A characterization of PQI interval orders

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Abstract

We provide an answer to an open problem concerning the representation of preferences by intervals. Given a finite set of elements and three relations on this set (indifference, weak preference and strict preference), necessary and sufficient conditions are provided for representing the elements of the set by intervals in such a way that 1) two elements are indifferent when the interval associated to one of them is included in the interval associated to the other; 2) an element is weakly preferred to another when the interval of the first is "more to the right" than the interval of the other, but the two intervals have a non empty intersection; 3) an element is strictly preferred to another when the interval of the first is "more to the right" than the interval of the other and their intersection is empty.

Key words: Intervals, Interval Orders, Indifference, Weak Preference, Strict Preference.

1 Introduction

Comparing intervals is a frequently encountered problem in preference modelling and decision aid. This is due to the fact that the comparison of alternatives (outcomes, objects, candidates, ...) generally are realized through their evaluations on numerical scales, while such evaluations often are imprecise or uncertain. A well known preference structure, in this context, is the semi order (see Luce, 1956 and for a comprehensive presentation Pirlot and Vincke, 1997) and more generally the interval order (see also Fishburn, 1985). An interval order is obtained when one considers that an alternative is preferred to another iff it's interval is "completely to the right" of the other (hereafter we assume that the larger an evaluation of an alternative is on a numerical scale the better the alternative is), while any two alternatives the intervals of which have a non empty intersection are considered indifferent. Such a model has a strict probabilistic interpretation, since the intervals associated to each alternative can be viewed as the extremes of the probability distributions of the evaluations of the alternatives. Under such an interpretation a "sure preference" occurs only if the distributions have an empty intersection. A second implicit assumption in this frame is that if there is no preference of an alternative over the other then they are indifferent.

It is easy however to notice that if, in the previous frame, we want to establish a "sure indifference", it is much more natural to consider that two alternatives are indifferent if their associated intervals (or distributions) are embedded. In such a case we obtain a preference relation which is known to be a partial order of dimension 2 (a partial order obtained from the intersection of exactly two linear orders; see Roubens and Vincke, 1985).

Practically we observe that we have three situations:
- a "sure indifference": when the intervals associated to two alternatives are embedded;
- a "sure preference": when the interval associated to one alternative is "more to the right" with respect to the interval associated to the other alternative and the two intervals have an empty intersection;
- an "hesitation between indifference and preference" which we denote as weak preference: when the interval associated to one alternative is "more to the right" with respect to the interval associated to the other alternative and the two intervals have a non empty intersection.

Such an interpretation fits better in the case we have qualitative uncertainties or imprecision and is consistent with the use of specific relations in order to represent situations of hesitation in preference modelling (see Tsoukiàs and Vincke, 1997). However, such a preference structure (hereafter called PQI interval order) lacked any characterization as mentioned for instance in Vincke, 1988 (by characterization we mean the determination of a list of properties concerning the three preference relations which are necessary and sufficient conditions in order to be able to represent them by intervals as mentioned before).

In this paper we present an answer for this problem. Section 2 provides the basic notations and definitions. In section 3 we recall some results concerning conventional interval orders. The main result is presented, demonstrated and discussed in section 4. Finally section 5 presents an algorithm for the detection of a PQI interval order on a set A.

2 Notations and definitions

In this paper we consider binary relations defined on a finite set A, that is subsets of $A \times A$ (the quantifiers apply therefore always to such a domain). Further on we will use the following notations for any binary relations S, T. If S is a binary relation on A we denote by $S(x,y)$ the fact that $(x,y) \in S$. $\neg$, $\wedge$ and $\vee$ denote the usual negation, conjunction and disjunction operations.

$S^{-1} = \ \{(x,y) : \ S(y,x)\}$
$S^{c} = \neg S = \ \{(x,y) : \ \neg S(x,y)\}$
$S^{d} = \neg S^{-1} = \ \{(x,y) : \ \neg S(y,x)\}$
$S \subset T : \ \forall x,y \ S(x,y) \rightarrow T(x,y)$
$S.T = \ \{(x,y) : \exists z \ S(x,z) \wedge T(z,y)\}$
$S^{2} = \ \{(x,y) : \exists z \ S(x,z) \wedge S(z,y)\}$
$S \cup T = \ \{(x,y) : \ S(x,y) \vee T(x,y)\}$
$S \cap T = \ \{(x,y) : \ S(x,y) \wedge T(x,y)\}$

We recall some well known definitions from the literature (our terminology follows Roubens and Vincke, 1985).

Definition 2.1 *A relation S on a set A is said to be:*
- reflexive: iff $\forall x \ S(x,x)$
- irreflexive: iff $\forall x \ \neg S(x,x)$
- symmetric: iff $\forall x,y \ S(x,y) \rightarrow S^{-1}(x,y)$
- asymmetric: iff $\forall x,y \ S(x,y) \rightarrow S^{d}(x,y)$
- complete: iff $\forall x,y, \ x \neq y, \ \ S(x,y) \vee S^{-1}(x,y)$
- transitive: iff $\forall x,y,z \ S(x,y) \wedge S(y,z) \rightarrow S(x,z)$
- negatively transitive: iff $\forall x,y,z \ \neg S(x,y) \wedge \neg S(y,z) \rightarrow \neg S(x,z)$

Definition 2.2 *A binary relation S is:*
- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence iff it is reflexive, symmetric and transitive.

In this paper we will consider relations representing strict preference, weak preference and indifference situations. We will denote them P, Q, I respectively. Moreover, such relations are expected to satisfy some "natural" properties of the type announced in the following two definitions.

Definition 2.3 *A $\langle P, I \rangle$ preference structure on a set A is a couple of binary relations, defined on A, such that:*
- I is reflexive and symmetric;

- P is asymmetric;
- $I \cup P$ is complete;
- P and I are mutually exclusive ($P \cap I = \emptyset$).

Definition 2.4 *A $\langle P, Q, I \rangle$ preference structure on a set A is a triple of binary relations, defined on A, such that:*
- I is reflexive and symmetric;
- P and Q are asymmetric;
- $I \cup P \cup Q$ is complete;
- P, Q and I are mutually exclusive.

Finally we introduce an equivalence relation as follows:

Definition 2.5 *The equivalence relation associated to a $\langle P, Q, I \rangle$ preference structure is the binary relation E, defined on the set A, such that, $\forall x, y \in A$:*

$$E(x,y) \quad \textit{iff} \quad \forall z \in A : \begin{cases} P(x,z) \Leftrightarrow P(y,z) \\ Q(x,z) \Leftrightarrow Q(y,z) \\ I(x,z) \Leftrightarrow I(y,z) \\ Q(z,x) \Leftrightarrow Q(z,y) \\ P(z,x) \Leftrightarrow P(z,y) \end{cases}$$

Remark 2.1 *In this paper we consider that two different elements of A are never equivalent for the given $\langle P, Q, I \rangle$ preference structure. This is not restrictive as it suffices to consider the quotient of A by E to satisfy the assumption. Under such an assumption we will use in the numerical representation of the preference relations only strict inequalities without any loss of generality.*

3 Interval Orders

In this section we recall some definitions and theorems concerning conventional interval orders and semi orders.

Definition 3.1 *A $\langle P, I \rangle$ preference structure on a set A is a PI interval order iff $\exists\, l, r : A \mapsto \mathcal{R}^+$ such that:*
$\forall\, x :\; r(x) > l(x)$
$\forall\, x, y : P(x,y) \;\Leftrightarrow\; l(x) > r(y)$
$\forall\, x, y : I(x,y) \;\Leftrightarrow\; l(x) < r(y)$ *and* $l(y) < r(x)$

Definition 3.2 *A $\langle P, I \rangle$ preference structure on a set A is a PI semi order iff $\exists\, l : A \mapsto \mathcal{R}^+$ and a positive constant k such that:*

$\forall\, x, y : P(x, y) \;\Leftrightarrow\; l(x) > l(y) + k$
$\forall\, x, y : I(x, y) \;\Leftrightarrow\; |l(x) - l(y)| < k$

Such structures have been extensively studied in the literature (see for example Fishburn, 1985). We recall here below the two fundamental results which characterize interval orders and semi orders.

Theorem 3.1 *A $\langle P, I\rangle$ preference structure on a set A is a PI interval order iff $P.I.P \subset P$.*

Proof. See Fishburn, 1985.

Theorem 3.2 *A $\langle P, I\rangle$ preference structure on a set A is a PI semi order iff $P.I.P \subset P$ and $I.P.P \subset P$.*

Proof. See Fishburn, 1985.

4 $\langle P, Q, I\rangle$ interval orders

As mentioned in the introduction, we are interested in situations where, comparing elements evaluated by intervals, one wants to distinguish three situations: indifference if one interval is included in the other, strict preference if one interval is completely "to the right" of the other and weak preference when one interval is "to the right" of the other, but they have a non empty intersection. Definition 4.1 precisely states this kind of situation, $l(x)$ and $r(x)$ respectively representing the left and right extremities of the interval associated to any element $x \in A$.

Definition 4.1 *A $\langle P, Q, I\rangle$ preference structure on a finite set A is a PQI interval order, iff there exist two real valued functions l and r such that, $\forall x, y \in A, x \neq y$:*
- $r(x) > l(x)$;
- $P(x, y) \;\Leftrightarrow\; r(x) > l(x) > r(y) > l(y)$;
- $Q(x, y) \;\Leftrightarrow\; r(x) > r(y) > l(x) > l(y)$;
- $I(x, y) \;\Leftrightarrow\; r(x) > r(y) > l(y) > l(x)$ or $r(y) > r(x) > l(x) > l(y)$.

The reader will notice that the above definition immediately follows Definition 3.1, since a preference structure characterized as a PI interval order can always be seen as a PQI interval order also. We give now necessary and sufficient conditions for which such a preference structure exists.

Theorem 4.1 *A $\langle P, Q, I\rangle$ preference structure on a finite set A is a PQI interval order, iff there exists a partial order I_l such that:*
i) $I = I_l \cup I_r \cup I_o$ where $I_o = \{(x, x),\ x \in A\}$ and $I_r = I_l^{-1}$;

ii) $(P \cup Q \cup I_l)P \subset P$;
iii) $P(P \cup Q \cup I_r) \subset P$;
iv) $(P \cup Q \cup I_l)Q \subset P \cup Q \cup I_l$;
v) $Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r$;

Proof.

We first give an outline of necessity demonstration which is the easy part of the theorem. If $\langle P, Q, I \rangle$ is a PQI interval order, then defining
- $I_l(x, y) \Leftrightarrow l(y) < l(x) < r(x) < r(y)$
- $I_r(x, y) \Leftrightarrow l(x) < l(y) < r(y) < r(x)$

we obtain two partial orders satisfying the desired properties. As an example we demonstrate property (v):

$Q(x, y)$ and $(P \cup Q \cup I_r)(y, z)$ imply $r(x) > r(y)$ and $r(y) > r(z)$, hence $r(x) > r(z)$, so that $(P \cup Q \cup I_r)(x, z)$.

Conversely let us assume the existence of I_l satisfying the properties of the theorem. Define a set A' isomorphic to A and denote by x' the image of $x \in A$ in A'. In the set $A \cup A'$ let us define the relation S as follows: $\forall\, x, y \in A, x \neq y$
- $S(x', x)$
- $S(x, y) \Leftrightarrow (P \cup Q \cup I_l)(x, y)$
- $S(x', y') \Leftrightarrow (P \cup Q \cup I_r)(x, y)$
- $S(x, y') \Leftrightarrow P(x, y)$
- $S(x', y) \Leftrightarrow \neg P(y, x)$

We demonstrate now that S is a linear order (irreflexive, complete and transitive relation) in $A \cup A'$.

Irreflexivity results from irreflexivity of P, Q, I_l and I_r.

To demonstrate completeness of S remark that for $x \neq y$:

$$
\begin{aligned}
\neg S(x, y) &\Leftrightarrow \neg(P \cup Q \cup I_l)(x, y) \\
&\Leftrightarrow (P \cup Q \cup I_l)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \\
&\Leftrightarrow S(y, x) \\
\neg S(x', y') &\Leftrightarrow \neg(P \cup Q \cup I_r)(x, y) \\
&\Leftrightarrow (P \cup Q \cup I_r)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \\
&\Leftrightarrow S(y', x') \\
\neg S(x, y') &\Leftrightarrow \neg P(x, y) \\
&\Leftrightarrow S(y', x)
\end{aligned}
$$

$$\begin{aligned}\neg S(x',y) &\Leftrightarrow P(y,x)\\ &\Leftrightarrow S(y,x')\end{aligned}$$

We demonstrate now that S is transitive.

- $S(x,y)$ and $S(y,z)$ imply $(P\cup Q\cup I_l)(x,y)$ and $(P\cup Q\cup I_l)(y,z)$. From conditions ii) and iv) of the theorem, we know that $(P\cup Q\cup I_l)(x,y)$ and $(P\cup Q)(y,z)$ imply $(P\cup Q\cup I_l)(x,z)$, hence $S(x,z)$. From transitivity of I_l we have that $I_l(x,y)$ and $I_l(y,z)$ imply $I_l(x,z)$, hence $S(x,z)$. Finally, if $(P\cup Q)(x,y)$ and $I_l(y,z)$ then $(P\cup Q\cup I_l)(x,z)$ because, if not, we would have $(P\cup Q\cup I_l)(z,x)$ which with $I_l(y,z)$ would give $(P\cup Q\cup I_l)(y,x)$ (by conditions ii) and iv) and transitivity of I_l), contradiction. So we get $S(x,z)$.
- $S(x,y)$ and $S(y,z')$ imply $(P\cup Q\cup I_l)(x,y)$ and $P(y,z)$, which, by condition ii), give $P(x,z)$, hence $S(x,z')$.
- $S(x,y')$ and $S(y',z)$ imply $P(x,y)$ and $\neg P(z,y)$. If $\neg S(x,z)$, then $(P\cup Q\cup I_l)(z,x)$ which, with $P(x,y)$ and by condition ii) would give $P(z,y)$, a contradiction. Thus $S(x,z)$. This reasoning applies also in the case $y=z$.
- $S(x,y')$ and $S(y',z')$ imply $P(x,y)$ and $(P\cup Q\cup I_r)(y,z)$, which, by condition iii), give $P(x,z)$, hence $S(x,z')$.
- $S(x',y')$ and $S(y',z)$ imply $(P\cup Q\cup I_r)(x,y)$ and $\neg P(z,y)$. If $\neg S(x',z)$, then $P(z,x)$ which, with $(P\cup Q\cup I_r)(x,y)$ and by condition iii) would give $P(z,y)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y=z$.
- $S(x',y')$ and $S(y',z')$ imply $(P\cup Q\cup I_r)(x,y)$ and $(P\cup Q\cup I_r)(y,z)$. From conditions iii) and v) of the theorem, we know that $(P\cup Q)(x,y)$ and $(P\cup Q\cup I_r)(y,z)$ imply $(P\cup Q\cup I_r)(x,z)$, hence $S(x',z')$. From transitivity of I_r we have that $I_r(x,y)$ and $I_r(y,z)$ imply $I_r(x,z)$, hence $S(x',z')$. Finally, if $I_r(x,y)$ and $(P\cup Q)(y,z)$ then $(P\cup Q\cup I_r)(x,z)$ because, if not, we would have $(P\cup Q\cup I_r)(z,x)$ which with $I_r(x,y)$ would give $(P\cup Q\cup I_r)(z,y)$ (by condition iii) and v) and transitivity of I_r), contradiction. So we get $S(x',z')$.
- $S(x',y)$ and $S(y,z)$ imply $\neg P(y,x)$ and $(P\cup Q\cup I_l)(y,z)$ If $\neg S(x',z)$, then $P(z,x)$ which, with $(P\cup Q\cup I_l)(y,z)$ and by condition ii) would give $P(y,x)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y=x$.
- $S(x',y)$ and $S(y,z')$ imply $\neg P(y,x)$ and $P(y,z)$. If $\neg S(x',z')$, then $(P\cup Q\cup I_r)(z,x)$ which, with $P(y,z)$ and by condition iii) would give $P(y,x)$, a contradiction. Thus $S(x',z')$. This reasoning applies also in the case $y=x$.

Since S is a linear order on $A\cup A'$, there exists a real valued function u such that, $\forall\, x,y\in A$:
- $S(x,y) \Leftrightarrow u(x)>u(y)$;
- $S(x',y') \Leftrightarrow u(x')>u(y')$;

- $S(x,y') \Leftrightarrow u(x) > u(y')$;
- $S(x',y) \Leftrightarrow u(x') > u(y)$.

We define $\forall\, x \in A$, $l(x) = u(x)$ and $r(x) = u(x')$ and we obtain:

- $\forall\, x : r(x) > l(x)$, since $S(x',x)$.
- $\forall\, x, y :\ P(x,y) \Leftrightarrow S(x,y') \Leftrightarrow l(x) > r(y)$.
- $\forall\, x, y :\ Q(x,y) \Leftrightarrow S(x,y) \wedge S(x',y') \wedge \neg P(x,y) \Leftrightarrow$ $l(x) > l(y)$ and $r(x) > r(y)$ and $r(y) > l(x)$, equivalent to: $r(x) > r(y) > l(x) > l(y)$.
- $\forall\, x, y :\ I(x,y) \Leftrightarrow$ $r(x) > r(y) > l(y) > l(x)$ or $r(y) > r(x) > l(x) > l(y)$ since $I(x,y)$ holds in all the remaining cases. $\square$

We can complete the investigation providing a characterization of PQI semi orders.

Definition 4.2 *A PQI semi order is a PQI interval order such that $\exists\, k > 0$ constant for which $\forall x :\ \ r(x) = l(x) + k$*

In other words, a PQI semi order is a $\langle P, Q, I\rangle$ preference structure for which there exists a real valued function $l : A \mapsto \mathcal{R}$ and a positive constant k such that $\forall\, x, y$:
- $P(x,y) \Leftrightarrow l(x) > l(y) + k$;
- $Q(x,y) \Leftrightarrow l(y) + k > l(x) > l(y)$;
- $I(x,y) \Leftrightarrow l(x) = l(y)$; (in fact I reduces to I_o).

For such preference structures the following theorem holds.

Theorem 4.2 *A $\langle P, Q, I\rangle$ preference structure is a PQI semi order iff:*
i) I is transitive
ii) $PP \cup PQ \cup QP \subset P$;
iii) $QQ \subset P \cup Q$;

Proof

Necessity is trivial. We give only the sufficiency proof. Since I is an equivalence relation, we consider the relation $P \cup Q$ on the set A/I. Such a relation is clearly a linear order (irreflexivity and completeness result from definition 2.4 and transitivity from conditions ii) and iii) of the theorem). Therefore we can index the elements of A/I by $i = 1, 2 \cdots n$ in such a way that $\forall\, x_i, x_{i+1} \in A/I$: $(P \cup Q)(x_{i+1}, x_i)$.

Choosing an arbitrary positive value k, we define function l as follows:
$l(x_1) = 0$ and for $i = 2, 3, \cdots n$
$l(x_{i+1}) > l(x_i)$
$l(x_i) > l(x_j) + k \;\; \forall\, j < i$ such that $P(x_i, x_j)$
$l(x_i) < l(x_m) + k \;\; \forall\, m < i$ such that $Q(x_i, x_m)$.

This is always possible because $P(x_i, x_j)$ and $Q(x_i, x_m)$ imply $(P \cup Q)(x_m, x_j)$ (if not, we would have $(P \cup Q)(x_j, x_m)$ which, with $P(x_i, x_j)$ and by condition ii) would give $P(x_i, x_m)$, hence $m > j$ and $l(x_m) > l(x_j)$). By construction the function l satisfies the numerical representation of a PQI semi order. $\square$

5 Detection of a PQI interval order

The problem is the following:
Given a set A and a $\langle P, Q, I\rangle$ preference structure on it, verify whether it is a PQI interval order. The difficulty resides in the fact that the theorem previously announced contains a second order condition which is the existence of the partial order I_l. For this purpose we give two propositions which show the difficulties in detecting such a structure.

Proposition 1 *There exist $\langle P, Q, I\rangle$ preference structures which are $P\hat{I}$-interval orders (where $\hat{I} = Q \cup I \cup Q^{-1}$), but are not PQI interval orders.*

Proof Consider the following case.
- $A = \{a, b, c, d, e\}$;
- $P = \{(a, c), (d, e), (a, e)\}$;
- $Q = \{(d, c), (a, b), (b, e)\}$;
- $I = \{(a, d), (c, e), (b, d), (b, c), (d, a), (e, c), (d, b), (c, b)\} \cup I_o$

On the one hand if we consider the relation $\hat{I} = Q \cup I \cup Q^{-1}$ it is easy to observe that the $\langle P, \hat{I}\rangle$ preference structure is a PI interval order ($P\hat{I}P \subset P$ holds). On the other hand if we accept that the given $\langle P, Q, I\rangle$ preference structure is a PQI interval order then we have (by the definition 4.1 and the theorem 4.1) that:
- $I(a, d)$ has to be $I_l(a, d)$ because of c;
- $I(d, b)$ has to be $I_l(d, b)$ because of e;
therefore by transitivity we should have $I_l(a, b)$, while we have $Q(a, b)$ which is impossible. Therefore we can conclude that for this particular case the PQI interval order representation is impossible. $\square$

Proposition 2 *There exist $\langle P, Q, I\rangle$ preference structures which have more* *than one PQI interval order representation.*

Proof Consider the following case.
- $A = \{a, b, c\}$;
- $P = \emptyset$;
- $I = \{(a, c), (b, c), (c, a), (c, b)\} \cup I_o$;
- $Q = \{(a, b)\}$

It is easy to observe that both $I_l(a, c), I_l(b, c)$ and $I_l(c, a), I_l(c, b)$ are possible, thus allowing two different PQI interval orders: one in which the interval of c is included in the intervals of both a and b and the other where the intervals of b and a are included in the interval c. Both representations are correct, although incompatible with each other. $\square$

In order to detect if a $\langle P, Q, I\rangle$ preference structure is a PQI interval order we propose the following algorithm which we present in terms of pseudo-code.

Step 1 For all x, y verify that $P^2 \subset P$, $P.Q \subset P$, $Q.P \subset P$ and $Q^2 \subset P \cup Q$.
Step 2 $\forall x, y, z \;\; I(x, y) \wedge P(x, z) \wedge Q(y, z) \rightarrow I_l(x, y)$
Step 3 $\forall x, y, z \;\; I(x, y) \wedge P(z, x) \wedge Q(z, y) \rightarrow I_l(x, y)$
Step 4 $\forall x, y, z \;\; I(x, y) \wedge I(y, z) \wedge P(x, z) \rightarrow I_l(x, y) \wedge I_l(z, y)$
Step 4bis $\forall x, y, z \;\; I(x, y) \wedge I(y, z) \wedge Q(x, z) \rightarrow (I_l(x, y) \wedge I_l(z, y)) \vee (I_l(y, x) \wedge I_l(y, z))$
Step 5 $\forall x, y, z \;\; I_l(x, y) \wedge I_l(y, z) \rightarrow I_l(x, z)$
Step 6 For a x, y such that $I(x, y)$ and I_l has not been established, choose arbitrary $I_l(x, y)$ and go to step 5.

The algorithm succeeds if it arrives to assign all elements of relation I to the relation I_l or to the relation I_r without any contradiction, that is without assigning to a relation a couple already assigned to another relation.

Proposition 3 *If the above algorithm succeeds, then the $\langle P, Q, I\rangle$ preference structure is a PQI interval order.*

Proof

We have to demonstrate that the conditions of Theorem 4.1 are verified.

(1) Exists a partial order I_l such that $I = I_l \cup I_o \cup I_l^{-1}$. By construction of I_l.
(2) $(P \cup Q \cup I_l).P \subset P$.
$P.P \subset P$ by step 1;
$Q.P \subset P$ by step 1;
$I_l.P \subset P$. Suppose that:
$\exists x, y, z : \;\; I_l(x, y) \wedge P(y, z) \wedge P(z, x)$.
Impossible since it implies $P(y, x)$ step 1

$\exists x, y, z: \quad I_l(x,y) \wedge P(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x)$ step 1
$\exists x, y, z: \quad I_l(x,y) \wedge P(y,z) \wedge I_l(z,x)$.
Impossible since it implies $I_l(z,y)$ step 5
$\exists x, y, z: \quad I_l(x,y) \wedge P(y,z) \wedge I_l(x,z)$.
Impossible since it implies $P(z,y)$ step 4
$\exists x, y, z: \quad I_l(x,y) \wedge P(y,z) \wedge Q(x,z)$.
Impossible since it implies $I_l(y,x)$ step 2.

(3) $P.(P \cup Q \cup I_l^{-1}) \subset P$.
$P.P \subset P$ by step 1;
$P.Q \subset P$ by step 1;
$P.I_l^{-1} \subset P$. Suppose that:
$\exists x, y, z: \quad P(x,y) \wedge I_l^{-1}(y,z) \wedge P(z,x)$.
Impossible since it implies $P(z,y)$ step 1
$\exists x, y, z: \quad P(x,y) \wedge I_l^{-1}(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x)$ step 1
$\exists x, y, z: \quad P(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(z,x)$.
Impossible since it implies $P(y,x)$ step 4
$\exists x, y, z: \quad P(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(x,z)$.
Impossible since it implies $I_l(x,y)$ step 5
$\exists x, y, z: \quad P(x,y) \wedge I_l^{-1}(y,z) \wedge Q(x,z)$.
Impossible since it implies $I_l(y,z)$ step 3.

(4) $(P \cup Q \cup I_l).Q \subset P \cup Q \cup I_l$.
$P.Q \subset P$ by step 1;
$Q.Q \subset P \cup Q$ by step 1;
$I_l.Q \subset P \cup Q \cup I_l$. Suppose that:
$\exists x, y, z: \quad I_l(x,y) \wedge Q(y,z) \wedge P(z,x)$.
Impossible since it implies $P(y,x)$ step 1
$\exists x, y, z: \quad I_l(x,y) \wedge Q(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x) \vee Q(y,x)$ step 1
$\exists x, y, z: \quad I_l(x,y) \wedge Q(y,z) \wedge I_l(z,x)$.
Impossible since it implies $I_l(z,y)$ step 5.

(5) $Q.(P \cup Q \cup I_l^{-1}) \subset P \cup Q \cup I_l^{-1}$.
$Q.P \subset P$ by step 1;
$Q.Q \subset P \cup Q$ by step 1;
$Q.I_l^{-1} \subset P \cup Q \cup I_l^{-1}$. Suppose that:
$\exists x, y, z: \quad Q(x,y) \wedge I_l^{-1}(y,z) \wedge P(z,x)$.
Impossible since it implies $P(z,y)$ step 1
$\exists x, y, z: \quad Q(x,y) \wedge I_l^{-1}(y,z) \wedge Q(z,x)$.
Impossible since it implies $P(y,x) \vee Q(y,x)$ step 1
$\exists x, y, z: \quad Q(x,y) \wedge I_l^{-1}(y,z) \wedge I_l(x,z)$.
Impossible since it implies $I_l(x,y)$ step 5. $\square$

How difficult is it to verify whether a PQI preference structure is a PQI interval order? In other terms, what is the complexity of the previous algorithm? The reader may notice that in Step 6 we make an arbitrary choice. If after such a choice the algorithm reaches a contradiction normally we have to backtrack and try with a new choice. Actually we have a tree structure defined by the branches created by each arbitrary choice. The exploration of such a tree normally is in NP. However, our conjecture is that the introduction of Step 4bis (which is useless for the demonstration of the correctness of the algorithm) reduces the complexity of the algorithm to polynomial time, since a failure (reaching a contradiction) will be independent from any arbitrary choice previously done. This is the subject of a forthcoming paper (see also Ngo The, 1998).

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