \ \forall x$ in A
- irreflexivity:	$\mathbf{F}V(x, x)\ \ \forall x$ in A
- symmetry:	$\mathbf{T}V(x, y) \rightarrow \mathbf{T}V(y, x) \forall x, y$ in A
- asymmetry:	$\mathbf{T}V(x, y) \rightarrow \mathbf{F}V(y, x) \forall x, y$ in A
- completeness:	$\mathbf{T}V(x, y) \vee \mathbf{T}V(y, x) \forall x, y$ in A
- transitivity:	$\mathbf{T}V(x, y) \wedge \mathbf{T}V(y, z) \rightarrow \mathbf{T}V(x, z) \forall x, y, z$ in A

- A preference structure will now be defined as a quadruple $< P, Q, I, R >$ of relations such that:

- P, Q, I, R are non contradictory;
- P, Q are asymmetric;
- I, R are symmetric;
- I is reflexive;
- P, Q, R are irreflexive;
- $P \cup Q \cup I \cup R$ is complete;
- P, Q, I, R are two by two disjoint
- if $(a, b) \in P$ then $(b, a) \neg\subset Q$
- if $(a, b) \in Q$ then $(b, a) \neg\subset P$

- This preference structure can be characterized by a preference relation S defined as follows:

$$S^t = P \cup I$$
$$S^k = Q \cup I$$
$$S^u \cup S^f = P^{-1} \cup Q^{-1} \cup R$$

- It is then demonstrated that (see Tsoukiàs, 1991):

$$P = S^t \cap (S^{-1^u} \cup S^{-1^f})$$
$$Q = S^k \cap (S^{-1^u} \cup S^{-1^f})$$
$$I = (S^t \cup S^k) \cap (S^{-1^t} \cup S^{-1^k})$$
$$R = (S^f \cup S^u) \cap (S^{-1^f} \cup S^{-1^u})$$

We will call S an "outranking relation" and P, Q, I, R as strict preference, weak preference, indifference and incomparability. It is easy to see that they correspond exactly to the definitions given by Roy in his methodology for his "foundamental system of preference relations". In the following figure all the possible combinations of S and S^{-1} on the space $A \times A$ are presented

Figure 1: the space $A \times A$ and the outranking relation.

$A \times A$	S^{-1t}	S^{-1f}	S^{-1u}	S^{-1k}
S^t	I	P	P	I
S^f	P^{-1}	R	R	Q^{-1}
S^u	P^{-1}	R	R	Q^{-1}
S^k	I	Q	Q	I

It can be observed that the conventional preference theory can be derived from this approach as a particular case deciding that S can have only traditional truth values, that is true or false.

From an operational point of view it is now possible to give a formalization to the concepts of concordance and discordance also. The concordance test should be interpreted as an interrogation if (a, b) is "inside" or "not inside" the relation S and the discordance test as an interrogation if (a, b) is "outside" or "not outside" the relation S. By these definitions and by the relations P, Q, I, R previously defined it is possible to build now a new family of methods where this approach is implemented. The advantages should be more flexibility, a richer representation of the different situations with a clear distinction among them, an enhancement of the possible aggregation procedures and the possibility to use non-compensatory aggregations. On the other hand it has to be observed that in this case it is definitely impossible to work with real valued functions representing the relations.

6 Conclusions

In this survey we tried to give a comprehensive view of the problems that are open in the field of preference modeling. From the limits of the classic approach we have the stimulus to explore other possibilities moving from simple relaxations in the foundamental theorems of utility theory (assumption of transitivity) towards completely new approaches such as the use of fuzzy operators or the explicit use of incomparabilities down until the level of the truth values of your logic, enabling more rich preference models.

We want to definitely state that it does not exist a "best" approach. The eventual choice of one is a multicriteria problem and therefore has no optimal solution. For if you want an easy numerical model you have to make strong and perhaps unrealistic assumptions; if you want rich and flexible models you have to give up strong axiomatization, if you want a direct representation of ambiguity you have to give up truth functionality, if you want a strong logic you have to accept a limited declarative power, if you want complete comparability you cannot model a lot of real situations, if you accept partiality you cannot use traditional mathematics and so on.

There are of course a lot of open fields of research as:
- the analysis of different scaling methods;
- the nontransitive preference relations;
- the noncompensatory aggregation procedures,
- the aggregation of partial orders;
- the minimal representation of partial orders;
- the study of fuzzy binary relations and partitions;
- the axiomatization of aggregation procedures;
- the study of incomparability;
- the development of a new mathematical structure for the outranking approach;

related to other fields and disciplines as algebra, logic, cognitive science, measurement theory etc. and a lot of contributions are to come in the future. We would like to make however two observations based on the fact that preference modeling is usually used for decision aid purposes.

- As an analyst you can choose any approach when involved in a decision aid process. What we think is very important is to clearly know what are the limits of the approach you use and when you cannot apply it. From this point of view we think that clear axiomatizations of the different approaches are very useful (not only as mathematical exercises) because they provide you exactly this information.
- We do not want to hide our preference for these approaches that adapt the mathematics to the decision process (and the reality) and not viceversa. For we think that in decision aid you are asked to help your client and not to teach him how to be "rational". Finally if a decision has to be taken based on your work the responsibility is for the client. So listen to him. Rationality can always be rediscussed, errors sometimes not.

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Abstract

In the paper a survey on non conventional preference modeling is presented. Some philosophical discussion on the logic and foundations of human preferences is presented as the first attempt to provide an axiomatization of the subject. The use of preference models in decision aid however introduces such parameters to put serious questions to the possibility to build a complete normative axiomatization of this discipline. The conventional preference models are discussed both from a mathematical and the decision aid point of view underlining the limitations and impossibilities of this approach towards complex and particular problem situations. The extensions of the conventional theory while enhance its potentialities fail however to give a satisfactory answer to some problems as uncertainty, ambiguity and imprecision of preference models mainly when multidimensional preferences have to be considered (collective and/or multicriteria preferences). Two non conventional approaches are presented, fuzzy preference models and the concept of outranking relation. In the first case an account of the theoretical foundations of valued preference relations is presented and some problems with axiomatization are discussed. In the second case, while the operational potentialities of the approach are underlined, the lack of a strong theoretical foundation is emphasized. A new four-valued logic is considered as a tool enabling to give such a theoretical support. The future research on the subject concludes the paper.

Riassunto

Nel lavoro è presentata una survey sulla modellizzazione delle preferenze non convenzionale. Un primo tentativo di fornire una assiomatizzazione dell'argomento è presentata sotto forma di una discussione filosofica sulla logica e le fondamenta delle preferenze umane. L'uso però dei modelli di preferenze nell'aiuto alla decisione introduce tali e tanti parametri da rendere impossibile la definizione di una base normativa completa per questa disciplina. L'approccio convenzionale viene quindi discusso sia dal punto di vista matematico che dell'aiuto alla decisione mettendo in evidenza i suoi limiti verso situazioni problematiche complesse e particolari. Le estensioni dell'approccio convenzionale, mentre aumentano le sue potenzialità, non permettono però di affrontare in modo soddisfacente problemi come l'incertezza, l'ambiguità e l'imprecisione nei modelli di preferenze soprattutto quando essi sono costruiti in presenza di preferenze multidimensionali (preferenze collettive e/o multicriteri). Vengono presentati due approcci non convenzionali, i modelli di preferenze sfocati e l'uso del concetto di relazione di surclassamento. Nel primo caso vengono presentate le fondamenta delle relazioni di preferenza valutate e discussi alcuni problemi legati alla loro assiomatizzazione. Nel secondo caso, mentre vengono sottolineate le potenzialità operative dell'approccio, viene enfatizzata l'assenza di un chiaro supporto teorico. In questo senso viene considerata una nuova logica a quattro valori che può essere lo strumento teorico con il quale costruire un'adeguata teoria di supporto. La ricerca futura sull'argomento conclude il lavoro.