Dynamic-R: a "Challenge-free" method for rating problem statements

Oussama Raboun[1], Eric Chojnacki[2] and Alexis Tsoukiàs[1*]

[1*]LAMSADE, CNRS, PSL, Université Paris Dauphine, France.
[2]IRSN, Cadarache, France.

*Corresponding author(s). E-mail(s): tsoukias@lamsade.dauphine.fr;
Contributing authors: o.raboun@gmail.com; eric.chojnacki@irsn.fr;

Abstract

In this paper, we are interested in decision aiding problems, aiming at rating a set of objects with respect to several dimensions, called criteria. A rating problem statement consists on partitioning a set of objects into predefined ordered equivalence classes, called categories, identified by ratings. Rating problems are widely studied in the literature, either based on the utility theory, rough sets or the majority principle. The existing methods based on the majority principle present some disadvantages potentially leading to an unconvincing rating because challenged by contradictory pairwise comparisons. In this work, we present a new method providing a "convincing" (challenge-free) rating over a set of studied objects, based on the aggregation of positive and negative reasons, respectively supporting and opposing a rating. The method exploits comparisons among the objects and the profiles characterizing the categories as well as comparisons among the objects.

Keywords: Multiple criteria decision analysis, Rating problem statements, Decision support systems, Algorithmic Decision Theory

1 Introduction

In this paper, we propose a new MCDA (Multiple Criteria Decision Analysis) method aiming at providing a "challenge-free" or "convincing" rating to a set of objects, here after named A, such as geographic units, financial products, clients in an insurance company, to name but a few, evaluated by ordinal information under at least one dimension. The work has been developed within the framework of a larger research project aiming at informing the local authorities about consequences (and decisions to be taken) in case of a major nuclear accident near the French Mediterranean coast (see Raboun et al (2020)).

A rating problem statement (see Colorni and Tsoukiàs, 2013) consists in partitioning A into predefined and ordered equivalence classes, called categories, identified by ratings. Since we are dealing with objects evaluated under several dimensions, called criteria, we will consider rating problem statements in the context of MCDA. By "convincing", we refer to the following claims:

Claim 1. *No object is assigned to a category worse than the categories to which are assigned the objects to which this object is better if directly compared.*

Claim 2. *There are no unrated objects (complete rating).*

Several MCDA methods have been developed to deal with rating problems. These methods can be partitioned into three categories:

1. methods based on the majority principle, called *outranking* methods, see (see Almeida-Dias et al, 2010, 2012; Fernandez et al, 2017; Leroy et al, 2011; Vincke, 1999; Yu, 1992);
2. methods based on the assessment of utility functions, (see Bugera et al, 2002; Dembczyński et al, 2006; Devaud et al, 1980; Greco et al, 2010; Köksalan and Özpeynirci, 2009);
3. methods based on rough sets, (see Dembczyński et al, 2009; Greco et al, 2001, 2002a,b).

In this work, we are interested in the same type of problems for which outranking methods fit. Outranking methods, in the context of rating problems, are based on preference relations established between the set A and reference profiles without considering comparisons among objects. Because of this feature, outranking methods may lead to non-convincing ratings, because of cycles of preferences or because of incomparabilities. This is because outranking relations do not have any remarkable ordering properties, (see Bouyssou, 1996). Consider the following example:

Example 1. *(Non convincing rating due to the Condorcet Paradox)*

Let us consider a rating problem characterized by three necessary and sufficient criteria, i.e. the three are exhaustive and none of them is a dictator. This comes to considering any coalition of two criteria as a decisive coalition. We consider that each criterion evaluates the set A on an ordinal scale: $\{B, A, A^+\}$ such that $B \prec A \prec A^+$, $\prec$ being a strict preference relation. In this problem we aim at assigning two objects $x = (A^+, A, B)$ and $y = (A, B, A^+)$ into two predefined ordered categories $\mathcal{C}_1$ (rate 1) and $\mathcal{C}_2$ (rate 2) such that $\mathcal{C}_1$ is the best. The two categories are separated by a lower bound of $\mathcal{C}_1$: $p = (B, A^+, A)$.

Using the majority rule to rate x and y, we obtain: $y \succ p$ and $p \succ x$. Thus, y will be rated 1 while x will be rated 2. The decision maker might be not convinced by the result: indeed x is strictly preferred to y (assuming the same majority rule).

The originality of this work consists in handling this type of inconsistencies, through a new "dynamic" and "convincing" MCDA rating method, named "Dynamic-R", for problems characterized by ordinal information under at least one criterion. In order to obtain a "convincing" rating (see our claims 1 and 2) we make an explicit use of clear positive and negative reasons, respectively supporting and opposing a rating, and we solve any potential contradiction. The dynamic aspect of the method is related to the rating procedure associated to the method: the rated objects are added to the profiles characterizing the categories and are used when new objects are considered for rating. Hence, the positive and negative reasons will be updated in order to take into account preferential information coming from these just-rated objects. In order to obtain a "convincing" rating, we address the following features:

- We allow comparison among elements in the set A;
- We allow both limiting and typical profiles;
- We separate positive and negative reasons;
- We distinguish the positive and negative reasons for or against an outranking from the positive and negative reasons for or against a rating;
- We provide a complete rating, as a result of our procedure.

The paper is organized as follows. Section 2, reviews the relevant literature. Section 3, introduces notations used all along the paper. In section 4, we give an overall overview of the method presenting the central ideas within it, without describing how these are implemented. Section 5, introduces four basic concepts: minimal requirements, positive and negative reasons for a binary relation (outranking), distance of an object with respect to a profile, incompatibility relation. Section 6, introduces the notion of positive and negative reasons for a rating, shows which are the properties satisfied by such definitions, introduces our principal constraint: a "convincing property" to be satisfied by ratings (see claim 1) and finally introduces how positive and negative reasons are updated in order to pass a consistency checking. Section 7, describes how the rating is constructed and which properties are satisfied. Section 8, then shows that our procedure indeed satisfies our "convincing property" and provides some information about the method's performance. We discuss the results and we conclude in Section 9.

2 Related Literature

Updating preferences is not really a new topic. It has been already introduced in Falmagne (1996) from a general perspective and has been extensively studied in marketing (for a presentation related to conjoint measurement see

Ben-Akiva et al, 2019). Incremental preference modelling has been considered also in Greco et al (2011) and more recently in Khannoussi et al (2021), Baarslag and Gerding (2015), Liu (2015), Perny et al (2016). However, it has never been used for the purpose presented in this paper and, as far as we know, it always requires an interaction protocol with a decision maker, while in our case is an "automatic" procedure.

Several rating methods have been developed aiming at rating a set of objects with respect to a consistency rule. For example, in Rocha and Dias (2008) the PASA (Progressive Assisted Sorting Algorithm) method has been proposed, respecting the following consistency principle: an object cannot be assigned to a category in case it is outranked by any example (reference profile) assigned to a worse category. This principle seems very close to our work since we also characterize the categories by a set of reference profiles and we have a consistency rule. However, this method presents also many limitations such as:

- the order of the selected objects for rating might bias the ratings of the next selected objects;
- in case of an imprecise rating, either the decision maker is needed or the rating is postponed;
- forcing the consistency might lead to bad quality of rating: objects involved in cycles are placed in the same category (the worse category among the ones to which objects can be assigned).

The THESEUS method (Fernandez and Navarro, 2011) is another rating method, aiming at providing a rating minimizing inconsistencies with respect to a learning set (reference profiles in our case). This method is based on an original approach, transforming a rating problem into a ranking problem. Such transformation consists on associating to each non rated object x, new alternatives x_k: "assign x to the category k". The generated alternatives x_k are assessed under the following criteria: inconsistencies with respect to the strict preference, the weak preference, and the indifference. Hence, the problem of rating x, comes to a ranking problem associated to selecting the best x_k, minimizing the inconsistencies. We address the following limitations of THESEUS method:

- The provided rating minimizes inconsistencies. However, it does not prevent an inconsistent rating;
- The dependency on the learning set: both small and very big learning sets may lead to a poor rating either because of incomparabilities or the high number of inconsistencies.

It is true that methods based on decision rules such as DRSA (Greco et al, 2001, 2002a,b), provide a rating respecting the convincing claims. However, these methods require a large learning set. In many decision aiding problems, all what we can have are few assignment examples given by the decision maker (the client). Methods using value functions or any other measure of "difference of preferences" may also be able to satisfy our claims, but this comes at the price of more information which may be "costly" to get (for more about differences of preference see Bouyssou et al, 2006).

3 Notations and concepts

All along this document we will use the following notation:

- $T = \{1, 2, 3, ...\}$: a set of time steps. Generally we will use $t \in T$ to refer to a time step in the process. For simplicity we will use the term step to refer to a time step in which a new set of objects is considered for rating.
- $A^t = \{x, y, z, w, ...\}$: a set of studied objects considered for rating at each step $t \in T$. The set A^t can be either known previously, or elicited during an interactive process between a decision analyst and a client. This set is traditionally called in the literature associated to decision sciences, alternatives or actions (Bouyssou et al, 2006).
- $\llbracket\ ;\ \rrbracket$: the mathematical notation to refer to the integer interval.
- $\mathcal{C} = \{\mathcal{C}_1, ..., \mathcal{C}_q\}$, $q \geq 2$: a set of predefined ordered categories, where $\mathcal{C}_k$ refers to a category where all objects are rated k. Without loss of generality, we assume that,$\forall k \in \llbracket 1\ ;\ q-1 \rrbracket : \mathcal{C}_k$ is better than $\mathcal{C}_{k+1}$. Hence, $\mathcal{C}_1$ is the best category.
- $Z^t = \{Z_1^t, ..., Z_q^t\}$: Reference profiles. At a given time step $t \in T$ we have a set of sets of reference profiles, such that any element Z_h^t is a set of $\{z_{h,k}$ elements where $k = 1, ..., i_{h,t}\}$, is an index depending from the category h and the time step t, $i_{h,t} \geq 1$. Each Z_h^t represents the set of reference profiles characterizing the category $\mathcal{C}_h$, at the step t. The initial set of reference profiles Z^0 is used as a learning set to generate the preferential information. At the end of each step $t \in T$, objects in A^t will be assigned to the sets of reference profiles associated to the corresponding categories. We will use also the notation: $\forall j, k \in \llbracket 1\ ,\ q \rrbracket, j < q : Z_{j,k}^t$ to refer to $\underset{i \in \llbracket j\ ,\ k \rrbracket}{\cup} Z_i^t$.
- $\mathcal{F} = \{1, ..., m\}$ with $m \geq 3$: a family of criteria under which objects are evaluated. We associate to each criterion $j \in \mathcal{F}$ a weak order $\succcurlyeq_j$ upon $\mathcal{A}^t$.
- $\mathcal{B} = \{b_1, ..., b_q\}$: a set of minimal requirements, characterizing categories where performances of the profile $b_k = (\underline{b}_{j,k})_{j \in \mathcal{F}}$ characterizing $\mathcal{C}_k$, are the minimal performances in order to be admissible in $\mathcal{C}_k$. These minimum requirements are characterized by the following condition: We assume that $\forall j \in \mathcal{F}, \forall k \in \llbracket 1\ ;\ q-1 \rrbracket : b_k \succcurlyeq_j b_{k+1}$. The profile b_k should not be confused with a limiting profile since it does not necessarily belong to $\mathcal{C}_k$.
- $\mathcal{A}^t = \cup_k Z_k^t \cup A^t \cup \mathcal{B}$: the set of all objects considered at the step t of the rating aggregation procedure.
- w: importance of coalitions of criteria. It is a capacity defined as: $w : 2^{\mathcal{F}} \rightarrow [0, 1]$. By definition of capacity we have $w(\mathcal{F}) = 1$, $w(\emptyset) = 0$, and for all $A, B \in 2^{\mathcal{F}}$ such that $A \subseteq B$, $w(A) \leq w(B)$. To simplify notations, we will use w_j to refer to $w(\{j\})$.

- $\mathcal{V}$: importance of the discordant criteria to reject a preference relation. It is a capacity defined as: $\mathcal{V} : 2^{\mathcal{F}} \rightarrow [0,1]$. By definition of capacity we have $\mathcal{V}(\mathcal{F}) = 1$ (all criteria reject a given preference), $\mathcal{V}(\emptyset) = 0$, and for all $A, B \in 2^{\mathcal{F}}$ such that $A \subseteq B$, $\mathcal{V}(A) \leq \mathcal{V}(B)$.
- λ: the majority, considered sufficient, enabling a coalition to be decisive, called concordance threshold.
- v: the veto threshold.
- $U_k^- \forall k \in [\![1 \; ; \; q-1]\!]$: the set of objects having negative reasons to be rated k or better, based on the comparison with reference profiles. The notation U_0^- will be used to refer to the set of objects not having negative reasons, against being rated 1, based on the comparison with reference profiles: $U_0^- = A^t \cap \neg(U_1^-)$.
- U_k^+: the set of objects having positive reasons to be rated k or worse, based on the comparison with reference profiles.
- U_k^{e-}: the set of objects for which negative reasons are enriched, due to the comparison with the other objects in A^t, to a worse category k.
- U_k^{r-}: the set of objects for which negative reasons are withdrawn, due to the comparison with the other objects in A^t, to a better category k.
- U_k^{e+}: the set of objects for which positive reasons are enriched, due to the comparison with the other objects in A^t, to a better category k.
- L_k^t: the set of objects for which the worst possible rating is k (without taking into account the way objects compare to each other).
- H_k^t: the set of objects for which the best possible rating is k (without taking into account the way objects compare to each other).
- $L_{u,k}^t$: the set of objects for which the worst possible rating is k, with respect to reference profiles and objects in A^t.
- $H_{u,k}^t$: the set of objects for which the best possible rating is k, with respect to reference profiles and objects in A^t.
- U_k^{2+}: the set of objects in $H_{u,h}^t \cap L_{u,l}^t$ (for $h \leq k$ and $l \geq k$) rated k, based on a distance from reference profiles.

4 General overview of Dynamic-R

The existing outranking based rating procedures use a majority principle applied on aggregating positive reasons (typically known as concordance relation). This relation is bounded by a minority principle (typically known as discordance relation, usually a veto condition) which can invalidate the aggregation of the positive reasons. Positive reasons are typically obtained comparing objects either to limiting profiles (a vector or a set of vectors) separating categories, or to typical profiles (a vector or a set of vectors) characterising the categories. In the first case we make use of asymmetric comparisons (intuitively an object x is rated k if it is better than the profile separating category k from category $k+1$), while in the second case we make use of symmetric comparisons (intuitively an object x is rated into category k if it is similar to a typical profile of such category). In both approaches objects are never compared to each other.

Dynamic-R introduces three new ideas:

1. it does not make any distinction between limiting and typical profiles since both of them might be available and provide positive or negative reasons about the rating of a given object x;
2. it explicitly introduces the concept of minimal requirements, a disjunctive constraint among the criteria, providing strong evidence that an object CANNOT be rated to a certain category (because it fails to satisfy a requirement on any of the criteria), without the vector of minimal requirements being a profile of any category;
3. it accumulates reference profiles since objects, that are rated at step t, are used as profiles both within step t as consistency checking, thus allowing comparisons among objects, and at step $t+1$.

Dynamic-R is a MCDA rating method extending the use of the concordance/discordance principles through the use of generalised positive and negative reasons (see Tsoukiàs et al, 2002) for which a given object can belong to a given category. The main inputs required by the method are: the set of partitions of reference profiles characterizing the categories Z^t, and the set of minimum requirements $\mathcal{B}$. At the basic level, the developed rating procedure, at each time step $t \in T$, is based on the assessment of:
- on the one hand, subsets of objects $U_k^+ \subseteq A^t$, $k \in [\![1\ ,\ q]\!]$, having reasons supporting their rating at most k (k or worse). Such set is based on the presence of a sufficient majority of criteria, not disqualified by a veto, in favor of an object in A^t, compared to a reference profile characterizing the category k;
- on the other hand, subsets of objects $U_k^- \subseteq A^t$, $k \in [\![1\ ,\ q-1]\!]$, having reasons opposing their rating at least k (opposing a rating to k or better). Such negative reasons might come either from the incompatibility with category k due to the violation of the minimum requirements, or the dominance or the strict preference (depending on the way negative reasons are defined) in favor of a reference profile characterizing a worse category. The concept of minimum requirements consists on profiles representing the minimum acceptable performances, under each criterion, regardless the global performance, in order to be admissible in a category.

Example 2. *(Example of positive reasons)* A new student in a school, might have positive reasons to be in the category of excellent students, if he is better, according to a majority of criteria, than a former excellent student. The reader will note that having positive reasons to be in the category of excellent students implies having positive reasons to be in any worse category, such as the one of good or even bad students.

Example 3. *(Example of negative reasons)* A student cannot be in the category of excellent students if an average student is preferred to him/her. Regardless the global mark, a student cannot be considered a good student if he performs worse than 7/20 in any of the lectures. Minimum requirements should not be confused with limiting profiles: the vector $(7/20, ..., 7/20)$ is the minimum requirement associated to the category of good students, however, a student performing 7/20 in all the lectures "$(7/20, ..., 7/20)$" is not a good student.

Remark 1. *In case the set of minimum requirements $\mathcal{B}$ is not empty, and the number of objects to be rated and reference profiles is important, it is better to not consider the strict preference relation in negative reasons, for two reasons:*

1. *The negative discrimination power due to vetoes, with respect to limiting profiles, might be substituted by the minimum requirements in the case where reference profiles are not necessarily limiting profiles. This substitution provide many advantages as the assessment of minimum requirements is directly related to the categories while their might exist a very high number of limiting profiles and thus an object discriminated by a limiting profile might be not discriminated by another.*
2. *Negative reasons based on strict preference might influence badly the quality of the obtained rating, due to non-transitivity: discriminating the assignment of an object to a category due to a strict preference in favor of a reference profile might be criticized since we might have cycles.*

Hence, the use of the strict preference in negative reasons will be limited to the cases where $\mathcal{B} = \emptyset$ and the number of objects to be assigned is low. Here after, negative reasons will be treated in two cases, whether strict preference is considered or not.

When the decision maker or the quality of the rating problem require taking into account the way objects compare to each other, new positive and negative reasons might appear, and some reference profiles might need to be updated. Considering the way objects compare to each other may lead to either enriching negative reasons, in case strict preference is used in the assessment of negative reasons, or enriching positive reasons, or withdrawing negative reasons.

The rating process associated to Dynamic-R, at a step t, can be structured as follows:

1. For each object $x \in A^t$, we compute for each category k, the sets of objects having respectively positive and negative reasons to be rated k: U_k^+ and U_k^-.
2. We revise the positive and negative reasons for each object, and the reference profile based on the way they compare to each other. The possible updates lead to

(a) a set of objects U_k^{e-} not having initially negative reasons opposing rating k(not in U_k^-), but for which their negative reasons were enriched to oppose rating k.
(b) a set of objects U_k^{e+} not having initially positive reasons supporting rating k(not in U_k^+), but for which their positive reasons were enriched to support rating k.
(c) a set of objects U_k^{r-} for which negative reasons opposing rating worse than k are withdrawn to oppose a rating k.

We then compute the updated reference profiles Z_u^t and the updated set of objects to be rated A_u^t.

3. We compute $H_{u,k}^t$ and $L_{u,k}^t$, $\forall k \in [\![1 \ ; \ q]\!]$. All objects in $H_{u,k}^t \cap L_{u,k}^t$ will be assigned to Z_k^{t+1}. We distinguish two cases:
(a) Objects belonging to any among the sets $H_{u,1}^t \cap L_{u,1}^t, ..., H_{u,q}^t \cap L_{u,q}^t$. In other terms, objects having the same higher and lower rating. These objects are rated k.
(b) Objects have different higher and lower rating $(A_u^t \setminus \cup_k (H_{u,k}^t \cap L_{u,k}^t))$; we can consider them as interval rated. In such a case, we compute a distance between objects and reference profiles characterizing the possible categories and we choose the "nearest" one. This is done through the use of U_k^{2+}. The distance is computed (see definition 8 later on in subsection 4.2) first over objects in $H_{u,1}^t$, then over $H_{u,2}^t$,..., ending with objects in $H_{u,q}^t$. Each time an object is rated based on the distance, we assign it to the corresponding set in Z_u^{t+1} and we update positive reasons for objects in worse categories. This procedure is repeated until all objects are rated.

Example 4. Imagine the situation of two students x and y, such that, one the one hand, x might be either exceptional, or excellent, or good student. On the other hand, y might be either excellent or good student. In case there is a sufficient majority of criteria in favor of y, with respect to x, and x is close to former exceptional students, x will provide y by positive reasons to be assigned to the category of exceptional students. However, since the best possible rating for y is excellent student, then y will be rated as excellent student without computing his distance with former students in each category (thanks to x). We will note U_k^{2+} the set of objects close to a category k, for which there are neither positive reasons nor valid negative reasons.

Remark 2. *In case the strict preference relation is not considered in the assessment of negative reasons, these will not be enriched:* $\forall k \in [\![1 \ , \ q-1]\!]$: $U_k^{e-} = \emptyset$. *This is due to the transitivity of both the dominance relation and the non violation of minimum requirements (more details will be provided in section 6.5).*

The order of the assessment of the updated sets of positive and negative reasons is important. The following example illustrates the case.

Example 5. let's consider three new students x, y and s, such that: x, y and s have positive and no negative reasons to be considered as a good student, an excellent student and an average student respectively. Let's assume that according to a sufficient majority of criteria, the student x is at least as good as y. Hence, based on this information, positive reasons will be enriched in order to support rating x as an excellent student. However, in case the student s is strictly preferred (better) to y, and there are negative reasons against being considered as an excellent student, the student y cannot be considered anymore as excellent, this corresponds to the enrichment of negative reasons. As a result of enriching negative reasons of y, the enrichment of positive reasons of x is no more valid. For this reason, $\forall k \in [\![1\ ,\ q-1]\!] : U_k^{e-}$ should be computed before $\forall k \in [\![1\ ,\ q]\!] : U_k^{e+}$. Furthermore, withdrawing negative reasons takes into account the enriched negative reasons, and can be generated by the enriched positive reasons. Let's consider that the student s is considered as average because his performance is strictly worst, according to a sufficient majority of criteria, than the performance of a former average student z (a reference profile in the category of average students). In this case, the rating of z has no negative reasons against being in the category of excellent students, and his positive reasons were enriched based on his comparison with the new students. Thus, the rating of z will be improved leading to withdrawing negative reasons against s. This will have an impact upon all students who are preferred to z, but are rated not better than z.

For all $k \in [\![1\ ,\ q]\!]$, the sets H_k^t and L_k^t might give an idea about the quality of the rating, by drawing a distribution of the precision of the rating. We can also provide the decision maker by statistics such as the median and the mode of the rated objects among the categories, or the percentage of objects rated at this level: In other terms the cardinality of $H_k^t \cap L_k^t$ for all $k \in [\![1\ ,\ q]\!]$, might be a good indicator for the quality of the rating.

The reader should note that Dynamic-R is a whole rating process, rather a simple rating procedure. The flowchart of Dynamic-R, is displayed in Figure 1, representing the main operations in the rating procedure.

5 Basic concepts within Dynamic-R

Dynamic-R is a method based on defining and aggregating positive and negative reasons respectively supporting or opposing an outranking and positive and negative reasons supporting or opposing a rating. These reasons are based on some concepts used in different MCDA methods, and the new concept of minimum requirements. These concepts will be used at the basic level. In this section, we will present the way to define the set of minimum requirements and the basic tools used in order to assess positive and negative reasons for or against an outranking.

5.1 Methodology for assessing the minimum requirements

The minimum requirements represent the minimum performance, that can be taken by an object, under each criterion and regardless on its performance on the other criteria, in order to be admissible in a category. Hence, the profiles in the set of minimum requirements $\mathcal{B}$ have to be dominated by an object x to be admissible in a category: Let's consider $b_k = (\underline{b}_{j,k})_{j\in\mathcal{F}}$, characterizing the category $\mathcal{C}_k$, an object $x = (x_j)_{j\in\mathcal{F}}$ cannot be rated k if $\exists j \in \mathcal{F} : \underline{b}_{j,k} \succ_j x_j$. In this section, we will present a methodology to assess the minimum requirements characterizing each category.

Let's name x^* the ideal object: an object consisting on the best possible performance under each criterion $x^* = (x^*_j)_{j\in\mathcal{F}}$, where x^*_j is the best possible performance under the scale of the criterion j (the best if j needs to be maximized, and the lowest if j needs to be minimized). In this section, we will use the following notation: $x = (x_j, x_{-j})$ where x_j is the performance of x under the criterion j, and x_{-j} the performance of x under the criteria $\mathcal{F} \setminus \{j\}$.

In case the number of categories and criteria is not very important, the developed procedure consists on asking the decision maker the following questions: "What is the worst performance that can be taken by x_j to rate the vector (x_j, x^*_{-j}), at category k?". Asking this question m (m being the cardinal of $\mathcal{F}$) times leads to determining the minimum requirements to be rated k: $b_k = (x_1, ..., x_m)$. This procedure requires $(k-1)\times m$ questions, to assess the minimum requirements of all the categories (we assume that the worst category does not require a minimum requirement by its nature).

Example 6. Imagine a headhunter aiming at performing an ordinal classification of candidates (graduated students) for a client based on the following criteria:

1. The global mark: to be maximized, assessed on a cardinal scale $[0, 20]$, representing the general mark of the degree;
2. Assiduity: to be maximized, assessed on an ordinal scale $\{1, ..., 10\}$, 1: refers to a not serious student and 10: refers to a very serious student;
3. The physical aptitude: to be maximized, assessed on an ordinal scale $\{1, ..., 5\}$, such that 1: not able to move, 2: bad health, 3: average health, 4: good health, 5: very high aptitude.
4. The requested annual salary: to be minimized, assessed on a cardinal scale $[40k, 70k]$ euros.

The headhunter aims at partitioning the candidates into three categories: Good opportunities for the client; opportunities that need to be discussed with the client; bad candidates for the client.
In this example the ideal candidate is characterized by the following performance vector $x^* = (20, 10, 5, 40k)$. In order to assess the minimum requirements, the headhunter might submit to the client the following "ceteris

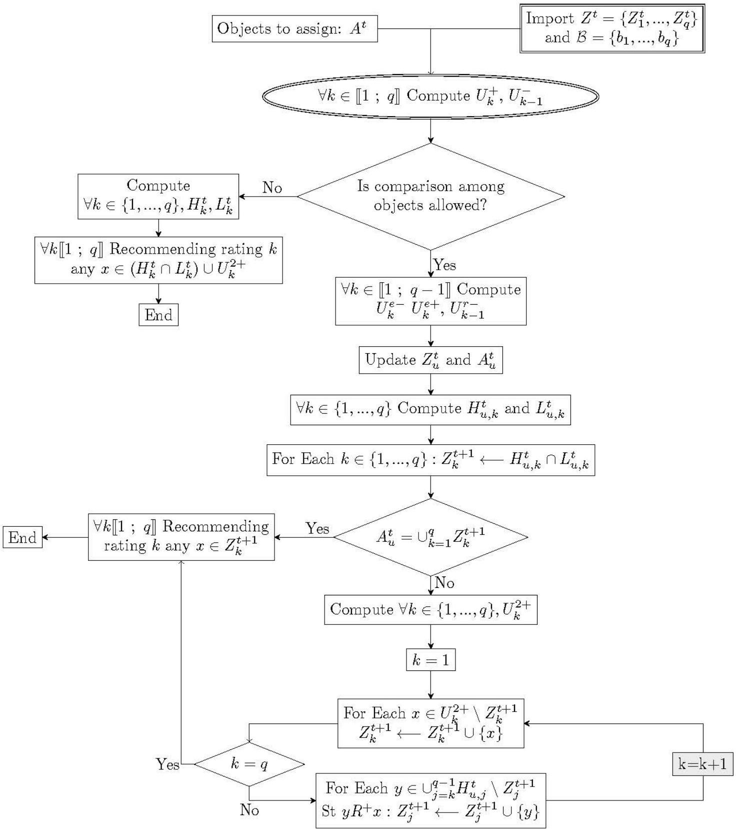

Figure 1: Dynamic-R rating algorithm.

paribus" questions:
- what is the minimum acceptable mark ($\underline{m}$) such that $x = (\underline{m}, 10, 5, 40k)$ is rated "good opportunity"?
- what is the minimum acceptable mark ($\underline{m}$) such that $x = (\underline{m}, 10, 5, 40k)$ is rated "opportunity to be discussed"?
- what is the minimum acceptable assiduity ($\underline{d}$) such that $x = (20, \underline{d}, 5, 40k)$ is rated "good opportunity"?
- what is the minimum acceptable assiduity ($\underline{d}$) such that $x = (20, \underline{d}, 5, 40k)$ is rated "opportunity to be discussed"?
- the same type of questions apply as far as a the minimum acceptable physical aptitude ($\underline{p}$) and the maximum acceptable wage ($\overline{w}$) are concerned.

Suppose the results to these questions provide the vector $(12, 7, 3, 50k)$ as minimum requirements for being a good candidate. This means that a candidate having less than 12 for Mark could never be considered good, the same reasoning applies for candidates with less than 7 for Assiduity, less than 3 for Physical aptitude, and more than 50k for wage.

5.2 Basic definitions

Definition 1. *(Positive Reasons for an outranking)*

Positive reasons for outranking relations are binary relations R^+ defined on $(\mathcal{A}^t)^2$ representing the capacity of a sufficient coalition of criteria, to influence the relative preference between two objects. This can be expressed as:

$$xR^+y \iff w(\{j \in \mathcal{F} : x \succeq_j y\}) \geq \lambda \tag{1}$$

where λ is the majority threshold

Remark 3. *Recall that $\mathcal{A}^t = \cup_k Z_k^t \cup A^t \cup \mathcal{B}$.*

Remark 4. *In case the measure associated with the decisive coalition of criteria is additive, the previous formulation would be:*

$$xR^+y \iff \sum_{j \in \mathcal{F}: x \succeq_j y} w_j \geq \lambda \tag{2}$$

Definition 2. *(Negative Reasons against an outranking)*

Negative reason against an outranking R^- is a binary relation defined on $(\mathcal{A}^t)^2$ displaying the capacity of a subset of criteria to reject a possible outranking in case its importance is greater than a veto v. This can be formulated by:

$$xR^-y \iff \mathcal{V}(\{j \in \mathcal{F} : y \succeq_j x\}) \geq v \tag{3}$$

Remark 5. *A negative reason in many outranking methods (Ostanello, 1985; Roy, 1991; Vincke, 1999), called discordance principle, is defined as the minimal difference v_j under each criterion $g_j \in \mathcal{F}$ not allowed to be compensated.*

Definition 3. *(Outranking relation)*

Outranking relation S_λ is a binary relation defined on $(\mathcal{A}^t)^2$. x outranks y can be interpreted as "x is at least as good as y". S_λ can be formulated as:

$$xS_\lambda y \iff xR^+y \wedge \neg(xR^-y) \tag{4}$$

Definition 4. *(Basic binary relations)*

Based on the Outranking relation, three possible binary relations might be defined: for $x, y \in \mathcal{A}^t$

- Strict Preference (P_λ): $xP_\lambda y \iff xS_\lambda y \wedge \neg(yS_\lambda x)$
- Indifference (I_λ): $xI_\lambda y \iff xS_\lambda y \wedge yS_\lambda x$
- Incomparability (J_λ): $xJ_\lambda y$ iff non of the previous binary relations hold.

Definition 5. *(Dominance relation)*

Dominance relation D is a binary relation defined on $(\mathcal{A}^t)^2$. For $x, y \in \mathcal{A}^t$, we say that xDy if x is at least as good as y under each criterion and strictly better than y under at least one criterion. This can be formulated by:

$$xDy \iff \exists i \in \mathcal{F}, \forall j \in \mathcal{F} : x \succeq_j y \wedge x \succ_i y \tag{5}$$

Remark 6. $xDy \implies xS_\lambda y$.

In this paper, many definitions involve binary relations between objects and the sets of reference profiles. We propose the following two definitions:

Definition 6. *(Binary relations used in positive and negative reasons)*
Consider the set A and a set of sets B. A binary relation $\mathcal{R} \subseteq A \times B$, such that $\forall (x, Y) \in A \times B : x\mathcal{R}Y$ should be read as "there are negative reasons opposing x to belong to Y", or "there are positive reasons for x belonging to Y".

Definition 7. *(Preference between $2^{A_u^t}$ and Z_u^t)*
Consider the power set 2^A and a set of sets B. A binary relation $\mathcal{R} \subseteq (2^A \times B) \cup (B \times 2^A)$, such that $\forall (X, Y) \in (2^A \times B) \cup (B \times 2^A) : X\mathcal{R}Y$ should be read as "The class X is at least as good as the class Y".

Remark 7. *In this work, we will consider only singletons in* $2^{A_u^t}$*.*

In assignment problems, the case where categories are not necessarily ordered, the assignment is based on a similarity index. This last can be seen as a distance between an object we aim to assign and a set of objects characterizing a class. We will adapt this idea to the context where the objects are described by ordinal information under at least one dimension.

Definition 8. *(Distance between an object and a set of characteristic profiles)*

Let Z_k^t be a set of reference profiles characterizing the category k at the step t. We define the distance of an object $x \in A^t$ from the set Z_k^t as:

$$dist(x, Z_k^t) = \min\left(\min_{z \in Z_k^t} \mid c(x,z) - c(z,x) \mid, \frac{1}{\mid Z_k^t \mid} \mid \sum_{z \in Z_k^t} c(x,z) - c(z,x) \mid \right) \tag{6}$$

where $c(x, y) = w(\{j \in \mathcal{F} : x \succeq_j y\})$.

This distance represents the relative position of an object within the attributes space with respect to a set of reference profiles. It computes the minimum between two values:

- on the one hand, the way the object compares to the closest reference profile;

- on the other hand, the way the object compares to all the reference profiles

The first component of the distance, $\min_{z \in Z_k^t} \mid c(x,z) - c(z,x) \mid$, represents the minimum of distances between "x" and each profile in Z_k^t. Intuitively, it can be seen as an answer to the question "is there any profile in Z_k^t close to x?". The second component of the distance, $\frac{1}{|Z_k^t|} \mid \sum_{z \in Z_k^t} c(x,z) - c(z,x) \mid$, represents the net flow evaluation: The difference between the total importance of criteria in favor of x compared to the profiles in Z_k^t and the total importance of criteria in favor of the reference profiles in Z_k^t compared to x. Figure 2 illustrates the defined distance. Objects $\circ$ are rated higher than objects $\bullet$ within the bi-attribute space f_1, f_2. The object $\star$ is rated as $\circ$ because the nearest object to $\star$ is a $\circ$. The object $*$ is rated as $\bullet$; the nearest objects are both a $\circ$ and a $\bullet$, but the set of $\bullet$ is globally nearest.

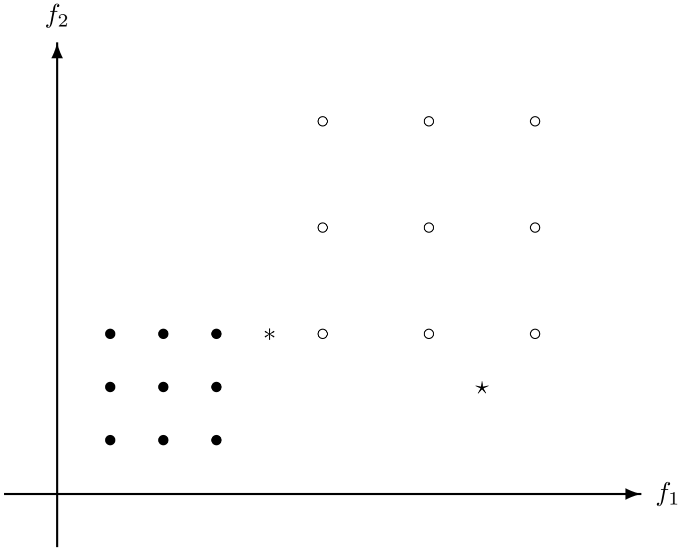

Figure 2: $\star$ is a $\circ$ and $*$ is a $\bullet$ because of their respective distances.

In case the set of minimum requirements is not empty, we define an incompatibility binary relation between categories and objects.

Definition 9. *(Incompatibility binary relation)*
Incompatibility binary relation $Incomp_{lower}$ defined on $A^t \times Z^t$, represents the non eligibility of an object to characterize a given category with respect to some minimum requirements. For $x \in A^t$, $Z_k^t \in Z^t$:

$$x Incomp_{lower} Z_k^t \iff \exists b_k \in \mathcal{B} : \neg(x D b_k) \tag{7}$$

This means that, at a given step $t \in T$, if for an object $x \in A^t, \exists j \in \mathcal{F} : x \prec_j \underline{b}_k$, the assignment of x to the category $\mathcal{C}_k$ should be "questioned", thus, x cannot be rated k. The incompatibility binary relation and the discordance index represent close concepts related to the respect of minority principle. It consists on the existence of strong reasons to not approve a preference relation, between two objects, even in the presence of a sufficient majority of concordant criteria. However, these two concepts are different: the discordance index characterizes the outranking between two objects, while the incompatibility binary relation characterizes the impossibility of an object to belong to a category.

6 Positive and negative reasons in Dynamic-R

As already mentioned, in this paper we generalise the concept of concordance and discordance through the use of positive and negative reasons (for a rating; see Tsoukiàs et al, 2002). In this section we define what negative reasons against a rating and positive reasons supporting a rating are, we define the "convincing property" to satisfy, we show how to construct the sets U_l^t and U_h^- without consistency checking (comparing objects to rates to profiles, but not among them) and then how to update positive and negative reasons once we allow to compare rated objects among them.

6.1 Negative reasons against a rating

Negative reasons represent information or premises against a rating. In our approach, negative reasons represent, either the "inconsistency" of a rating, or the incompatibility of an object with a category.

The "inconsistency" should be considered as a situation where an object being potentially rated k is either weakly dominated or strictly preferred by a reference profile of rate $k+1$ which is worse. Incompatibility should be understood as the situation where an object being potentially rated k fails to meet one of the minimal requirements of category $\mathcal{C}_k$.

In order to assess the negative reasons against a rating, the assignment of objects to a given category will depend on the relative position of the non assigned objects with reference profiles using the dominance and either the strict preference relations or the absence of the incompatibility of objects with the categories.

Definition 10. *(U_k^-, For $k \in [\![1\ ;\ q-1]\!]$)*

The set of objects having negative reasons against being assigned to a given category k, U_k^- can be formulated as

$$U_k^- = \{x \in A^t \cup Z_{1,q}^t, xR^- Z_k^t\}, \forall k \in [\![1\ ;\ q-1]\!] \quad (8)$$

where R^- is a binary relation defined on $(A^t \cup Z^t_{1,q}) \times Z^t$. $xR^-Z^t_k$ should be read as: "there are negative reasons against rating x, k": For $x \in A^t \cup Z^t_{1,q}$, $Z^t_k \in Z^t$:

- Case using the incompatibility and the strict preference relation:

$$xR^-Z^t_k \iff \exists h \in [\![k+1\ ;\ q]\!], \exists z \in Z^t_h : zP_\lambda x \vee xIncomp_{lower}Z^t_k. \tag{9}$$

- Case using the incompatibility and the dominance:

$$xR^-Z^t_k \iff \exists h \in [\![k+1\ ;\ q]\!], \exists z \in Z^t_h : zDx \vee xIncomp_{lower}Z^t_k. \tag{10}$$

Remark 8. *Objects* $\underline{b}_1, ..., \underline{b}_q$ *do not necessarily belong respectively to the categories* $\mathcal{C}_1, ..., \mathcal{C}_q$*.*

If Definition 10 holds then:

Proposition 1. *(Monotonicity of negative reasons)*

1. If there exist negative reasons against assigning an object to a given category then there exist negative reasons against assigning it to any better category:

$$\forall x \in A^t \cup Z^t_{1,q}, \forall Z^t_h \in Z^t :\ xR^-Z^t_h \implies \forall k \in [\![1\ ;\ h]\!] : xR^-Z^t_k; \tag{11}$$

2. If there are no negative reasons to assign an object to a given category then there are no negative reasons to assign it to any worse category:

$$\forall x \in A^t \cup Z^t_{1,q}, \forall Z^t_h \in Z^t :\ \neg(xR^-Z^t_h) \implies \forall k \in [\![h\ ;\ q]\!] : \neg(xR^-Z^t_k). \tag{12}$$

Proof. (properties of negative reason). Obvious, by construction of R^- in definition 10.

■

Corollary 1. The monotonicity of negative reasons can also be formulated as:

$$\forall k \in [\![2\ ;\ q-1]\!] : U^-_k \subseteq U^-_{k-1} \tag{13}$$

Proof. Direct consequence of Proposition 1. ■

Negative reasons prevent a rating that can be challenged. To confirm a rating, we need to verify the existence of reasons supporting an assignment to categories for which no negative reasons are involved. In the next subsection, we will define and discuss the forms of the reasons supporting a rating, called positive reasons.

6.2 Positive reasons supporting a rating

Positive reasons represent information or premises supporting a rating. These reasons are built with respect to the "positive consistency" of the rating, that could be understood as the situation where an object can be rated k because it is at least as good as at least one reference profile belonging to $\mathcal{C}_k$.

Definition 11. *(U_k^+, For $k \in [\![1\ ;\ q]\!]$)*
The set of objects having positive reasons supporting the assignment to a given category k, named U_k^+, can be formulated as:

$$\forall k \in [\![1\ ;\ q]\!] : U_k^+ = \{x \in A^t \cup Z_{1,q}^t, xR^+Z_k^t\} \tag{14}$$

where R^+ is a binary relation defined on $(A^t \cup Z_{1,q}^t) \times Z^t$ representing the possibility to be at least as good as reference profiles characterizing a category. R^+ can be formulated as: For $x \in A^t \cup Z_{1,q}^t$, $Z_k^t \in Z^t$:

$$xR^+Z_k^t \iff \exists h \leq k, \exists z \in Z_h^t : xS_\lambda z. \tag{15}$$

Proposition 2. *(Monotonicity of positive reasons)*

1. If there exist positive reasons supporting the assignment of an object to a given category then there exist positive reasons supporting its assignment to any worse category:

$$\forall x \in A^t \cup Z_{1,q}^t, \forall Z_h^t \in Z^t :\ xR^+Z_h^t \implies \forall k \in [\![h\ ;\ q]\!] : xR^+Z_k^t; \tag{16}$$

2. If there are no positive reasons to assign an object to a given category then there are no positive reasons to assign it to any better category:

$$\forall x \in A^t \cup Z_{1,q}^t, \forall Z_h^t \in Z^t :\ \neg(xR^+Z_h^t) \implies \forall k \in [\![1\ ;\ h]\!] : \neg(xR^+Z_k^t). \tag{17}$$

Proof. Obvious, by construction of R^+ in Definition 11. ∎

Corollary 2. The monotonicity of positive reasons can also be formulated as:

$$\forall k \in [\![2\ ;\ q]\!] : U_{k-1}^+ \subseteq U_k^+ \tag{18}$$

Proof. Direct consequence of Proposition 2. ∎

6.3 Convincing Property

The aim of this paper is to provide a "convincing" rating. Hence, at all steps in T of the process, the set of reference profiles should respect the following "convincing" condition:

Definition 12. *"Convincing" property*

$$\forall y \in Z^t_k, \nexists z \in Z^t_h(k > h) : yS_\lambda z \wedge y \notin U^-_h \tag{19}$$

This property being central for our reasoning we try to present it extensively. A rating satisfies the "convincing property" if
For all y rated k at time step t ($\forall y \in Z^t_k$)
there is no z in profiles rated better than k ($\neg\exists z \in Z^t_h$ $(k > h)$; remember the lower the rate the better)
such that y is at least as good as z ($yS_\lambda z$)
and y is not among the objects having negative reasons for being rated h or better($y \notin U^-_h$).
In other terms, we guarantee that if y is rated k there are sufficient reasons for doing so.
In the next two subsections, we will present the aggregation procedure of positive and negative reasons first without considering the way objects compare to each other (without a consistency checking) and then checking whether there are reasons for updating the rating because of inconsistent pairwise comparisons among rated objects.

6.4 Aggregating of U^+_l, U^-_h, without consistency checking

Let's assume that the decision maker only needs a rating without any consistency checking. We need to aggregate the sets U^+_l and U^-_h, for all $l \in [\![1 \ ; \ q]\!]$ $h \in [\![1 \ ; \ q-1]\!]$.

Rating an object comes to assigning it to the best possible category, for which there are no negative reasons. Thus, the aggregation is made in a "lexicographical" way: We first verify the absence of negative reasons, then the existence of positive ones. Under this principle, we will assess two partitions of A^t: H^t_h, for all $h \in [\![1 \ ; \ q]\!]$, representing the objects for which the best possible rating is h; and L^t_l, for all $l \in [\![1 \ ; \ q]\!]$, representing the objects for which the worst possible rating is l. These assessments are based only on the way objects compare to reference profiles.

Definition 13. *(H^t_h and L^t_l, for $h, l \in [\![1 \ ; \ q]\!]$)*

For a given $t \in T$, the partitions of A^t, H^t_h and L^t_l, for which the best and the worst possible ratings are respectively $h, l \in [\![1 \ ; \ q]\!]$, can be formulated as:

$$H^t_h = U^-_{h-1} \setminus U^-_h \tag{20}$$

$$L^t_l = U^+_l \setminus \big(U^-_l \cup (U^+_{l-1} \setminus U^-_{l-1})\big) \tag{21}$$

Proposition 3. *(properties of H^t_h and L^t_l)*

For a given $t \in T$, the sets $H^t_1, ..., H^t_q$ and $L^t_1, ..., L^t_q$, are two partitions of A^t.

Proof. For a given $t \in T$, let's prove that

1. $H_1^t, ..., H_q^t$ is a partition of A^t:
   For all $h, k \in [\![1\ ;\ q]\!]$, we have $H_h^t \cap H_k^t = \emptyset$, $(h < k)$, since: Using the Definition 13, we have $H_h^t \cap H_k^t = (U_{h-1}^- \setminus U_h^-) \cap (U_{k-1}^- \setminus U_k^-)$. Due to the monotonicity of negative reasons (see Corollary 1), $U_{k-1}^- \subseteq U_h^-$. Hence:

$$H_h^t \cap H_k^t = \emptyset \tag{22}$$

   It is also easy to check that

$$\cup_h\, H_h^t = A^t \tag{23}$$

   since $U_0^- = A^t \setminus U_1^-$ and $U_q^- = \emptyset$. Hence: $\cup_{h=1}^q H_h^t = A^t \setminus U_q^- = A^t$. ‘
2. $L_1^t, ..., L_q^t$ is a partition of A^t:
   For all $l, k \in [\![1\ ;\ q]\!]$, we have $L_l^t \cap L_k^t = \emptyset$, $(k < l)$, since:

$$\begin{aligned} L_l^t \cap L_k^t &= \left[U_l^+ \setminus \left(U_l^- \cup (U_{l-1}^+ \setminus U_{l-1}^-)\right)\right] \cap \left[U_k^+ \setminus \left(U_k^- \cup (U_{k-1}^+ \setminus U_{k-1}^-)\right)\right] \\ &= \left((U_l^+ \setminus U_l^-) \setminus (U_{l-1}^+ \setminus U_{l-1}^-)\right) \cap \left((U_k^+ \setminus U_k^-) \setminus (U_{k-1}^+ \setminus U_{k-1}^-)\right) \end{aligned}$$

   Based on Corollary 1: $U_{l-1}^- \subseteq U_k^-$. And based on Corollary 2: $U_k^+ \subseteq U_{l-1}^+$.
   Hence: $U_k^+ \setminus U_k^- \subseteq U_{l-1}^+ \setminus U_{l-1}^-$.
   Thus:

$$L_l^t \cap L_k^t = \emptyset \tag{24}$$

   It is easy to check that

$$\cup_l\, L_l^t = A^t \tag{25}$$

   Since:

$$\begin{aligned} \cup_l L_l^t =& \cup_l\, (U_l^+ \setminus U_l^-) \setminus (U_{l-1}^+ \setminus U_{l-1}^-) \\ =& (U_q^+ \setminus U_q^-) \setminus (U_,^+ 0 \setminus U_0^-) \end{aligned}$$

   $U_0^+ \setminus U_0^- = \emptyset$, $U_q^+ = A^t$ and $U_q^- = \emptyset$. Thus, $\cup_l L_l^t = A^t$.

■

Having computed the two series of sets H_h^t and L_l^t we can identify the objects for which the best possible rating and the worst possible rating is the same $(H_k^t \cap L_k^t,\ \ \forall k \in [\![1\ ;\ q]\!])$. It is clear that after performing this step there will exist objects for which the best possible rating and the worst possible rating do not coincide. We have two options here:

- either present an "interval rating" (x is rated between l and h);
- or try to reduce this imprecision by computing the "distance" of x with respect to all such possible categories, as defined in Definition 8, and choosing the rate $k = \arg\min_j dist(x, Z_j^t)$.

6.5 Updating positive and negative reasons; consistency checking

For $t \in T$, the assessment of U_h^- and U_l^+, for $l \in [\![1\ ;\ q]\!]$, $h \in [\![1\ ;\ q-1]\!]$ is based on the sets of reference profiles. These last were updated in the previous step. In this section, we will discuss and analyse two major features: the way objects in A^t might modify the sets of reference profiles (for example by enriching their positive reasons); and the way objects in A^t might change positive and negative reasons supporting or against a given rating. Analysing preferential information originated by A^t leads to three possible treatments: enriching negative reasons (only when strict preference is considered in the assessment of negative reasons), enriching positive reasons and withdrawing negative reasons. In the following we will enhance our notation: U_k^{e-} will stand for enriched negative reasons, U_k^{r-} will stand for withdrawn negative reasons, and U_k^{e+} will stand for enriched positive reasons.

Definition 14. *(U_k^{e-}, For $k \in [\![h\ ;\ q-1]\!]$; enriching negative reasons)*
For a given $k \in [\![1\ ;\ q-1]\!]$, the set of objects, U_k^{e-}, for which negative reasons were enriched to prevent a rating $k \in [\![1\ ;\ q-1]\!]$, can be formulated as

$$U_k^{e-} = \{x \in Z_{1,q}^t \cup A^t : (xR^{e-}Z_k^t) \wedge \neg(xR^{e-}Z_{k+1}^t)\} \tag{26}$$

where R^{e-} is binary relation representing enriched negative reasons against a rating. R^{e-} can be formulated as:
For $x \in U_h^- \setminus U_{h+1}^-$:

$$xR^{e-}Z_k^t \iff \exists y \in \cup_{j=k}^q U_j^{e-} \cup U_k^- : yP_\lambda x \wedge \neg(yIncomp_{lower}Z_{h+1}^t). \tag{27}$$

Definition 14, represents the assessment of the sets of objects or reference profiles for which the negative reasons were enriched to prevent a rating to a worse category. The intuitive idea of this definition is simple: if there is an object y (in A^t or in Z_k^t) strictly preferred to or dominating x (the object to rate), the category to which y is rated (let's say k) becomes an upper bound for the category to which x can be rated. However, this upper bound will not hold in case y is rated k because incompatible with better than k categories. In such cases the category of x will be bounded by the best category with which y is not incompatible.

The following proposition presents a characteristic of the binary relation used in the assessments of $U_1^{e-}, ..., U_{q-1}^{e-}$.

Proposition 4. *(properties of R^{e-})*
For $t \in T$, for $x \in U_h^- \setminus U_{h+1}^-$, if there are enriched negative reasons opposing x being rated k then there are enriched negative reasons opposing any rating between k and h. This can be formulated as:

$$xR^{e-}Z_k^t \implies \forall j \in [\![h\ ;\ k]\!] : xR^{e-}Z_j^t \tag{28}$$

Proof.
Obvious since, $\forall j \in [\![h\ ;\ k]\!]$ we have $\cup_{h=k}^{q} U_h^{e-} \cup U_k^- \subseteq \cup_{h=j}^{q} U_h^{e-} \cup U_j^-$.
($\cup_{h=k}^{q} U_h^{e-} \subseteq \cup_{h=j}^{q} U_h^{e-}$ and $z \in U_k^- \implies z \in U_j^-$, see equation 2) ∎

Remark 9. *Under the hypothesis that at a given step $t \in T$, reference profiles are "convincing", the enrichment of positive reasons or the withdrawn of negative reasons cannot lead to the enrichment of negative reasons related to any object or reference profile. This is justified by the fact that improving the assignment of an object to a given category $\mathcal{C}_k$ cannot influence negatively the assignment of any object in a better category (see proposition 2). For this reason, enriching negative reasons will be processed first.*

Definition 15. *(U_k^{e+}, For $k \in [\![1\ ;\ q-1]\!]$; enriching positive reasons).*
For a given $k \in [\![1\ ;\ q-1]\!]$, the set of objects, U_k^{e+}, for which positive reasons were enriched to support a rating k, can be formulated as:

$$U_k^{e+} = \{x \in Z_{1,q}^t \cup A^t : xR^{e+}Z_k^t \wedge \neg(xR^{e+}Z_{k-1}^t)\} \tag{29}$$

where R^{e+} is a binary relation representing enriched positive reasons supporting a rating. R^{e+} can be formulated as:
For $x \in U_l^+ \setminus U_{l-1}^+$, $k < l$:

$$xR^{e+}Z_k^t \iff \exists y \in (\cup_{j=1}^{k} U_j^{e+} \cup U_k^+) \setminus (\cup_{j=k}^{q-1} U_j^{e-} \cup U_k^-) : xS_\lambda y \tag{30}$$

Definition 15, represents the assessment of the sets of objects for which positive reasons were enriched to support the assignment to a better category. Enriching positive reasons for a given x is mainly due to the presence of $y \in (\cup_{j=1}^{k} U_j^{e+} \cup U_k^+) \setminus (\cup_{j=k}^{q-1} U_j^{e-} \cup U_k^-)$, having positive and no negative reasons to be assigned to a category better than x, such that xR^+y. Hence, y will provide x by new positive reasons that will potentially improve its possible rating. In other terms if there is an object y rated better than x (let's say k) and x is at least as good as y then we can improve the rating of x at most at the k level.

The following proposition presents a characteristic of the binary relation used in the assessments of $U_1^{e+}, ..., U_q^{e+}$.

Proposition 5. *(properties of R^{e+})* For $t \in T$, for $x \in U_l^+ \setminus U_{l-1}^+$, for a given k better than l ($k < l$), if there are enriched positive reasons supporting x being rated k then there are enriched positive reasons supporting any rating between l and k. This can be formulated as:

$$xR^{e+}Z_k^t \implies \forall j \in [\![k\ ;\ l-1]\!] : xR^{e+}Z_j^t \tag{31}$$

Proof.
Obvious since $\forall j \in [\![k\ ;\ t-1]\!]$ we have

$(\cup_{h=1}^{k} U_h^{e+} \cup U_k^{+}) \setminus (\cup_{h=k}^{q-1} U_h^{e-} \cup U_k^{-}) \subseteq (\cup_{h=1}^{j} U_h^{e+} \cup U_j^{+}) \setminus (\cup_{h=j}^{q-1} U_h^{e-} \cup U_j^{-})$, since:

$(\cup_{h=1}^{k} U_h^{e+} \cup U_k^{+}) \subseteq (\cup_{h=1}^{j} U_h^{e+} \cup U_j^{+})$ and $(\cup_{h=j}^{q-1} U_h^{e-} \cup U_j^{-}) \subseteq (\cup_{h=k}^{q-1} U_h^{e-} \cup U_k^{-})$. ■

Definition 16. *(U_k^{r-}, For $k \in [\![1 \; ; \; q-1]\!]$)*

For a given $k \in [\![1 \; ; \; q-1]\!]$, the set of objects, U_k^{r-}, for which negative reasons are withdrawn to prevent a better rating k, can be formulated as:

$$U_j^{r-} = \{x \in Z_{1,q}^t \cup A^t : xR^{r-} Z_j^t \wedge \neg(xR^{r-} Z_{j+1}^t)\} \tag{32}$$

where R^{r-} is binary relation representing withdrawn negative reasons against a rating. R^{r-} can be formulated as:

For $x \in U_h^{e-} \cup (U_h^{-} \setminus U_{h+1}^{-})$, $k < h$:

- case of negative reasons with strict preference

$$xR^{r-} Z_k^t \iff \begin{cases} \forall z \in (U_h^{-} \cup U_h^{e-}) \setminus \cup_{j=1}^{h-1} U_j^{r-} & : \neg(zP_\lambda x \vee zDx) \\ \text{And} & \\ \exists y \in \left[\cup_{j=1}^{h} U_j^{e+} \cup U_h^{+}\right] \cap U_{h,k}^{-} & : yP_\lambda x \vee xIncomp_{lower} Z_k^t \end{cases} \tag{33}$$

- case of negative reasons without strict preference

$$xR^{r-} Z_k^t \iff \begin{cases} \forall z \in (U_h^{-}) \setminus \cup_{j=1}^{h-1} U_j^{r-} & : \neg(zDx) \\ \text{And} & \\ \exists y \in \left[\cup_{j=1}^{h} U_j^{e+} \cup U_h^{+}\right] \cap U_{h,k}^{-} & : yDx \vee xIncomp_{lower} Z_k^t \end{cases} \tag{34}$$

with $U_{h,k}^{-} = \cup_{j=k}^{h-1} (U_j^{e-} \cup U_j^{r-}) \cup U_k^{-} \setminus \left(\cup_{j=h}^{q-1} U_j^{e-} \cup (\cup_{j=1}^{k-1} U_j^{r-}) \cup U_h^{-}\right)$ representing objects with valide negative reasons against ratings between $h-1$ and k.

Definition 16, represents the assessment of the sets of objects for which negative reasons were withdrawn to prevent a rating to a better category. The binary relation R^{r-} associated to these sets, and characterizing the operation of withdrawing negative reasons for a given x from a worse rating l, to a better rating h, are defined by two conditions:

1. Eligibility for withdrawing negative reasons for an object x: The existence of another object or reference profile y, having valid negative reasons against being rated $l-1$, and either strictly preferred to x or dominating x (in case strict preference is not considered), will invalidate the ability of x to improve its position (x will still have valid negative reasons against being rated l).

2. New negative reasons against a rating k: the improvement of the rating of x will be at most limited by the improvement of the object or reference profile, let's name it y, at the origin of x's negative reasons. The limitation might also come from an other element strictly preferred or dominating x, limiting its improvement to at most $k+1$ (since the withdrawn negative reasons will oppose being rated k). It is also possible that the withdrawn of x's negative reasons will not be limited by any object or reference profile, but by its own performance not dominating the minimum requirement b_k.

In other terms, if x is bounded to a category k because there is an object y strictly preferred or dominating x rated at most k, then if for some reason y improves its rating this will also improve the rating of x (withdrawing the reasons for which x was bounded to k). Such "improvements" can be limited because of the incompatibility relation (in case it holds) for x and y.

Proposition 6. *(properties of R^{r-})*
For $t \in T$, $\forall x \in U_h^{e-} \cup (U_h^- \setminus U_{h+1}^-)$, if there are withdrawn negative reasons opposing x being rated k then there are withdrawn negative reasons opposing any rating better than k. This can be formulated as:

$$xR^{r-}Z_k^t \implies \forall j \in [\![1 \ ; \ k]\!] : xR^{r-}Z_j^t \tag{35}$$

Proof.
Suppose that for a given object $x \in U_h^{e-} \cup (U_h^- \setminus U_{h+1}^-)$ there exists a $k \in [\![1 \ ; \ h-1]\!]$ such that $xR^{r-}Z_k^t$:
$\forall j \in [\![1 \ ; \ k]\!]$, we have: $U_{h,k}^- \subseteq U_{h,j}^-$ since $U_k^- \subseteq U_j^-$; $\cup_{l=k}^{h-1}(U_l^{e-} \cup U_l^{r-}) \subseteq \cup_{l=j}^{h-1}(U_l^{e-} \cup U_l^{r-})$; and $\cup_{l=1}^{j-1} U_l^{r-} \subseteq \cup_{l=1}^{k-1} U_l^{r-}$.
Hence $\exists y \in \left[\cup_{l=1}^{h} U_l^{e+} \cup U_h^+\right] \cap U_{h,k}^- \subseteq \left[\cup_{l=1}^{h} U_l^{e+} \cup U_h^+\right] \cap U_{h,j}^-$ such that $yP_\lambda x \vee yDx \vee xIncomp_{lower} Z_k^t$
Thus, $xR^{r-}Z_j^t$. ■

Remark 10. *The updates of negative and positive reasons leads to a change of some reference profiles, either to a better or to a worse category. In case the strict preference relation is not used in the assessment of negative reasons, positions of reference profiles cannot change to a worse category. The updated sets of reference profiles can be formulated as:*

$$Z_{uk}^t = Z_k^t \setminus \left(\cup_{j=1}^{k-1} U_j^{e+} \cap (\cup_{j=1}^{k-2} U_j^{r-})\right) \cup \\ \left(\cup_{j=1}^{k-1} U_j^{e+} \setminus U_{k-1}^-\right) \cup \left(\cup_{j=k}^{q-1} U_j^{e-} \setminus (\cup_{j=1}^{k-1} U_j^{r-})\right) \tag{36}$$

Since the sets of reference profiles are updated, we have more objects to rate. Thus, we note A_u^t the new set of objects that need to be rated at the *current step* $t \in T$. A^t_u *can be formulated as:*

$$A^t_u = A^t \cup \left(Z^t_{1,q} \setminus (\cup_{j=1}^{q-1} Z^t_{uj}) \right) \tag{37}$$

7 Recommendation construction

Once the procedure of consistency checking terminated and the positive and negative reasons updated we have the sets U^+_k, U^{e+}_k, U^-_k, U^{e-}_k, and U^{r-}_k, for all k. These are the basis for constructing a "challenge-free" rating of the set A^t_u (see equation 37). The reader will recall that we already defined H^t_k (L^t_k) as the sets of objects for which the best (worst) rating is k before consistency checking (using the set A^t). Updating the set to rate we need to construct the new partition $H^t_{u,1}, ..., H^t_{u,q}$, and $L^t_{u,1}, ..., L^t_{u,q}$.

We proceed through four steps. We first define the binary relation allowing to construct the partition. Then we show that this binary relation satisfies the expected monotonicity properties. We then define the sets H^t_{uj} and L^t_{uj} and we show that these are exactly the partition we are looking for. At the end of the section, in order to refine the obtained partition we introduce the set U^{2+}_k exploiting the notion of distance already introduced in Section 4.

Definition 17. $\succcurlyeq^t$ *is a weak order built on* $(2^{A^t_u} \times Z^t_u) \cup (Z^t_u \times 2^{A^t_u})$, *representing the preference between a subset of* A^t_u *and sets in* Z^t_u. $\forall t \in T, \succcurlyeq^t$, *defined as follows:*

1. *On* $Z^t_u \times 2^{A^t_u}$: $\forall x \in A^t_u, \exists k \in [\![1 \ ; \ q]\!]$ *such that:*

$$Z^t_{u,k} \succcurlyeq^t \{x\} \iff \begin{cases} x \in \left(\cup_{j=k-1}^{q-1} U^{e-}_j \cup U^-_{k-1} \right) \setminus \cup_{j=1}^{k-2} U^{r-}_j & k > 2; \\ x \in \cup_{j=k-1}^{q-1} U^{e-}_j \cup U^-_{k-1} & k \leq 2; \end{cases} \tag{38}$$

2. *On* $2^{A^t_u} \times Z^t_u$: $\forall x \in A^t_u, \exists k \in [\![1 \ ; \ q]\!]$ *such that:*

$$\{x\} \succcurlyeq^t Z^t_{u,k} \iff \begin{cases} \neg(Z^t_{u,k+1} \succcurlyeq^t \{x\}) \wedge (x \in \cup_{j=1}^{k} U^{e+}_j \cup U^+_k) & k \neq q \\ x \in \cup_{j=1}^{q} U^{e+}_j \cup U^+_q \end{cases} \tag{39}$$

The binary relation $\succcurlyeq^t$, (to be used for the assessment of $H^t_{u,h}$ and $L^t_{u,l}$ for all $h, l \in [\![1 \ ; q]\!]$), is characterized by the following proposition:

Proposition 7. *(properties of* $\succcurlyeq^t$*)* For a given $t \in T$, $x \in A^t_u$, $Z^t_{u,k} \in Z^t_u$, we have the following properties:

1. If a set of reference profiles characterizing a rating k is at least as good as x than any set of reference profiles characterizing a better rating is at least as good as x. This can be formulated as:

$$Z_{u,k}^t \succsim^t \{x\} \implies \forall s \in [\![1\ ;\ k]\!], Z_{u,s}^t \succsim^t \{x\} \tag{40}$$

2. If x is as good as a set of reference profiles characterising the rating k then x is at least as good as any reference profile characterising a worse rating (from k to q).

$$\{x\} \succsim^t Z_{u,k}^t \implies \forall s \in [\![k\ ;\ q]\!], \{x\} \succsim^t Z_{u,s}^t \tag{41}$$

Proof. For a given step $t \in T$, and $x \in A_u^t$,

1. Let us assume that $\exists Z_{u,k}^t \in Z_u^t$ such that $Z_{u,k}^t \succsim^t \{x\}$. We aim to prove that $\forall s \in [\![1\ ;\ k]\!], Z_{u,s}^t \succsim^t \{x\}$.
   Since for $s \in [\![1\ ;\ k]\!]$, we have: $\cup_{j=k-1}^{q-1} U_j^{e-} \subseteq \cup_{j=s-1}^{q-1} U_j^{e-}$; $U_{k-1}^- \subseteq U_{s-1}^-$; $\cup_{j=1}^{s-1} U_j^{r-} \subseteq \cup_{j=1}^{k-1} U_j^{r-}$, then $x \in (\cup_{j=s-1}^{q-1} U_j^{e-} \cup U_{s-1}^-) \setminus (\cup_{j=1}^{s-1} U_j^{r-})$. Hence, $Z_{u,s}^t \succsim^t \{x\}$.
2. Let us assume that $\exists Z_{u,k}^t \in Z_u^t$ such that $\{x\} \succsim^t Z_{u,k}^t$. We aim to prove that $\forall s \in [\![k\ ;\ q]\!], \{x\} \succsim^t Z_s^t$.
   Since for $s \in [\![k\ ;\ q]\!]$, we have: $\neg(Z_{u,k+1}^t \succsim^t \{x\}) \implies \neg(Z_{u,s+1}^t \succsim^t \{x\})$ (justified by 40); Also $\cup_{j=1}^{k} U_j^{e+} \subseteq \cup_{j=1}^{s} U_j^{e+}$; $U_k^+ \subseteq U_s^+$ (see proposition 2 "1."). Hence, $\{x\} \succsim^t Z_{u,k}^t \implies \neg(Z_{u,s+1}^t \succsim^t \{x\}) \wedge (x \in \cup_{j=1}^{s} U_j^{e+} \cup U_s^+)$. Thus, $\{x\} \succsim^t Z_{u,s}^t$. ■

We are now able to introduce the definition of the rating partition of the set A_u^t.

Definition 18. *($H_{u,h}^t$ and $L_{u,l}^t$, for $h, l \in [\![1\ ; q]\!]$)*
For a given $t \in T$, the partitions of A_u^t, $H_{u,h}^t$ and $L_{u,l}^t$, for which the best and the worse possible ratings are respectively $h, l \in [\![1\ ; q]\!]$, can be formulated as:

$$\begin{cases} H_{u,h}^t = \{x \in A_u^t, Z_{u,h}^t \succsim^t \{x\} \wedge \neg(Z_{u,h+1}^t \succsim^t \{x\})\} & h \neq q \\ H_{u,q}^t = \{x \in A_u^t, Z_{u,q}^t \succsim^t \{x\}\} \end{cases} \tag{42}$$

$$\begin{cases} L_{u,l}^t = \{x \in A_u^t, \{x\} \succsim^t Z_{u,l}^t \wedge \neg(\{x\} \succsim^t Z_{u,l-1}^t)\} & l \neq 1 \\ L_{u,1}^t = \{x \in A_u^t, \{x\} \succsim^t Z_{u,1}^t\} \end{cases} \tag{43}$$

Definition 18, represents the assessment of the two partitions of A_u^t: $H_{u,h}^t$ and $L_{u,l}^t$ for all $h, l \in [\![1\ ; q]\!]$. A set $H_{u,h}^t$ contains objects for which the best possible rating is h. In other terms the best category for which x has no valid negative reasons is h.

Valid negative reasons preventing being rated $h-1$ or better are formulated as:

$$\left(\cup_{j=h-1}^{q-1} U_j^{e-} \cup U_{h-1}^-\right) \setminus \cup_{j=1}^{h-2} U_j^{r-}$$

. To detail this formula, negative reasons against a rating $h-1$ or better contains:

- the negative reasons against being rated $h-1$ or better unless they were withdrawn to a better category $U^{-}_{h-1} \setminus \cup_{j=1}^{h-2} U_j^{r-}$.
- the enriched negative reasons to a worse category than $h-1$ they were withdrawn to a better category $\cup_{j=h-1}^{q-1} U_j^{e-} \setminus \cup_{j=1}^{h-2} U_j^{r-}$

$L^t_{u,l}$ contains objects for which the worst possible rating is l. The worst possible rating for an object x is the best category for which x has no valid negative reasons and valid positive reasons for being rated l. The absence of valid negative reasons are represented by $\neg(Z^t_{u,k+1} \succcurlyeq^t \{x\})$. Valid positive reasons are presented by $\cup_{j=1}^{k} U_j^{e+} \cup U_k^{+}$. At this point we only need to show that the above defined sets form indeed a partition of the set A^t_u.

Proposition 8. For a given $t \in T$, the sets $H^t_{u,1}, ..., H^t_{u,q}$ and $L^t_{u,1}, ..., L^t_{u,q}$, are two partitions of A^t_u.

Proof. At a given step $t \in T$:

1. Let's prove that $H^t_{u,1}, ..., H^t_{u,q}$ is a partition of A^t_u.
   Since, based on Proposition 7, for all $h, j \in [\![1\ ; q]\!]$, $h < j$, $Z^t_{u,j} \succcurlyeq^t \{x\} \implies Z^t_{u,h+1} \succcurlyeq^t \{x\}$, we have:
$$H^t_{u,h} \cap H^t_{u,j} = \emptyset \tag{44}$$
   For all $x \in A^t_u$, we have $Z^t_{u,1} \succcurlyeq^t \{x\}$ (even in case x has no valid negative reasons $x \in U^-_0 = A^t \setminus U^-_1$) Hence:
$$\begin{aligned} x \in A^t_u \implies & Z^t_{u,1} \succcurlyeq^t \{x\} \\ \implies & \left(Z^t_{u,1} \succcurlyeq^t \{x\} \wedge \neg(Z^t_{u,2} \succcurlyeq^t \{x\})\right) \vee ... \\ & \vee \left(Z^t_{u,q-1} \succcurlyeq^t \{x\} \wedge \neg(Z^t_{u,q} \succcurlyeq^t \{x\})\right) \vee Z^t_{u,q} \succcurlyeq^t \{x\} \\ \implies & \cup_{j=1}^{q} H^t_{u,j} \end{aligned}$$
   Also since for all $h \in [\![1\ ;\ q]\!]$: $H^t_{u,h} \subseteq A^t_u$, we have:
$$\cup_{h=1}^{q} H^t_{u,h} = A^t_u \tag{45}$$
   From 44 and 45, $H^t_{u,1}, ..., H^t_{u,q}$ is a partition of A^t_u.
2. Let's prove that $L^t_{u,1}, ..., L^t_{u,q}$ is a partition of A^t_u. Based on Proposition 7, for all $l, j \in [\![1\ ; q]\!]$, $j < l$, $\{x\} \succcurlyeq^t Z^t_{u,j} \implies \{x\} \succcurlyeq^t Z^t_{u,l-1}$, we have:
$$L^t_{u,l} \cap L^t_{u,j} = \emptyset \tag{46}$$
   For all $x \in A^t_u$, we have $\{x\} \succcurlyeq^t Z^t_{u,q}$ (since the way positive and negative reasons are assessed, we will always have valid positive reasons and no valid

negative reasons to be in the worst category). Hence:

$$
\begin{aligned}
x \in A^t_u \implies & \{x\} \succcurlyeq^t Z^t_{u,q} \\
\implies & \big(\{x\} \succcurlyeq^t Z^t_{u,q} \wedge \neg(\{x\} \succcurlyeq^t Z^t_{u,q-1})\big) \vee ... \\
& \vee \big(\{x\} \succcurlyeq^t Z^t_{u,2} \wedge \neg(\{x\} \succcurlyeq^t Z^t_{u,1})\big) \vee \{x\} \succcurlyeq^t Z^t_{u,1} \\
\implies & \cup^q_{j=1} L^t_{u,j}
\end{aligned}
$$

Since for all $l \in [\![1\ ;\ q]\!]$: $L^t_{u,l} \subseteq A^t_u$, we have:

$$\cup^q_{l=1} L^t_{u,l} = A^t_u \tag{47}$$

From 46 and 47, $L^t_{u,1}, ..., L^t_{u,q}$ is a partition of A^t_u. ■

A first rating might be established based on this partition. This rating concerns objects for which the best and worst possible rating lead to the same category: objects in $H^t_{u,k} \cap L^t_{u,k}$, for all k. However, the rating of objects is not always precise: objects in $A^t_u \setminus (\cup^q_{k=1} H^t_{u,k} \cap L^t_{u,k})$. Such objects require additional information in order to be rated. This information can be seen as additional positive reasons supporting a rating to one of the categories located between the best and the worst possible categories. For this aim, we define a symmetric binary relation based in the distance function *dist*, see definition 8. This function represents a similarity measure evaluating how close is an object from an updated set of reference profiles Z^t_u.

Definition 19. *(U^{2+}_k, for $k \in [\![1\ ;\ q]\!]$)*
U^{2+}_k, for $k \in [\![1\ ;\ q]\!]$, refers to the set of objects for which the rating is not precise and the closest updated reference profiles are the ones rated k. for $k \in [\![1\ ;\ q]\!]$, U^{2+}_k can be formulated as:

$$U^{2+}_k = \{x \in \Big((\cup^k_{j=1} H^t_{u,j}) \cap (\cup^q_{j=k} L^t_{u,j})\Big) \setminus (H^t_{u,k} \cap L^t_{u,k}); x R^+_{2r} Z^t_{u,k}\} \tag{48}$$

where R^+_{2r} is a binary relation defined on $A^t_u \times Z^t_u$, that can be interpreted for $(x, Z^t_{u,k})$ as "x is as good as reference profiles characterizing $\mathcal{C}_k$". For $x \in A^t_u$, R^+_{2r} can be formulated as:

$$x R^+_{2r} Z^t_{u,k} \implies Z^t_{u,k} = \arg \min_{Z \in K_x \subseteq Z^t_u} dist(x, Z) \tag{49}$$

where $K_x = \{Z^t_{u,k} \in Z^t_u; x \in \Big((\cup^k_{j=1} H^t_{u,j}) \cap (\cup^q_{j=k} L^t_{u,j})\Big) \setminus (H^t_{u,k} \cap L^t_{u,k})\}$.

K_x consists on sets of reference profiles characterizing categories for which the rating of the object x is not precise based on $\succcurlyeq^t$.

The use of the second level of positive reasons may lead to the violation of the convincing "condition". In order to avoid that, Algorithm 3, starts by

Algorithm 1: Rating Algorithm

Input: $\forall s \in [\![1\ ;\ q]\!] : H^t_{u,s}, U^{2+}_s, Z^t_{u,s}, L^t_{u,s}$;
Output: $\forall s \in [\![1\ ;\ q]\!] : Z^{t+1}_s$;

Function `Rating algorithm`($\forall s \in [\![1\ ;\ q]\!] : H^t_{u,s}, U^{2+}_s, Z^t_{u,s}, L^t_{u,s}$):
    $Z_{2r} \longleftarrow \emptyset$;
    $Z^{t+1}_s \longleftarrow \emptyset$;
    for *s=1 to q* **do**
        $Z^{t+1}_s \longleftarrow Z^t_{u,s} \cup (L^t_{u,s} \cap H^t_{u,s})$;
        $Z_{2r} \longleftarrow Z^{t+1}_s$;
    end
    for *s=1 to q-1* **do**
        foreach $x \in (U^{2+}_s \cup Z^{t+1}_s) \setminus Z_{2r}$ **do**
            $Z^{t+1}_s \longleftarrow Z^{t+1}_s \cup \{x\}$;
            $Z_{2r} \longleftarrow Z_{2r} \cup \{x\}$;
            for *j=s+1 to q* **do**
                foreach $y \in H^t_j \setminus Z^{t+1}_j$ *st:* yR^+x **do**
                    $Z^{t+1}_j \longleftarrow Z^{t+1}_j \cup \{y\}$;
                end
            end
            for *j=1 to s* **do**
                foreach $y \in H^t_j \setminus Z^{t+1}_s$ *st:* yR^+x **do**
                    $Z^{t+1}_s \longleftarrow Z^{t+1}_s \cup \{y\}$;
                end
            end
        end
    end
    return $\forall s \in [\![1\ ;\ q]\!] : Z^{t+1}_s$;
End Function

Figure 3: Rating Algorithm

rating objects for which the rating is precise, then the ones for which the rating requires using the distance. The assignment of objects for which the rating is not precise is computed from the best to the worst category. This direction of rating is followed because each object x rated k based on the second level of positive reasons lead to enriching positive reasons of other objects in worse categories: an object $y \in H^t_{u,s}$ with $k < s$ (worse than k), such that yR^+x will be rated s and thus assigned to Z^{t+1}_s. Also, the objects for which the best possible rating, $H^t_{u,j}$ with $j \leq k$ (obviously their worst possible rating l is worst then k: $k < l$, otherwise they would be previously rated) will be assigned to Z^{t+1}_s. x will be then removed from the considered objects (will be added to the set Z_{2r} see Algorithm 3) and we will move to the next object having the best second level of positive reasons.

8 Performance of Dynamic-R

In this section, we will show that the obtained rating is convincing in case the initial set of reference profiles Z^0 is convincing. In other terms we show that our method satisfies the two claims (1 and 2) presented in the Introduction. We will also provide statistics about the precision of the rating before using the symmetric relation *dist*.

8.1 The respect of the convincing condition

Obtaining a "convincing" rating is guaranteed by the following theorem:

Theorem 1. For $t \in T$, if Z^t respects the convincing condition then Z^{t+1} respects the convincing condition.

Proof. For $t \in T$, for $k \in [\![1\ ;\ q]\!]$, let x be a reference profile in Z_k^{t+1}.

Let us consider that there exists a reference profile $y \in Z_s^{t+1}$ characterizing a category worse than k ($k < s$) such that such that yR^+x and $y \notin U_k^-$ at beginning of the step $t+1$.

We have: $x \in Z_k^{t+1} \implies \exists h \in [\![1\ ;\ k]\!], l \in [\![k\ ;\ q]\!] : x \in H_{u,h}^t \cap L_{u,l}^t$

We distinguish two cases:

Case 1 $h = l$**:** In such case, since yR^+x, x would provide positive reasons to y supporting its rating k. Hence:

$$y \in U_k^+ \cup (\cup_{j=1}^k U_j^{e+}) \tag{50}$$

Based on Definition 18, we have:

$$\neg(yR^- Z_k^{t+1}) \implies \exists j \geq k : y \in H_{u,j}^t \tag{51}$$

Thus from 50 and 51, we have $\exists j \leq k : y \in L_{u,j}^t$ (better than k). Absurd since y was assigned to a worst category $s > k$.

Case 2: $h < l$. In such case, since x was assigned by Algorithm 3 to a category better than the one to which y was assigned, then x will provide positive reasons to y (because yR^+x) to be assigned to the best possible (for which it has no valid negative reasons) category worse than k. Since $y \notin U_k^-$, at the beginning of the step $t+1$, then y had no valid negative reasons preventing being rated k at the end of the step t. Hence, it would be assigned by the algorithm to at least Z_k^{t+1}. Absurd since y was assigned to a set of reference profiles Z_s^{t+1} characterizing a worse category. ■

Theorem 1 guarantees that the obtained rating is convincing at each step. A direct deduction of this theorem is the following:

Corollary 3. *If Z^0 is convincing, then for all $t \in T$, Z^t is convincing*

Proof. Obvious: direct conclusion of Theorem 1. ■

At the end of each step, the obtained rating is complete. This is formulated in the following proposition.

Theorem 2. *For $t \in T$, the resulting rating of Dynamic-R is complete:* $Z^{t+1}_{1,q} = Z^t_{1,q} \cup A^t$.

Proof. By construction we have $Z^{t+1}_{1,q} \subseteq Z^t_{1,q} \cup A^t$. Let's consider $z \in Z^t_k$. In case neither positive nor negative reasons were updated, z will be in $Z^t_{u,k}$ and thus in Z^{t+1}_k. Otherwise z will be in A^t_u. By construction we have $A^t \subseteq A^t_u$. Using Proposition 8: $H^t_{u,1}, ..., H^t_{u,q}$ and $L^t_{u,1}, ..., L^t_{u,q}$, are two partitions of A^t_u. Also R^+_{2r} is computed for all objects in $A^t_u \setminus \cup_k((H^t_{u,k} \cap L^t_{u,k}))$. Hence $\cup_k(H^t_{u,k} \cup L^t_{u,k})) = \cup_k(U^{2+}_k \cup (H^t_{u,k} \cap L^t_{u,k}))$. ■

8.2 Statistics about the precision of the rating

In order to derive statistics about the precision of Dynamic-R before using R^+_{2r}, we will define a fitness index for each object. The fitness index represents the precision of a given rating associated to an object. For instance, the best fitness index corresponds to the case where the best and the worst possible rating for any object refers to the same category, while the worst fitness index corresponds to the case where for all objects the best possible rating is 1 and the worst is q.

Definition 20. *(fitness index)*

The fitness index, for $t \in T$, is a function $f_t : A^t_u \to [1/q, 1]$ assessing the precision level of rating associated to objects based on best and the worst possible rating. "f_t" is defined as follow:

$$\forall t \in T, \forall x \in A^t_u, f_t(x) = \frac{q + h_t(x) - l_t(x)}{q} \tag{52}$$

where $h_t, l_t : A^t_u \to [\![1 \;;\; q]\!]$, and $h_t \leq l_t$, being respectively the best and the worst ratings that can be taken by an object at a step $t \in T$.

Remark 11. $\forall l, h \in [\![1 \;;\; q]\!], x \in L^t_l \cap H^t_h \implies (h_t(x) = h) \wedge (l_t(x) = l)$

Definition 20 concerns the case where some objects might be interval rated between a lower category l and a higher category h. Depending on how distant these two categories are we establish the rating precision or fitness. There might be several such objects which, for several different reasons, are not precisely rated, but interval rated. For this reason we might define equivalence classes of objects having the same fitness. These equivalence classes can be defined as follow:

Definition 21. *(The class of objects with an equivalent priority)*

The class of objects with equivalent priority B_j^t, at the step t, is an equivalence class where all objects have the same fitness value. Such equivalence class can be defined as follow: $\forall j \in [\![0\ ;\ q-1]\!]$,

$$B_j^t = \{x \in A_u^t; f_t(x) = \frac{q-j}{q}\} \tag{53}$$

Remark 12. B_j^t represents the set of objects for which the imprecision is $\frac{j}{q}$: for an object $x \in B_j^t$, $j = l_t(x) - h_t(x)$.

The set of equivalence classes can be used to describe the quality of the rating based on the previously defined positive and negative reasons. Based on the cardinality of B_j^t, $H_{u,j}^t$ and $L_{u,j}^t$ for all j, we can draw a distribution function related to the precision and the diversity of the rating (based on $H_{u,j}^t$ and $L_{u,j}^t$ for all j) before computing a symmetric binary relation. The mode, the median and the mean can be provided to the decision maker.

These distributions can be also indicators about the quality of the reference profiles and the objects to be rated: In case the number of objects rated with a high precision is important, and the cardinalities of $H_{u,1}^t \cap L_{u,1}^t, ..., H_{u,q}^t \cap L_{u,q}^t$ converge to a discrete uniform distribution, this means that the set of reference profiles and the objects to be rated are very rich.

9 Conclusion and Discussion

This paper is aimed at presenting a new MCDA rating method, named Dynamic-R which satisfies two important properties generally missing by other similar at scope methods. Dynamic-R is “complete”: each object to be rated can potentially be rated to a single category and no object can remain unrated at the end of a finite number of iterations of the procedure. Dynamic-R is “convincing”: when an object x is definitely rated at a certain category k there is no object y rated worst which is also preferred to x in direct comparison ($y \succ x$). In other terms Dynamic-R, despite being based on ordinal information and the majority principle manages to avoid the well know Condorcet paradox due to the intransitivity of the ordering relation. This result is obtained because the method dynamically updates the positive and negative reasons thanks to which ratings are computed. Such updates take into account each single object as soon as it is rated, which means that ratings are dynamically updated as well until the whole set rating becomes stable.

More precisely the procedure uses a set of reference profiles which are either typical objects of each category or boundary objects. We also introduce “minimal requirements” which are not reference profiles for the categories, but allow to establish negative reasons for certain ratings. Each rated object becomes on its turn a reference profile, implicitly allowing direct comparisons among the objects to be rated. Ratings are computed using positive (supporting a rating) and negative reasons (adverse to a rating), which are dynamically updated. We compute the best and worst rating for all objects until ratings become stable and in case of "interval assignments" we choose a precise category using a distance function.

There are a number of features to which attention is needed. First, the sequence of computing and updating the rating reasons is not free. We need to start from the negative reasons before turning our attention to the positive ones. Second, it appears that the use of a dominance relation instead of a strict preference one provides more interesting results mainly when minimal requirements are necessary in order to belong to certain categories (minimal requirements not being profiles of the categories). Another point which needs attention is the fact that although Dynamic-R is independent from the rating sequence for any time step it is not formally proven that independence holds all along the time steps. We strongly conjecture it is the case (due to Theorem 1), but we still need to prove it (in a forthcoming paper).

We see two possible improvements of this method. The first, quite straightforward, considers the use of ordering relations among subsets of objects, this allowing to generalise the ideas introduced in this paper. This topic is the subject of a forthcoming paper. The second, concerns the use of the method within the framework of automatic rating devices (such as recommender systems, see Ricci et al, 2011; Rahutomo et al, 2018). The fact that the method satisfies completeness and provides clear reasons for any rating computed allows to use it automatically, although this may result in further challenges as far as learning, updating and revising is concerned.

There are two issues which may be worth discuss further. The first considers the "danger" in using a totally automated rating procedure. There are certainly advantages in using a procedure which satisfies a number of desirable properties (like the "convincing" property in our case), but excluding the client/decision maker is an issue to handle with care (see Tsoukiàs, 2021). It could be that a reasonable compromise consists in implementing a hybrid version where recommendations are submitted to the client/decision maker exploiting the fact that the reasons for such recommendations are "clear" and (following our definition) "convincing". The second issue consists in raising the question of why using majority based procedures which are subject to the Condorcet paradox, which means that in case of a preference cycle the alternatives will belong to the same category, unless minimal requirements allow to make a distinction. Dynamic-R overcomes the problem using further information (which is strictly necessary as shown in Vincke, 1982). However, if further information is available then why not using preference difference measures (such as value functions)? The issue opens interesting topics of investigation to be discussed in a forthcoming paper about rating robustness.

Acknowledgments and Declarations

The authors would like to thank the National Research Agency (AMORAD project, grant ANR-11-RSNR-0002) for supporting financially this work through French program "Programme Investissement d'Avenir". Two anonymous referees provided many valuable comments which helped improving the presentation of the paper. The authors have no competing interests to declare that are relevant to the content of this article.

The authors have no competing interests to declare that are relevant to the content of this article.

References

Almeida-Dias J, Figueira JR, Roy B (2010) ELECTRE TRI-C: A multiple criteria sorting method based on characteristic reference actions. European Journal of Operational Research 204(3):565–580

Almeida-Dias J, Figueira JR, Roy B (2012) A multiple criteria sorting method where each category is characterized by several reference actions: The ELECTRE TRI-nC method. European Journal of Operational Research 217(3):567–579

Baarslag T, Gerding E (2015) Optimal incremental preference elicitation during negotiation. In: Proceedings of IJCAI'15, pp 3–9

Ben-Akiva M, McFadden D, Train K (2019) Foundations of stated preference elicitation: Consumer behavior and choice-based conjoint analysis. Foundations and Trends in Econometrics 10:1–144

Bouyssou D (1996) Outranking relations: do they have special properties? Journal of Multi-Criteria Decision Analysis 5(2):99–111

Bouyssou D, Marchant T, Pirlot M, et al (2006) Evaluation and decision models with multiple criteria: Stepping stones for the analyst, 1st edn. Springer, Boston, URL http://www.springer.com/sgw/cda/frontpage/0,11855,4-40521-22-116132747-0,00.html

Bugera V, Konno H, Uryasev S (2002) Credit cards scoring with quadratic utility functions. Journal of Multi-Criteria Decision Analysis 11(4-5):197–211

Colorni A, Tsoukiàs A (2013) What is a decision problem? preliminary statements. In: Proceedings of ADT 2013, LNAI 8176. Springer, pp 139 – 153

Dembczyński K, Kotłowski W, Slowinski R (2006) Additive preference model with piecewise linear components resulting from dominance-based rough set approximations. In: In proceedings of ICAISC. Springer, pp 499–508

Dembczyński K, Greco S, Slowinski R (2009) Rough set approach to multiple criteria classification with imprecise evaluations and assignments. European

Journal of Operational Research 198(2):626–636

Devaud J, Groussaud G, Jacquet-Lagreze E (1980) UTADIS: Une methode de construction de fonctions d'utilite additives rendant compte de jugements globaux. In: European working group on MCDA, Bochum , Germany

Falmagne J (1996) A stochastic theory for the emergence and the evolution of preference structures. Mathematical Social Sciences 31:63—84

Fernandez E, Navarro J (2011) A new approach to multi-criteria sorting based on fuzzy outranking relations: The THESEUS method. European Journal of Operational Research 213(2):405–413

Fernandez E, Figueira JR, Navarro J, et al (2017) ELECTRE TRI-nB: a new multiple criteria ordinal classification method. European Journal of Operational Research 263(1):214–224

Greco S, Matarazzo B, Slowinski R (2001) Rough sets theory for multicriteria decision analysis. European journal of operational research 129(1):1–47

Greco S, Matarazzo B, Slowinski R (2002a) Rough approximation by dominance relations. International journal of intelligent systems 17(2):153–171

Greco S, Matarazzo B, Slowinski R (2002b) Rough sets methodology for sorting problems in presence of multiple attributes and criteria. European journal of operational research 138(2):247–259

Greco S, Mousseau V, Slowinski R (2010) Multiple criteria sorting with a set of additive value functions. European Journal of Operational Research 207(3):1455–1470

Greco S, Kadziński M, Mousseau V, et al (2011) ELECTRE GKMS robust ordinal regression for outranking methods. European Journal of Operational Research 214:118 – 135

Khannoussi A, Olteanu A, Labreuche C, et al (2021) Simple ranking method using reference profiles: incremental elicitation of the preference parameters. 4OR: A Quarterly Journal of Operations Research p in press

Köksalan M, Özpeynirci SB (2009) An interactive sorting method for additive utility functions. Computers & Operations Research 36(9):2565–2572

Leroy A, Mousseau V, Pirlot M (2011) Learning the parameters of a multiple criteria sorting method. In: Proceedings of ADT 2011, LNAI 6992. Springer Verlag, Berlin, pp 219–233

Liu X (2015) Modeling users' dynamic preference for personalized recommendation. In: Proceedings of IJCAI'15, pp 1785–1791

Ostanello A (1985) Outranking relations. In: Fandel G, Spronk J (eds) Multiple Criteria Decision Methods and Applications. Springer, p 41–60

Perny P, Viappiani P, Boukhatem A (2016) Incremental preference elicitation for decision making under risk with the rank-dependent utility model. In: Proceedings of UAI 2016. ACM Digital Libray, pp 597 – 606

Raboun O, Chojnacki E, Duffa C, et al (2020) Spatial risk assessment in case of multiple nuclear release scenarios. Socio-Economic Planning Sciences 70:100,721

Rahutomo F, Hidayati I, Saputra P, et al (2018) Automatic rating system for national delivery services. In: Proceedings of ICAST, pp 226 – 230

Ricci F, Rokach L, Shapira B, et al (2011) Recommender Systems Handbook. Springer Verlag, Berlin

Rocha C, Dias L (2008) An algorithm for ordinal sorting based on ELECTRE with categories defined by examples. Journal of Global Optimization 42(2):255–277

Roy B (1991) The outranking approach and the foundations of ELECTRE methods. Theory and Decision 31:49–73

Tsoukiàs A (2021) Social responsibility of algorithms: an overview. In: Papathanasiou J, Zaraté P, Freire de Sousa J (eds) EURO Working Group on DSS. Springer Nature, p 153 – 166, URL https://doi.org/10.1007/978-3-030-70377-6_9

Tsoukiàs A, Perny P, Vincke P (2002) From concordance/discordance to the modelling of positive and negative reasons in decision aiding. In: Bouyssou D, Jacquet-Lagrèze E, Perny P, et al (eds) Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy. Kluwer Academic, Dordrecht, p 147–174

Vincke P (1982) Arrow's theorem is not a surprising result. European Journal of Operational Research 10:22–25

Vincke P (1999) Outranking approach. In: Multicriteria Decision Making. Springer, p 305–333

Yu W (1992) Aide multicritère à la décision dans le cadre de la problématique du tri: Concepts, méthodes et applications. PhD thesis, Paris 9