Preference Handling

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Outline

Preferences

- Preferences are “rational” desires.
- Preferences are at the basis of any decision aiding activity.
- There are no decisions without preferences.
- Preferences, Values, Objectives, Desires, Utilities, Beliefs, ...

Decision Aiding

A client		An analyst
	A problem situation $\langle \mathcal{A}, \mathcal{O}, \mathcal{S} \rangle$	
	A problem formulation $\langle \mathbb{A}, \mathbb{V}, \Pi \rangle$	
	An evaluation model $\langle A, D, E, H, U, R \rangle$	
	A final recommendation	

Basic information

- A: a set of alternatives (enumerative, combinatorial, product space ...)
- D: a set of dimensions (attributes) describing A.
- E: the "scales" used for the attributes in D.
- H: **Preferential Information**
- U: uncertainties ....
- R: algorithms, procedures, protocols etc ...

What are the problems?

- How to learn preferences?
- How to model preferences?
- How to aggregate preferences?
- How to use preferences for recommending?

Binary relations

- $\succeq$: binary relation on a set (A).
- $\succeq \subseteq A \times A$ or $A \times P \cup P \times A$.
- $\succeq$ is reflexive.

What is that?

If $x \succeq y$ stands for x is at least as good as y, then the asymmetric part of $\succeq$ ($\succ$: $x \succeq y \;\wedge\; \neg(y \succeq x)$) stands for strict preference. The symmetric part stands for indifference ($\sim_1$: $x \succeq y \;\wedge\; y \succeq x$) or incomparability ($\sim_2$: $\neg(x \succeq y) \;\wedge\; \neg(y \succeq x)$).

More binary relations

- We can further separate the asymmetric (symmetric) part in more relations representing hesitation or intensity of preference.

$$\succ = \succ_1 \cup \succ_2 \cdots \succ_n$$

- We can get rid of the symmetric part since any symmetric relation can be viewed as the union of two asymmetric relations and the identity.
- We can also have valued relations such that: $v(x \succ y) \in [0, 1]$

Binary relations properties

Binary relations have specific properties such as:

- Irreflexive: $\forall x \quad \neg(x \succ x)$;
- Asymmetric: $\forall x, y \quad x \succ y \ \rightarrow \ \neg(y \succ x)$;
- Transitive: $\forall x, y, z \quad x \succ y \ \wedge \ y \succ z \ \rightarrow \ x \succ z$;
- Ferrers; $\forall x, y, z, w \quad x \succ y \ \wedge \ z \succ w \ \rightarrow \ x \succ w \ \vee \ z \succ y$;

Numbers

$$x \succeq y \quad \Leftrightarrow \quad \Phi(x,y) \geq 0$$

where:

$\Phi : A \times A \mapsto \mathbb{R}$. Simple case $\Phi(x,y) = f(x) - f(y)$; $f : A \mapsto \mathbb{R}$

N.B.

Likelihoods can also be expressed under form of binary relations and their numerical representations ($\omega_1 \succeq \omega_2$: event 1 is likely to occur at least as much as event 2).

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Consider sentences of the type:

- I like red shoes.
- I do not like brown sugar.
- I prefer Obama to McCain.
- I do not want tea with milk.
- Cost is more important than safety.
- I prefer flying to Athens than having a suite at Istanbul.

What do we learn out of such sentences?

- Basic hypotheses about the structure of the evaluation model.
- Binary relations.
- Numerical values (exact or imprecise).
- Importance Parameters.
- Inconsistencies.

Preference Structures

A preference structure

is a collection of binary relations $\sim_1, \cdots \sim_m, \succ_1, \cdots \succ_n$ such that:

- they are pair-disjoint;
- $\sim_1 \cup \cdots \sim_m \cup \succ_1 \cup \cdots \succ_n = A \times A$;
- $\sim_i$ are symmetric and $\succ_j$ are asymmetric;
- possibly they are identified by their properties.

$\sim_1, \sim_2, \succ$ Preference Structures

Independently from the nature of the set A (enumerated, combinatorial etc.), consider $x, y \in A$ as whole elements. Then:

If $\succeq$ is a weak order then:

$\succ$ is a strict partial order, $\sim_1$ is an equivalence relation and $\sim_2$ is empty.

If $\succeq$ is an interval order then:

$\succ$ is a partial order of dimension two, $\sim_1$ is not transitive and $\sim_2$ is empty.

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If $\succeq$ is a *PQI* interval order then:

$\succ_1$ is transitive, $\succ_2$ is quasi transitive, $\sim_1$ is asymmetrically transitive and $\sim_2$ is empty.

If $\succeq$ is a pseudo order then:

$\succ_1$ is transitive, $\succ_2$ is quasi transitive, $\sim_1$ is non transitive and $\sim_2$ is empty.

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What characterises such structures?

Characteristic Properties

Weak Orders are complete and transitive relations.
Interval Orders are complete and Ferrers relations.

Numerical Representations

w.o. $\Leftrightarrow \exists f : A \mapsto \mathbb{R} : x \succeq y \leftrightarrow f(x) \geq f(y)$
i.o. $\Leftrightarrow \exists f, g : A \mapsto \mathbb{R} : f(x) > g(x); x \succeq y \leftrightarrow f(x) \geq g(y)$

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More about structures

Characteristic Properties

PQI Interval Orders are complete and generalised Ferrers relations.
Pseudo Orders are coherent bi-orders.

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p.o. $\Leftrightarrow \exists f, t, g : A \mapsto \mathbb{R} : f(x) > t(x) > g(x); x \succ_1 y \leftrightarrow g(x) > f(y); x \succ_2 y \leftrightarrow g(x) > t(y)$

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What if *A* is multi-attribute described?

$$x = \langle x_1 \cdots x_n \rangle \quad y = \langle y_1 \cdots y_n \rangle$$

$$x \succeq y \quad \Leftrightarrow \quad \Phi([u_1(x_1) \cdots u_n(_n)], [u_1(y_1) \cdots u_n(y_n)] \geq 0$$

A special case is when Φ is increasing to its first n arguments and decreasing to the following n arguments: it then can be an additive function. See more in conjoint measurement theory.

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What is measuring?

Constructing a function from a set of “objects” to a set of “measures”.

Objects come from the real world.

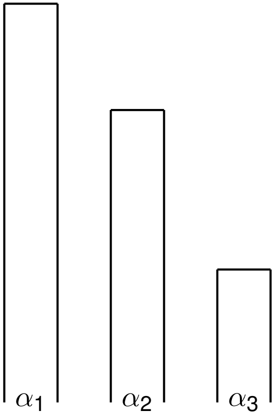
Measures come from empirical observations on some attributes of the objects.

The problem is: how to construct the function out from such observations?

Measurement

1. Real objects $(x, y, \cdots)$.
2. Empirical evidence comparing objects $(x \succeq y, \cdots)$.
3. First numerical representation $(\Phi(x, y) \geq 0)$.
4. Repeat observations in a standard sequence $(x \circ y \succeq z \circ w)$.
5. Enhanced numerical representation $(\Phi(x, y) = \Phi(x) - \Phi(y))$.

Example



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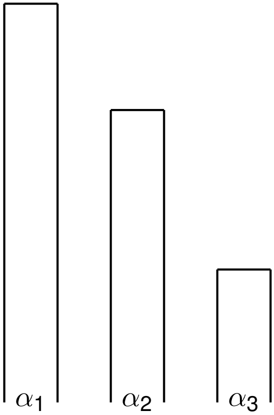

$$\alpha_1 \succ \alpha_2 \succ \alpha_3$$

α_1	α_2	α_3
10	8	6
97	32	12
3	2	1

Any of the above could be a numerical representation of this empirical evidence.
Ordinal Scale: any increasing transformation of the numerical representation is compatible with the EE.

Further Example

Consider putting together objects and observing:

$$\alpha_1 \circ \alpha_5 > \alpha_3 \circ \alpha_4 > \alpha_1 \circ \alpha_2 > \alpha_5 > \alpha_4 > \alpha_3 > \alpha_2 > \alpha_1$$

Consider now the following numerical representations:

	L_1	L_2	L_3
α_1	14	10	14
α_2	15	91	16
α_3	20	92	17
α_4	21	93	18
α_5	28	99	29

L_1, L_2 and L_3 capture the simple order among α_{1-5}, but L_2 fails to capture the order among the combinations of objects.

Further Example

NB

For L_1 we get that $\alpha_2 \circ \alpha_3 \sim \alpha_1 \circ \alpha_4$
while for L_3 we get that $\alpha_2 \circ \alpha_3 > \alpha_1 \circ \alpha_4$.
We need to fix a “standard sequence”.

Length

If we fix a “standard” length, a unit of measure, then all objects will be expressed as multiples of that unit.

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Scales

Ratio Scales

All proportional transformations (of the type αx) will deliver the same information. We only fix the unit of measure.

Interval Scales

All affine transformations (of the type $\alpha x + \beta$) will deliver the same information. Besides the unit of measure we fix an origin.

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More complicated

Consider a Multi-attribute space:

$$X = X_1 \times \cdot X_n$$

to each attribute we associate an ordered set of values:

$$X_j = \langle x_j^1 \cdots x_j^m \rangle$$

An object x will thus be a vector:

$$x = \langle x_1^l \cdots x_n^k \rangle$$

Generally speaking ...

$$x \succeq y$$

$$\Longleftrightarrow$$

$$\langle x_1^l \cdots x_n^k \rangle \succeq \langle y_1^i \cdots y_n^j \rangle$$

$$\Longleftrightarrow$$

$$\Phi(f(x_1^l \cdots x_n^k), f(y_1^i \cdots y_n^j)) \geq 0$$

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500

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a_2	25	80	C	700	$1500+\delta_2$

For what value of δ_2 a_1 and a_2 are indifferent?

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500
a_2	25	80	C	700	1700

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
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The trade-offs introduced with δ_1 and δ_2 allow to get $a \sim a_1 \sim a_2$

What do we get?

Standard Sequences

Length: objects having the same length allow to define a unit of length;

Value: objects being indifferent can be considered as having the same value and thus allow to define a “unit of value”.

Remark 1: indifference is obtained through trade-offs.
Remark 2: separability among attributes is the minimum requirement.

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The easy case

IF

1. restricted solvability holds;
2. at least three attributes are essential;
3. $\succeq$ is a weak order satisfying the Archimedean condition $\forall x, y \in \mathbb{R}, \exists n \in \mathbb{N} : ny > x$.

THEN

$$x \succeq y \Leftrightarrow \sum_j u_j(x) \geq \sum_j u_j(y)$$

General Usage

The above ideas apply also in

- Economics (comparison of bundle of goods);
- Decision under uncertainty (comparing consequences under multiple states of the nature);
- Inter-temporal decision (comparing consequences on several time instances);
- Social Fairness (comparing welfare distributions among individuals).

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