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Preference Modelling

by

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ABSTRACT

This paper provides the reader with a presentation of preference modelling fundamental notions as well as some recent results in this field. Preference modelling is an inevitable step in a variety of fields: economy, sociology, psychology, mathematical programming, even medicine, archaeology, and obviously decision analysis. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented at the beginning of the paper. We start by discussing different reasons for constructing a model or preference. We then go through a number of issues that influence the construction of preference models. Different formalisations besides classical logic such as fuzzy sets and non-classical logics become necessary. We then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It is relevant to have a numerical representation of preferences: functional representations, value functions. The concepts of thresholds and minimal representation are also introduced in this section. In section 7, we briefly explore the concept of deontic logic (logic of preference) and other formalisms associated with "compact representation of preferences" introduced for special purposes. We end the paper with some concluding remarks.

Introduction

The purpose of this paper is to present fundamental notions of preference modelling as well as some recent results in this field. Basic references on this issue can be considered: [75], [117], [163], [78], [165], [109], [183], [187], [159], [5].

The paper is organised as follows: The purpose for which formal models of preference and more generally of objects comparison are studied, is introduced in section 1. In section 2, we analyse the information used when such models are established and introduce different sources and types of uncertainty. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented in section 3. Besides classical logic, different formalisms can be used in order to establish a preference model, such as fuzzy sets and non-classical logics. These are discussed in section 4. In section 5, we then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It appears relevant to have a numerical representation of preferences: functional representations, value functions and intervals. These are discussed in section 6. The concepts of thresholds and minimal representation are also introduced in this section. Finally, after briefly exploring the concept of deontic logic (logic of preference) and other related issued in section 7, we end the paper with some concluding remarks

1 Purpose

Preference modelling is an inevitable step in a variety of fields. Scientists build models in order to better understand and to better represent a given situation; such models may also be used for more or less operational purposes (see [30]). It is often the case that it is necessary to compare objects in such models, basically in order to either establish if there is an order between the objects or to establish whether such objects are "near". Objects can be everything, from candidates to time intervals, from computer codes to medical patterns, from prospects (lotteries) to production systems. This is the reason why preference modelling is used in a great variety of fields such as economy ([9], [10], [11], [51]), sociology, psychology ([45], [42], [111], [110], [37]), political science ([15], [177]), artificial intelligence ([65]), computer science ([175],[187]), temporal logic (see [6]) and the interval satisfiability problem ([91, 148]) mathematical programming ([156], [155]), electronic business, medicine and biology ([22], [38], [113], [136], [107]), archaeology ([101]), and obviously decision analysis.

In this paper, we are going to focus on preference modelling for decision aiding purposes, although the results have a much wider validity.

Throughout this paper, we consider the case of somebody (possibly a decision-maker) who tries to compare objects taking into account different points of view. We denote the set of alternatives A, to be labelled $a, b, c, ...$ and the set of points of view J, labelled $j = 1, 2, ..., m$. In this framework, a data $g_j(a)$ corresponds to the evaluation of the alternative a from the point of view $j \in J$.

As already mentioned, comparing two objects can be seen as looking for one of the two following possible situations:

- object a is "before" object b, where "before" implies some kind of order between a and b, such

an order referring either to a direct preference (a is preferred to b) or being induced from a measurement and its associated scale (a occurs before b, a is longer, bigger, more reliable, than b);
- object a is "near" object b, where "near" can be considered either as indifference (object a or object b will do equally well for some purpose), or as a similarity, or again could be induced by a measurement (a occurs simultaneously with b, they have the same length, weight, reliability).

The two above-mentioned "attitudes" (see [139]) are not exclusive. They just stand to show what type of problems we focus on. From a decision aiding point of view we traditionally focus on the first situation. Ordering relations is the natural basis for solving ranking or choice problems. The second situation is traditionally associated with problems where the aim is to be able to put together objects sharing a common feature in order to form "homogeneous" classes or categories (a classification problem).

The first case we focus on is the ordering relation: given the set A, establishing how each element of A compares to each other element of A from a "preference" point of view enables to obtain an order which might be used to make either a choice on the set A (identify the best) or to rank the set A. Of course, we have to consider whether it is possible to establish such an ordering relation and of what type (certain, uncertain, strong, weak etc.) for all pairs of elements of A. We also have to establish what "not preference" represents (indifference, incomparability etc.). In the following sections (namely in section 5), we are going to see that different options are available, leading to different so called preference structures.

In the second case we focus on the "nearness" relation since the issue here is to put together objects which ultimately are expected to be "near" (whatever the concept of "near" might represent). In such a case, there is also the problem how to consider objects which are "not near". Typical situations in this case include the problems of grouping, discriminating and assigning ([97])). A further distinction in such problems concerns the fact that the categories with which the objects might be associated could already exist or not and the fact that such categories might be ordered or not. Putting objects into non pre-existing non ordered categories is the typical classification problem, conversely, assigning objects to pre-existing ordered categories is known as the "sorting" problem ([147, 219, 152]).

It should be noted that although preference relations have been naturally associated to ranking and choice problem statements, such a separation can be argued. For instance, there are sorting procedures (which can be seen as classification problems) that use preference relations instead of "nearness" ones ([214, 124, 134]). The reason is the following: in order to establish that two objects belong to the same category we usually either try to check whether the two objects are "near" or whether they are near a "typical" object of the category (see for instance [152]). If, however, a category is described, not through its typical objects, but through its boundaries, then, in order to establish if an object belongs to such a category it might make sense to check whether such an object performs "better" than the "minimum", or "least" boundary of the category and that will introduce the use of a preference relation.

Recently Ngo The ([139]) claimed that decision aiding should not exclusively focus on preference relations, but also on "nearness relations", since quite often the problem statement to work with in a problem formulation is that of classification (on the existence of different problem statements and their meaning the reader is referred to [203, 170, 171, 52].

2 Nature of Information

As already mentioned, the purpose of our analysis is to present the literature associated with objects comparison for either a preference or a nearness relation. Nevertheless, such an operation is not always as intuitive as it might appear. Building up a model from reality is always an abstraction (see [28]). This can always be affected by the presence of uncertainty due to our imperfect knowledge of the world, our limited capability of observation and/or discrimination, the inevitable errors occurring in any human activity etc. ([168]). We call such an uncertainty exogenous. Besides, such an activity might generate uncertainty since it creates an approximation of reality, thus concealing some features of reality. We call this an endogenous uncertainty (see [189]).

As pointed out by Vincke ([204]) preference modelling can be seen as either the result of direct comparison (asking a decision-maker to compare two objects and to establish the relation between them) from which it might be possible to infer a numerical representation, or as the result of the induction of a preference relation from the knowledge of some "measures" associated to the compared objects.

In the first case, uncertainty can arise from the fact that the decision-maker might not be able to clearly state a preference relation for any pair of actions. We do not care why this may happen, we just consider the fact that the the decision-maker may reply when asked if "x is preferred to y": yes, no, I do not know, yes and no, I am not sure, it might be, it is more preference than indifference, but ... etc.. The problem in such cases is how to take such replies into account when defining a model of preferences.

In the second case, we may have different situations such as: incomplete information (missing values for some objects), uncertain information (the value of an object lies within an interval to which an uncertainty distribution might be associated, but the precise value is unknown), ambiguous information (contradictory statements about the present state of an object). The problem here is how to establish a preference model on the basis of such information and to what extent the uncertainty associated with the original information will be propagated to the model and how.

Such uncertainties can be handled through the use of various formalisms (see section 4 of this paper). Two basic approaches can be distinguished (see also [71]).

1. Handling uncertain information and statements. In such a case, we consider that the concepts used in order to model preferences are well-known and that we could possibly be able to establish a preference relation without any uncertainty, but we consider this difficult to do in the present situation with the available information. A typical example is the following: we know that x is preferred to y if the price of x is lower than the price of y, but we know very little about the prices of x and y. In such cases we might use an uncertainty distribution (classical probability, ill-known probabilities, possibility distributions, see [75, 43, 106, 70]) in order to associate a numerical uncertainty with each statement.

2. Handling ambiguous concepts and linguistic variables. With such a perspective we consider that sentences such as "x is preferred to y" are ill-defined, since the concept of preference itself is ill-defined, independently from the available information. A typical example is a sentence of the type: "the largest the difference of price between x and y is, the strongest the preference is". Here we might know the prices of x and y perfectly, but the concept of preference is defined through a

continuous valuation. In such cases, we might use a multi-valued logic such that any preferential sentence obtains a truth value representing the “intensity of truth” of such a sentence. This should not be confused with the concept of “preference intensity”, since such a concept is based on the idea of “measuring” preferences (as we do with temperature or with weight) and there is no “truth” dimension (see [117, 163, 162, 116]). On the other hand such a subtle theoretical distinction can be transparent in most practical cases since often happens that similar techniques are used under different approaches.

3 Notation and Basic Definitions

The notion of binary relation appears for the first time in De Morgan’s study ([49]) and is defined as a set of ordered pairs in Peirce’s works ([149], [150], [151]). Some of the first work dedicated to the study of preference relations can be found in [72] and in [176] (more in general the concept of models of arbitrary relations will be introduced in [184], [185]). Throughout this paper, we adopt Roubens’ and Vincke’s notation ([165]).

Definition 3.1 (Binary Relation). *Let A be a finite set of elements (a, b, c,...,n), a binary relation R on the set A is a subset of the cartesian product A×A, that is, a set of ordered pairs (a, b) such that a and b are in A: R ⊆ A×A.*

For an ordered pair (a, b) which belongs to R, we indifferently use the notations:

$$(a, b) \in R \text{ or } aRb \text{ or } R(a, b)$$

Let R and T be two binary relations on the same set A. Some set operations are:

The Inclusion :	$R \subseteq T$ if $aRb \longrightarrow aTb$
The Union :	$a(R \cup T)b$ iff aRb or(inclusive) aTb
The Intersection :	$a(R \cap T)b$ iff aRb and aTb
The Relative product :	$a(R.T)b$ iff $\exists c \in A :$ aRc and cTb
	$(aR^2b$ iff $aR.Rb)$

When such concepts apply we respectively denote (R^a), (R^s), ($\hat{R}$) the asymmetric, the symmetric and the complementary part of binary relation R:

$$aR^ab \text{ iff } aRb \text{ and } not(bRa)$$
$$aR^sb \text{ iff } aRb \text{ and } bRa$$
$$a\hat{R}b \text{ iff } not(aRb) \text{ and } not(bRa)$$

The complement (R^c), the converse (the dual)($\overline{R}$) and the co-dual (R^{cd}) of R are respectively defined as follows:

$$aR^cb \text{ iff } not(aRb)$$
$$a\overline{R}b \text{ iff } bRa$$
$$aR^{cd}b \text{ iff } not(bRa)$$

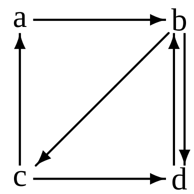

Figure 1: Graphical representation of R

The relation R is called

reflexive,	if	$aRa, \quad \forall a \in A$
irreflexive,	if	$aR^c a, \quad \forall a \in A$
symmetric,	if	$aRb \longrightarrow bRa, \quad \forall a, b \in A$
antisymmetric,	if	$(aRb, bRa) \longrightarrow a = b, \quad \forall a, b \in A$
asymmetric,	if	$aRb \longrightarrow bR^c a, \quad \forall a, b \in A$
complete,	if	$(aRb \quad \text{or} \quad bRa), \quad \forall a \neq b \in A$
strongly complete,	if	$aRb \quad \text{or} \quad bRa, \quad \forall a, b \in A$
transitive,	if	$(aRb, bRc) \longrightarrow aRc, \quad \forall a, b, c \in A$
negatively transitive,	if	$(aR^c b, bR^c c) \longrightarrow aR^c c, \quad \forall a, b, c \in A$
negatively transitive,	if	$aRb \longrightarrow (aRc \quad \text{or} \quad cRb), \quad \forall a, b, c \in A$
semitransitive,	if	$(aRb, bRc) \longrightarrow (aRd \quad \text{or} \quad dRc), \quad \forall a, b, c, d \in A$
Ferrers relation,	if	$(aRb, cRd) \longrightarrow (aRd \quad \text{or} \quad cRb), \quad \forall a, b, c, d \in A$

The equivalence relation E associated with the relation R is a reflexive, symmetric and transitive relation, defined by:

$$aEb \text{ iff } \quad \forall a \in A \begin{cases} aRc & \Longleftrightarrow & bRc \\ cRa & \Longleftrightarrow & cRb \end{cases}$$

A binary relation R may be represented by a direct graph (A, R) where the nodes represent the elements of A, and the arcs, the relation R. Another way to represent a binary relation is to use a matrix M^R; the element M^R_{ab} of the matrix (the intersection of the line associated to a and the column associated to b) is 1 if aRb and 0 if $not(aRb)$.

Example 3.1. *Let R be a binary relation defined on a set A, such that the set A and the relation R are defined as follows:*
$A = \{a, b, c, d\}$ *and* $R = \{(a, b), (b, d), (b, c), (c, a), (c, d), (d, b)\}$

The graphical and matrix representation of R are given in figures 1 and 2.

	a	b	c	d
a	0	1	0	0
b	0	0	1	1
c	1	0	0	1
d	0	1	0	0

Figure 2: Matrix representation of R

4 Languages

Preference models are formal representations of comparisons of objects. As such they have to be established through the use of a formal and abstract language capturing both the structure of the world being described and the manipulations of it. It seems natural to consider formal logic as such a language. However, as already mentioned in the previous sections, the real world might be such that classical formal logic might appear too rigid to allow the definition of useful and expressive models. For this purpose, in this section, we introduce some further formalisms which extend the expressiveness of classical logic, while keeping most of its calculus properties.

4.1 Classical Logic

The interested reader can use two references: [126, 199] as introductory books to the use and the semantics of classical logic. All classic books mentioned in this paper, implicitly or explicitly use classical logic, since binary relations are just sets and the calculus of sets is algebraically equivalent to truth calculus. Indeed the semantics of logical formulas as established by [184], [185], show the equivalence between membership of an element to a set and truth of the associate sentence.

Building a binary preference relation, a valuation of any proposition takes the values {0, 1}:

$$\mu(aRb) = 1 \text{ iff } aRb \text{ is true}$$
$$\mu(aRb) = 0 \text{ iff } aRb \text{ is false.}$$

The reader will note that all notations introduced in the previous section are based on the above concept. He/she should also note that when we write “a preference relation P is a subset of $A \times A$", we introduce a formal structure where the universe of discourse is $A \times A$ and P is the model of the sentence “x in relation P with y”, that is, P is the set of all elements of $A \times A$ (ordered pairs of x and y) for which the sentence is true.

The above semantic can be in sharp contrast with decision analysis experience. For this purpose we will briefly introduce two more semantics: fuzzy sets and four-valued logic.

4.2 Fuzzy Sets

In this section, we provide a survey of basic notions of fuzzy set theory. We present definitions of connectives and several valued binary relation properties in order to be able to use this theory in the field of decision analysis. Basic references for this section include [218], [70], [180], [84].

Fuzzy sets were first introduced by Zadeh ([216], [217]). The concept and the associated logics were further developed by other researchers: [92], [114], [115], [128], [129], [137], [142], [67].

Fuzzy measures can be introduced for two different uses: either they can represent a concept imprecisely known (although well defined) or a concept which is vaguely perceived such as in the case of a linguistic variable. In the first case they represent possible values, while in the second they are better understood as a continuous truth valuation (in the interval $[0, 1]$). To be more precise:
- in the first case we associate a possibility distribution (an ordinal distribution of uncertainty) to classical logic formulas;
- in the second case we have a multi-valued logic where the semantics allow values in the entire interval $[0, 1]$.
A fuzzy set can be associated either with the set of alternatives considered in a decision aiding model (consider the case where objects are represented by fuzzy numbers) or with the preference relations. In decision analysis we may consider four possibilities [1]:

- Alternatives with crisp values and crisp preference relations
- Alternatives with crisp values and fuzzy preference relations
- Alternatives with fuzzy values and crisp preference relations (defuzzification , [122] with gravity center, [213] with means interval)
- Alternatives with fuzzy values and fuzzy preference relations (possibility graphs, [69] four fuzzy dominance index, [166]); but in this paper we are going to focus on fuzzy preference relations

In the following we introduce the definitions required for the rest of the paper.

Definition 4.1 (Fuzzy Set). *A fuzzy set (or a fuzzy subset) F on a set Ω is defined by the result of an application:*

$$\mu_F : \Omega \longrightarrow [0, 1]$$

where $\forall x \in \Omega$, $\mu(x)$ is the membership degree of x to F.

[1] Lets take an example: Imagine that we have to choose one car between two. We have to know the performance of each car in order to establish the relation of preference:

- in the first case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 5$); the relation of preference is known too (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the second case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 7$); we are not sur about the preference relation that is why the relation of preference is fuzzy ($\mu(car1Pcar2) = 0.75$)
- in the third case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers ; in this case we can use triangular or trapezoidal fuzzy number to represent the performance); the relation of preference is crisp (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the fourth case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers); the relation of preference is fuzzy too (($\mu(car1Pcar2) = 0.75$))

Definition 4.2 (Negation). *A function n: [0, 1] $\longrightarrow$ [0, 1] is a negation if and only if it is non-increasing and :*
n(0) = 1 and n(1) = 0

If the negation n is strictly decreasing and continuous then it is called *strict*.

In the following we investigate the two basic classes of operators, the operators for the intersection (triangular norms called t-norms) and the union (triangular conorms called t-conorms or s-norms) of fuzzy sets:

Definition 4.3 (t-norm). *A function T: $[0, 1]^2 \longrightarrow [0, 1]$ is a triangular norm (t-norm), if and only if it satisfies the four conditions:*

Equivalence Condition: $T(1, x) = x \; \forall x \in [0, 1]$
T is commutative: $T(x, y) = T(y, x) \; \forall x, y \in [0, 1]$
T is nondecreasing in both elements: $T(x, y) \leq T(u, v)$ *for all* $0 \leq x \leq u \leq 1$ *and* $0 \leq y \leq v \leq 1$
T is associative: $T(x, T(y, z)) = T(T(x, y), z) \; \forall x, y, z \in [0, 1]$

The function T define a general class of intersection operators for fuzzy sets.

Definition 4.4 (t-conorm). *A function S: $[0, 1]^2 \longrightarrow [0, 1]$ is a (t-conorm), if and only if it satisfies the four conditions:*

Equivalence Condition: $S(0, x) = x \; \forall x \in [0, 1]$
S is commutative: $S(x, y) = S(y, x) \; \forall x, y \in [0, 1]$
S is nondecreasing in both elements: $S(x, y) \leq S(u, v)$ *for all* $0 \leq x \leq u \leq 1$ *and* $0 \leq y \leq v \leq 1$
S is associative: $S(x, S(y, z)) = S(S(x, y), z) \; \forall x, y, z \in [0, 1]$

T-norms and t-conorms are related by duality. For suitable negation operators[2] pairs of t-norms and t-conorms satisfy the generalisation of the De Morgan law:

Definition 4.5 (De Morgan Triples). *Suppose that T is a t-norm, S is a t-conorm and n is a strict negation.* $\langle T, S, n \rangle$ *is a De Morgan triple if and only if:*

$$n(S(x, y)) = T(n(x), n(y))$$

Such a definition extends De Morgan's law to the case of fuzzy sets. There exist different proposed De Morgan triples: [60], [89], [174], [212], [68], [209], [215].

The more frequent t-norms and t-conorms and negations are presented in table 1.

We make use of De Morgan's triplet $\langle T, S, n \rangle$ in order to extend the definitions of the operators and properties introduced above in crisp cases. First, we give the definitions of operators of implication I_T and equivalence E_T:

$$I_T(x, y) = sup\{z \in [0, 1] : T(x, z) \leq y\}$$
$$E_T(x, y) = T(I_T(x, y), I_T(y, x)$$

[2] a suitable one can be the complement operator defined: $n(\mu(x)) = 1 - \mu(x)$

Table 1: Principal t-norms, t-conorms and negations

Names	*t-norms*	*t-conorms*
Zadeh	$min(x, y)$	$max(x, y)$
probabilistic	$x * y$	$x + y - xy$
Lukasiewicz	$max(x + y - 1, 0)$	$min(x + y, 1)$
Hamacher($\gamma > 0$)	$(xy)/$	$(x + y + xy - (1 - \gamma)xy)/$
	$(\gamma + (1 - \gamma)(x + y - xy))$	$(1 - (1 - \gamma)xy)$
Yager($p > 0$)	$max(1 - ((1 - x)^p + (1 - y)^p)^{1/p}, 0)$	$min((x^p + y^p)^{1/p}, 1)$
Weber($\lambda > -1$)	$max((x + y - 1 + \lambda xy)/(1 + \lambda), 0))$	$min(x + y + \lambda xy, 1)$
drastic	x if $y = 1$	x if $y = 0$
	y if $x = 1$	y if $x = 0$
	0 ifnot	1 ifnot

Since preference modelling makes use of binary relations, we extend the definitions of binary relation properties to the valued case. For the sake of simplicity $\mu(R(x, y))$ will be denoted $R(x, y)$: a valued binary relation $R(x, y)$ is ($\forall a, b, c, d \in A$)

reflexive,	if	$R(a, a) = 1$
irreflexive,	if	$R(a, a) = 0$
symmetric,	if	$R(a, b) = R(b, a)$
T-antisymmetric,	if	$a \neq b \longrightarrow T(R(a, b), R(b, a)) = 0$
T-asymmetric,	if	$T(R(a, b), R(b, a)) = 0$
S-complete,	if	$a \neq b \quad \longrightarrow \quad S(R(a, b), R(b, a)) = 1$
S-strongly complete,	if	$S(R(a, b), R(b, a)) = 1$
T-transitive,	if	$T(R(a, c), R(c, b)) \leq R(a, b)$
negatively S-transitive,	if	$R(a, b) \leq S(R(a, c), R(c, b))$
T-S-semitransitive,	if	$T(R(a, d), R(d, b)) \leq S(R(a, c), R(c, b))$
T-S-Ferrers relation,	if	$T(R(a, b), R(c, d)) \leq S(R(a, d), R(c, b))$

Different instances of De Morgan triplets will provide different definitions for each property.

The equivalence relation is one of the most-used relations in decision analysis and is defined in fuzzy set theory as follows:

Definition 4.6 (Equivalence Relation). *A function E :[0, 1]^2 $\longrightarrow$ [0, 1] is an equivalence if and only if it satisfies:*
E(x, y) = E(y, x) $\forall$ x, y $\in$ [0, 1]
E(0, 1) = E(1, 0) = 0
E(x, x) = 1 $\forall$ x $\in$[0, 1]
x $\leq$x' $\leq$ y' $\leq$ y $\Longrightarrow$ E(x, y) $\leq$ E(x', y')

In section 5.3, some results obtained by the use of fuzzy set theory are represented.

4.3 Four-valued Logics

When we compare objects, it might be the case that it is not possible to establish precisely whether a certain relation holds or not. The problem is that such a hesitation can be due either to incomplete information (missing values, unknown replies, unwillingness to reply etc.) or to contradictory information (conflicting evaluation dimensions, conflicting reasons for and against the relation, inconsistent replies etc.). For instance, consider the query "is Anaxagoras intelligent?" If you know who Anaxagoras is you may reply "yes" (you came to know that he is a Greek philosopher) or "no" (you discover he is a dog). But if you know nothing you will reply "I do not know" due to your ignorance (on this particular issue). If on the other hand you came to know both that Anaxagoras is a philosopher and a dog you might again reply "I do not know", not due to ignorance, but to inconsistent information. Such different reasons for hesitation can be captured through four-valued logics allowing for different truth values for four above-mentioned cases. Such logics were first studied in [66] and introduced in the literature in [17] and [18]. Further literature on such logics can be found in [23], [73], [87], [83], [112], [186], [8], [190].

In the case of preference modelling, the use of such logics was first suggested in [188] and [54]. Such logics extend the semantics of classical logic through two hypotheses:
- the complement of a first order formula does not necessarily coincide with its negation;
- truth values are only partially ordered (in a bilattice), thus allowing the definition of a boolean algebra on the set of truth values.

The result is that using such logics, it is possible to formally characterise different states of hesitation when preferences are modelled (see [193], [194]. Further more, using such a formalism, it becomes possible to generalise the concordance/discordance principle (used in several decision aiding methods) as shown in [191] and several characterisation problems can be solved (see for instance [195]). Recently (see [157], [88]) it has been suggested to use the extension of such logics for continuous valuations.

5 Preference Structures

Definition 5.1 (Preference Structure). *A preference structure is a collection of binary relations defined on the set A and such that:*
- for each couple a, b in A; at least one relation is satisfied
- for each couple a, b in A; if one relation is satisfied, another one cannot be satisfied.

In other terms a preference structure defines a partition[3] of the set $A \times A$. In general it is recommended to have two other hypotheses with this definition (also denoted as fundamental relational system of preferences):

- Each preference relation in a preference structure is uniquely characterized by its properties (symmetry, transitivity, etc..)

[3] to have a partition of the set $A \times A$, the inverse of the asymmetric relation must be considered too.

- For each preference structure, there exists a unique relation from which the different relations composing the preference structure can be deduced. Any preference structure on the set A can thus be characterised by a unique binary relation R in the sense that the collection of the binary relations are be defined through the combinations of the epistemic states of this characteristic relation[4].

5.1 $\langle P, I\rangle$ Structures

The most traditional preference model considers that the decision-maker confronted with a pair of distinct elements of a set A, either:

-clearly prefers one element to the other,

or

-feels indifferent about them.

The subset of ordered pairs (a, b) belonging to $A \times A$ such that the statement "a is preferred to b" is true, is called *preference relation* and is denoted by P.

The subset of pairs (a, b) belonging to $A \times A$ such that the statement "a and b are indifferent" is true, is called *indifference relation* and is denoted by I (I being considered the complement of $P \cup P^{-1}$ with respect to $A \times A$).

In the literature, there are two different ways of defining a specific preference structure:
- the first defines it by the properties of the binary relations of the relation set;
- the second uses the properties of the characteristic relation. In the rest of the section, we give definitions in both ways.

Definition 5.2 (⟨ P, I ⟩ Structure). *A ⟨ P, I ⟩ structure on the set A is a pair $\langle P, I\rangle$ of relations on A such that:*

- *P is asymmetric,*
- *I is reflexive, symmetric.*

The characteristic relation R of a $\langle P, I\rangle$ structure can be defined as a combination of the relations P and I as :

$$aRb \text{ iff } a(P \cup I)b \tag{1}$$

In this case P and I can be defined from R as follows:

$$aPb \text{ iff } aRb \text{ and } bR^c a \tag{2}$$
$$aIb \text{ iff } aRb \text{ and } bRa \tag{3}$$

The construction of *orders* is of a particular interest, especially in decision analysis since they allow an easy operational use of such preference structures. We begin by representing the most

[4]While several authors prefer using both of them, there are others for which one is sufficient. For example Fishburn does not require the use of preference structures with a characteristic relation

elementary orders (weak order, complete order). To define such structures we add properties to the relations P and I (namely different forms of transitivity).

Definition 5.3 (Total Order). *Let R be a binary relation on the set A, R being a characteristic relation of ⟨ P, I ⟩, the following definitions are equivalent:*
i. R is a total order.
ii. R is reflexive, antisymmetric, complete and transitive

$$iii. \begin{cases} I = \{(a,a), \forall a \in A\} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

$$iv. \begin{cases} P \text{ is transitive} \\ P\,I \subset P \text{ (or equivalently } I\,P \subset P) \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

With this relation, we have an indifference between any two objects only if they are identical. The total order structure consists of an arrangement of objects from the best one to the worst one without any ex aequo.

In the literature, one can find different terms associated with this structure: total order, complete order, simple order or linear order.

Definition 5.4 (Weak Order). *Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:*
i. R is a weak order
ii. R is reflexive, strongly complete and transitive

$$iii. \begin{cases} I \text{ is transitive} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

This structure is also called complete preorder or total preorder. In this structure, indifference is an equivalence relation. The associated order is indeed a total order of the equivalence (indifference) classes of A.

The first two structures consider indifference as a transitive relation. This is empirically falsifiable. Literature studies on the intransitivity of indifference show this; undoubtedly the most famous is that of [123], which gives the example of a cup of sweetened tea.[5] Before him, [9], [90], [74], [96] and [160] already suggested this phenomenon. For historical commentary on the subject, see [82]. Relaxing the property of transitivity of indifference results in two well-known structures: semi-orders and interval orders.

[5] one can be indifferent between a cup of tea with n milligrams of sugar and one with $n+1$ milligrams of sugar, if one admits the transitivity of the indifference, after a certain step of transitivity, one will have the indifference between a cup of tea with n milligram of sugar and that with $n+N$ milligram of sugar with N large enough, even if there is a very great difference of taste between the two; which is contradictory with the concept of indifference

Definition 5.5 (Semiorder). *Let R be a binary relation on the set A, R being a characteristic relation of* $\langle P, I \rangle$*, the following definitions are equivalent:*
i. R is a semiorder
ii. R is reflexive, complete, Ferrers relation and semitransitive

$$iii. \begin{cases} P.I.P \subset P \\ P^2 \cap I^2 = \emptyset \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

$$iv. \begin{cases} P.I.P \subset P \\ P^2 I \subset P \text{ (or equivalently } IP^2 \subset P\text{)} \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

Definition 5.6 (Interval Order (IO)). *Let R be a binary relation on the set A, R being a characteristic relation of* $\langle P, I \rangle$*, the following definitions are equivalent:*
i. R is an interval order
ii. R is reflexive, complete and Ferrers relation

$$iii. \begin{cases} P.I.P \subset P \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

A detailed study of this structure can be found in [159], [130], [78]. It is easy to see that this structure generalizes all the structures previously introduced.

Can we relax transitivity of preference? Although it might appear counterintuitive there is empirical evidence that such a situation can occur: [125], [197]. Similar work can be found in: [77], [80], [79], [205], [29], [31], [33], [32].

5.2 Extended Structures

The $\langle P, I \rangle$ structures presented in the previous section neither take into account all the decision-maker's attitudes, nor all possible situations. In the literature, there are two non exclusive ways to extend such structures:

- Introduction of several distinct preference relations representing (or more) hesitation(s) between preference and indifference;

- Introduction of one or more situations of incomparability.

5.2.1 Several Preference Relations

One can wish to give more freedom to the decision-maker and allow more detailed preference models, introducing one or more intermediate relations between indifference and preference. Such relations might represent one or more zones of ambiguity and/or uncertainty where it is difficult to make a distinction between preference and indifference. Another way to interpret such "intermediate" relations is to consider them as different "degrees of preference intensity". From a technical point of view these structures are similar and we are not going to further discuss such semantics.

We distinguish two cases: one where only one such intermediate relation is introduced (usually called weak preference and denoted by Q), and another where several such intermediate relations are introduced.

1. $\langle P, Q, I\rangle$ preference structures. In such structures we introduce one more preference relation, denoted by Q which is an asymmetric and irreflexive binary relation. The usual properties of preference structures hold. Usually such structures arise from the use of thresholds when objects with numerical values are compared or, equivalently, when objects whose values are intervals are compared. The reader who wants to have more information on thresholds can go to section 6.1. where all definitions and representation theorems are given.

   $\langle P, Q, I\rangle$ preference structures have been generally discussed in [202]. Two cases are studied in the literature:
   - PQI interval orders and semi-orders (for their characterisation see [196]). The detection of such structures has been shown to be a polynomial problem (see [141]).
   - double threshold orders (for their characterisation see [202], [195]) and more precisely pseudo-orders (see [172], [173]).
   One of the difficulties of such structures is that it is impossible to define P, Q and I from a single characteristic relation R as is the case for other conventional preference structures.

2. $\langle P_1, \cdots P_n\rangle$ preference structures. Practically, such structures generalise the previous situation where just one intermediate relation was considered. Again, such structures arise when multiple thresholds are used in order to compare numerical values of objects. The problem was first introduced in [47] and then extensively studied in [164], [57], [59], see also [198], [133], [3], [58]. Typically such structures concern the coherent representation of multiple interval orders. The particular case of multiple semi-orders was studied in [55].

5.2.2 Incomparability

In the classical preference structures presented in the previous section, the decision-maker is supposed to be able to compare the alternatives (we can have aPb, bPa or aIb). But certain situations, such as lack of information, uncertainty, ambiguity, multi-dimensional and conflicting preferences, can create incomparability between alternatives. Within this framework, the partial structures use a third symmetric and irreflexive relation J ($aJb \Longleftrightarrow not(aPb), not(bPa), not(aIb), not(aQb), not(bQa)$), called incomparability, to deal with this kind of situation. To have a partial structure $\langle P, I, J\rangle$ or $\langle P, Q, I, J\rangle$, we add to the definitions of the preceding structures (total order, weak order, semi-order, interval order and pseudo-order), the relation of incomparability ($J \neq \emptyset$); and we obtain respectively partial order, partial preorder (quasi-order), partial semi-order, partial interval order and partial pseudo-order ([165]).

Definition 5.7 (Partial Order). *Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J\rangle$, the following definitions are equivalent:*
i. R is a partial order.

ii. R is reflexive, antisymmetric, transitive

iii. $\begin{cases} \text{P is asymmetric, transitive} \\ \text{I is reflexive, symmetric} \\ \text{J is irreflexive and symmetric} \\ I = \{(a,a), \forall a \in A\} \end{cases}$

Definition 5.8 (quasi-order). *Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J \rangle$, the following definitions are equivalent:*
i. R is a quasi-order.
ii. R is reflexive, transitive

iii. $\begin{cases} \text{P is asymmetric, transitive} \\ \text{I is reflexive, symmetric and transitive} \\ \text{J is irreflexive and symmetric} \\ (P.I \cup I.P) \subset P \end{cases}$

A fundamental result ([72], [78]) shows that every partial order (resp. partial preorder) on a finite set can be obtained as an intersection of a finite number of total orders (resp. total preorders, see [25]).

A further analysis of the concept of incomparability can be found in [193] and [194]. In these papers it is shown that the number of preference relations that can be introduced in a preference structure, so that it can be represented through a characteristic binary relation, depends on the semantics of the language used for modelling. In other terms, when classical logic is used in order to model preferences, no more than three different relations can be established (if one characteristic relation is used). The introduction of a four-valued logic allows to extend the number of independently defined relations to 10, thus introducing different types of incomparability (and hesitation) due to the different combination of positive and negative reasons (see [191]). It is therefore possible with such a language to consider an incomparability due to ignorance separately from one due to conflicting information.

5.3 Valued Structures

In this section, we present situations where preferences between objets are defined by a valued preference relation such that $\mu(R(a,b))$ represents either the intensity or the credibility of the preference of a over b[6] or the proportion of people who prefer a to b or the number of times that a is preferred to b. In this section, we make use of results cited in [84] and [153]. To simplify the notation, the valued relation $\mu(R(a,b))$ is denoted $R(a,b)$ in the rest of this section. We begin by giving a definition of a valued relation:

Definition 5.9 (Valued Relation). *:*
A valued relation R on the set A is a mapping from the cartesian product A×A onto a bounded subset of $\mathbb{R}$, often the interval [0,1].

[6] this value can be given directly by the decision-maker or calculated by using different concepts, such values (indices) are widely used in many mcda methods such as ELECTRE, PROMETHEE ([169],[35])

Remark 5.1. *A valued relation can be interpreted as a family of crisp nested relations. With such an interpretation, each α-cut level of a fuzzy relation corresponds to a different crisp nested relation.*

In this section, we show some results obtained by the use of fuzzy set theory as a language which is capable to deal with uncertainty. The seminal paper by Orlovsky ([145]) can be considered as the first attempt to use fuzzy set theory in preference modelling. Roy in [167] will also make use of the concept of fuzzy relations in trying to establish the nature of a pseudo-order. In his paper Orlovsky defines the strict preference relation and the indifference relation with the use of Lukasiewicz and min t-norms. After him, a number of researchers were interested in the use of fuzzy sets in decision aiding, most of these works are published in the journal Fuzzy Sets and Systems.

In the following we give some definitions of fuzzy ordered sets. We derive the following definitions from the properties listed in section 4.2:

Definition 5.10 (Fuzzy Total Order). *A binary relation R on the set A, is a fuzzy total order iff:*
- R is antisymmetric, strongly complete and T-transitive

Definition 5.11 (Fuzzy Weak Order). *A binary relation R on the set A is a fuzzy weak order iff:*
- R is strongly complete and transitive

Definition 5.12 (Fuzzy Semi-order). *A binary relation R on the set A is a fuzzy semi-order iff:*
- R is strongly complete, a Ferrers relation and semitransitive

Definition 5.13 (Fuzzy Interval Order (IO)). *A binary relation R on the set A is a fuzzy interval order iff:*
- R is a strongly complete Ferrers relation

Definition 5.14 (Fuzzy Partial Order). *A binary relation R on the set A is a fuzzy partial order iff:*
- R is antisymmetric reflexive and T-transitive

Definition 5.15 (Fuzzy Partial Preorder). *A binary relation R on the set A is a fuzzy partial preorder iff:*
- R is reflexive and T-transitive

All the definitions above are given in terms of the characteristic relation R. The second step is to define valued preference relations (valued strict preference, valued indifference and valued incomparability) in terms of the characteristic relation ([85], [86], [84], [146], [154]). For this, equations 1- 3 are interpreted in terms of fuzzy logical operations:

$$P(a,b) = T[R(a,b), nR(b,a)] \tag{4}$$

$$I(a,b) = T[R(a,b), R(b,a)] \tag{5}$$

$$R(a,b) = S[P(a,b), I(a,b)] \tag{6}$$

However, it is impossible to obtain a result satisfying these three equations using a De Morgan triplet. [7], [84] present this result as an impossibility theorem that proves the non-existence of a single, consistent many-valued logic as a logic of preference. A way to deal with this contradiction is to consider some axioms to define $\langle P, I, J\rangle$. Fodor, Ovchinnikov, Roubens propose to define three general axioms that they call Independence of Irrelevant Alternatives (IA), Positive Association (PA), Symmetry (SY). With their axioms, the following propositions hold:

Proposition 5.1 (Fuzzy Weak Order). *if $\langle P, I\rangle$ is a fuzzy weak order then:*
- P is a fuzzy strict partial order
- I is a fuzzy similarity relation (reflexive, symmetric, transitive)

Proposition 5.2 (Fuzzy Semi-order). *if $\langle P, I\rangle$ is a fuzzy semi-order then:*
- P is a fuzzy strict partial order
- I is not transitive

Proposition 5.3 (Fuzzy Interval Order (IO)). *if $\langle P, I\rangle$ is a fuzzy interval order then:*
- P is a fuzzy strict partial order
- I is not transitive

De Baets, Van de Walle and Kerre ([48], [200], [201]) define the valued preference relations without considering a characteristic relation:

P is T-asymmetric ($P \cap_T P^{-1}$)= $\emptyset$
I is reflexive and J is irreflexive ($I(a,a)$=1, $J(a,a)$=0 $\forall a \in A$)
I and J are symmetric ($I = I^{-1}$, $J = J^{-1}$)
$P \cap_T I = \emptyset$, $P \cap_T J = \emptyset$, $I \cap_T J = \emptyset$
$P \cup_T P^{-1} \cup_T I \cup_T$ = $A \times A$

With a continuous t-norm and without zero divisors, these properties are satisfied only in crisp case. To deal with this problem, we have to consider a continuous t-norm with zero divisor.

In multiple criteria decision aiding, we can make use of fuzzy sets in different ways. One of these helps to construct a valued preference relation from the crisp values of alternatives on each criteria. We cite the proposition of Perny and Roy ([154]) as an example here. They define a fuzzy outranking relation R from a real valued function θ defined on $\mathbb{R} \times \mathbb{R}$, such that $R(a,b) = \theta(g(a), g(b))$ verifies the following conditions for all a, b in A:

$$\forall y \in X, \quad \theta(x,y) \quad \text{is a nondecreasing function of } x \tag{7}$$
$$\forall x \in X, \quad \theta(x,y) \quad \text{is a nonincreasing function of } y \tag{8}$$
$$\forall z \in X, \quad \theta(z,z) = 1 \tag{9}$$

The resulting relation R is a fuzzy semi-order (i.e. reflexive, complete, semi-transitive and Ferrers fuzzy relation). Roy (1978) proposed in Electre III to define the outranking relation R

characterized by a function θ for each criterion as follows:

$$\theta(x, y) = \frac{p(x) - min\{y - x, p(x)\}}{p(x) - min\{y - x, q(x)\}}$$

Where $p(x)$ and $q(x)$ are thresholds of the selected criteria.

We may work with alternatives representing some imprecision or ambiguity for a criterion. In this case, we make use of fuzzy sets to define the evaluation of the alternative related to the criterion. In the ordered pair $\{x, \mu_j^a\}$, μ_j^a represents the grade of membership of x for alternative a related to the criterion j. The fuzzy set μ is supposed to be normal ($sup_x(\mu_j^a) = 1$) and convex ($\forall x, y, z \in \mathbb{R}$, $y \in [x, z]$, $\mu_j^a(y) \leq min\{\mu_j^a(x), \mu_j^a(z)\}$). The credibility of the preference of a over b is obtained from the comparison of the fuzzy intervals (normal, convex fuzzy sets) of a and b with some conditions:

- The method used should be sensitive to the specific range and shape of the grades of membership.

- The method should be independent of the irrelevant alternatives.

-The method should satisfy transitivity.

Fodor and Roubens ([84]) propose to use two procedures.

In the first one, the credibility of the preference of a over b for j is defined as the possibility that $a \geq b$:

$$\Pi_j(a \geq b) = \bigvee_{x \geq y} [\mu_j^a(x) \wedge \mu_j^b(y)] = sup_{x \geq y}[min(\mu_j^a(x), \mu_j^b(y))] \tag{10}$$

The credibility as defined by 10 is a fuzzy interval order (Π_j is reflexive, complete and a Ferrers relation) and

$$min(\Pi_j(a, b), \Pi_j(b, a)) = sup_x min(\mu_j^a(x), \mu_j^b(x))$$

In the case of a symmetrical fuzzy interval (μ^a), the parameters of the fuzzy interval can be defined in terms of the valuation $g_j(a)$ and thresholds $p(g_j(a)$ and $q(g_j(a)$. Some examples using trapezoidal fuzzy numbers can be found in the work of Fodor and Roubens.

The second procedure proposed by Fodor and Roubens makes use of the shapes of membership functions, satisfies the three axioms cited at the beginning of the section and gives the credibility of preference and indifference as follows:

$$P_j(a, b) = R_j^d(a, b) = 1 - \Pi_j(b \geq a) = N_j(a > b) \tag{11}$$

$$I_j(a, b) = min[\Pi_j(a \geq b), \Pi_j(b \geq a)] \tag{12}$$

Where Π (the possibility degree) and N (the necessity degree) are two dual distributions of the possibility theory that are related to each other with the equality: $\Pi(A) = 1 - N(A)$ (see [71] for an axiomatic definition of the theory of possibility).

6 Domains and Numerical Representations

In this section we present a number of results concerning the numerical representation of the preference structures introduced in the previous section. This is an important operational problem. Given a set A and a set of preference relations holding between the elements of A, it is important to know whether such preferences fit a precise preference structure admitting a numerical representation. If this is the case, it is possible to replace the elements of A with their numerical values and then work with these. Otherwise, when to the set A is already associated a numerical representation (for instance a measure), it is important to test which preference structure should be applied in order to faithfully interpret the decision-maker's preferences ([204]).

6.1 Representation Theorems

Theorem 6.1 (Total Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

i. R is a total order structure (see 5.3)

ii. $\exists$ g: $A \mapsto \mathbb{R}^+$ satisfying for $\forall a, b \in A$:
$$\begin{cases} aPb \quad \textit{iff} \quad g(a) > g(b) \\ a \neq b \quad \Longrightarrow \quad g(a) \neq g(b) \end{cases}$$

iii. $\exists$ g: $A \mapsto \mathbb{R}^+$ satisfying for $\forall a, b \in A$:
$$\begin{cases} aRb \quad \textit{iff} \quad g(a) > g(b) \\ a \neq b \quad \Longrightarrow \quad g(a) \neq g(b) \end{cases}$$

In the infinite not enumerable case, it can be impossible to find a numerical representation of a total order. For a detailed discussion on the subject, see [16]. The necessary and sufficient conditions to have a numerical representation for a total order are present in many works: [50], [75], [117], [36].

Theorem 6.2 (Weak Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

i. R is a weak order structure (see 5.4)

ii. $\exists$ g: $A \mapsto \mathbb{R}^+$ satisfying for $\forall a, b \in A$:
$$\begin{cases} aPb \quad \textit{iff} \quad g(a) > g(b) \\ aIb \quad \textit{iff} \quad g(a) = g(b) \end{cases}$$

iii. $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a, b \in A$*:*
aRb *iff* $g(a) \geq g(b)$

Remark 6.1. *Numerical representations of preference structures are not unique. All monotonic strictly increasing transformations of the function g can be interpreted as equivalent numerical representations*[7].

Intransitivity of indifference or the appearance of intermediate hesitation relations is due to the use of thresholds that can be constant or dependent on the value of the objects under comparison (in this case values of the threshold might obey further coherence conditions).

Theorem 6.3 (Semi-Order). *Let* $R = \langle P, I \rangle$ *be a binary relation on a finite set A, the following definitions are equivalent:*

i. R is a semi-order structure (see 5.5)

ii. $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a, b \in A$*:*

$$\begin{cases} aPb & \textit{iff} \quad g(a) > g(b) + q \\ aIb & \textit{iff} \quad |g(a) - g(b| \leq q \end{cases}$$

iii. $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a, b \in A$*:*
aRb *iff* $g(a) \geq g(b) - q$

iv. $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *and* $\exists q : \mathbb{R} \mapsto \mathbb{R}^+$ *satisfying* $\forall a, b \in A$*:*

$$\begin{cases} aRb & \textit{iff} & g(a) \geq g(b) - q(g(b)) \\ (g(a) > g(b)) & \longrightarrow & (g(a) + q(g(a)) \geq g(b) + q(g(b))) \end{cases}$$

For the proofs of these theorems see [176], [78], [182], [159].

The threshold represents a quantity for which any difference smaller than this one is not significant for the preference relation. As we can see, the threshold is not necessarily constant, but if it is not, it must satisfy the inequality which defines a coherence condition.

Here too, the representation of a semi-order is not unique and all monotonic increasing transformations of g appear as admissible representations provided the condition that the function q also obeys the same transformation [8].

Theorem 6.4 (PI Interval Order). *Let* $R = \langle P, I \rangle$ *be a binary relation on a finite set A, the following definitions are equivalent:*

i. R is an interval order structure (see 5.6)

ii. $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *satisfying* $\forall a, b \in A$*:*

$$\begin{cases} aPb & \textit{iff} \quad g(a) > g(b) + q(b) \\ aIb & \textit{iff} \quad \begin{matrix} g(a) \leq g(b) + q(b) \\ g(b) \leq g(a) + q(a) \end{matrix} \end{cases}$$

[7] the function g defines an ordinal scale for both structures

[8] but in this case the scale defined by g is more complex than an ordinal scale

It should be noted that the main difference between an interval order and a semi-order is the existence of a coherence condition on the value of the threshold. One can further generalise the structure of interval order, by defining a threshold depending on both of the two alternatives. As a result, the asymmetric part appears without circuit: [3], [4],[181], [2], [53], [5]. For extensions on the use of thresholds see [81], [132], [98]. For the extension of the numerical representation of interval orders in the case A is infinite not denumerable see [76], [40], [36], [144], [138].

We can now see the representation theorems concerning preference structures allowing an intermediate preference relation (Q). Before that, let us mention that numerical representations with thresholds are equivalent to numerical representations of intervals. It is sufficient to note that associating a value $g(x)$ and a strictly positive value $q(g(x))$ to each element x of A is equivalent to associating two values: $l(x) = g(x)$ (representing the left extreme of an interval) and $r(x) = g(x) + q(g(x))$ (representing the right extreme of the interval to each x; obviously: $r(x) > l(x)$ always holds).

Theorem 6.5 (PQI Interval Orders). *Let $R = \langle P, Q, I\rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a PQI interval Order

ii.There exists a partial order L such that:
1) $I = L \cup R \cup I_d$ where $I_d = \{(x, x), x \in A\}$ and $R = L^{-1}$;
2) $(P \cup Q \cup L).P \subset P$; *3) $P.(P \cup Q \cup R) \subset P$;*
4) $(P \cup Q \cup L).Q \subset P \cup Q \cup L$; *5) $Q.(P \cup Q \cup R) \subset P \cup Q \cup R$.*

iii. $\exists$ l,r: $A \mapsto \mathbb{R}^+$ satisfying:

$$\begin{cases} r(a) \geq l(a) & \\ aPb & \textit{iff} \quad l(a) > r(b) \\ aQb & \textit{iff} \quad r(a) > r(b) \geq l(a) \geq l(b) \\ aIb & \textit{iff} \quad r(a) \geq r(b) \geq l(a) \quad \textit{or} \quad r(b) \geq r(a) \geq l(a) \geq l(b) \end{cases}$$

For proofs, further theory on the numerical representation and algorithmic issues associated with such a structure see [196], [141], [140].

Theorem 6.6 (Double Threshold Order). *Let $R = \langle P, Q, I\rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a double Threshold Order (see [202])

ii. $$\begin{cases} Q.I.Q \subset Q \cup P \\ P.I.P \subset P \\ Q.I.P \subset P \\ P.Q^{-1}.P \subset P \end{cases}$$

iii. $\exists$ *g,q,p:* $A \mapsto \mathbb{R}^+$ *satisfying:*

$$\begin{cases} aPb & \textit{iff} \quad g(a) > g(b) + p(b)) \\ aQb & \textit{iff} \quad g(b) + p(b) \geq g(a) > g(b) + q(b) \\ aIb & \textit{iff} \qquad g(b) + q(b) > g(a) > g(b) - q(a) \end{cases}$$

Theorem 6.7 (pseudo-order). *Let $R = \langle P, Q, I \rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a pseudo-order

$$ii. \begin{cases} \textit{is a double threshold order} \\ \langle (P \cup Q), I \rangle \textit{ is a semi-order} \\ \langle P, (Q \cup I \cup Q^{-1}) \rangle \textit{is a semi-order} \\ P.I.Q \subset P \end{cases}$$

$$iii. \begin{cases} \textit{is a double threshold order} \\ g(a) > g(b) \Longleftrightarrow \qquad g(a) + q(a) > g(b) + q(b) \\ \qquad\qquad\qquad\qquad g(a) + p(a) > g(b) + p(b) \end{cases}$$

A pseudo-order is a particular case of double threshold order, such that the thresholds fulfil a coherence condition. It should be noted however, that such a coherence is not sufficient in order to obtain two constant thresholds. This is due to different ways in which the two functions can be defined (see [59]). For the existence of multiple constant thresholds see [55].

For partial structures of preference, the functional representations admit the same formulas, but equivalences are replaced by implications. In the following, we present a numerical representation of a partial order and a quasi-order examples:

Theorem 6.8 (Partial Order). *If $\langle P, I, J \rangle$ presents a partial order structure,then* $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *such that:*

$$\left\{ \quad aPb \quad \Longrightarrow \quad g(a) > g(b) \right.$$

Theorem 6.9 (Partial Weak order). *If $\langle P, I, J \rangle$ presents a partial weak order structure, then* $\exists$ *g:* $A \mapsto \mathbb{R}^+$ *such that:*

$$\begin{cases} aPb \implies g(a) > g(b) \\ aIb \implies g(a) = g(b) \end{cases}$$

The detection of the dimension of a partial order [9] is a NP hard problem ([57], [78]).

Remark 6.2. *In the preference modelling used in decision aiding, there exist two different approaches: In the first one, the evaluations of alternatives are known (they can be crisp or fuzzy)*

[9]when it is a partial order of dimension 2, the detection can be made in a polynomial time

and we try to reach conclusions about the preferences between the alternatives. For the second one, the preferences between alternatives (pairwise comparison) are given by an expert (or by a group of experts), and we try to define an evaluation of the alternatives that can be useful. The first approach uses the inverse implication of the equivalences presented above (for example for a total order we have $g(a) > g(b) \longrightarrow aPb$); and the second one the other implication of it (for the same example, we have $aPb \longrightarrow g(a) > g(b)$)

Remark 6.3. *There is a body of research on the approximation of a preference structure by another one; here we cite some studies on the research of a total order with a minimum distance to a tournament (complete and antisymmetric relation): [179], [24], [131], [14], [13], [39], [105],*

6.2 Minimal Representation

In some decision aiding situations, the only available preferential information can be the kind of preference relation holding between each pair of alternatives. In such a case we can try to build a numerical representation of each alternative by choosing a particular functional representation of the ordered set in question and associating this with the known qualitative relations.

This section aims at studying some minimal or parsimonious representations of ordered sets, which can be helpful for this kind of situation. Particularly, given a countable set A and a preference relation $R \subseteq A \times A$, we are interested to find a numerical representation $\hat{f} \in \mathcal{F} = \{f : A \mapsto \mathbb{R},\ f$ homomorph to $R\}$, such that for all $x \in A$, $\hat{f}$ is minimal.

6.2.1 Total Order, Weak order

The way to build a minimal representation for a total order or a weak order is obvious since the preference and the indifference relations are transitive: The idea is to minimize the value of the difference $g(a) - g(b)$ for all a, b in A. To do this we can define a unit $k = \min_{a,b \in A}(g(a) - g(b))$ and the minimal evaluation $m = \min_{a \in A}(g(a))$. The algorithm will be:

- Choose any value for k and m , e.g. $k = 1$, $m = 0$;
- Find the alternative i which is dominated by all the other alternatives j in A and evaluate it by $g(i) = m$
- For all the alternatives l for which we have lIi, note $g(l) = g(i)$
- Find the alternative i' which is dominated by all the alternatives j' in $A - \{i\}$ and evaluate it by $g(i') = m + k$
- For all the alternatives l' for which we have $l'Ii'$, note $g(l') = g(i')$
- Stop when all the alternatives are evaluated

6.2.2 Semi-order

The first study on the minimal representation of semi-orders was done in [158] who proved its existence and proposed an algorithm to build it. One can find more information about this in [56], [127], [159] and [139]. Pirlot uses an equivalent definition of the semi-order which uses a second positive constant: *Total Semi-order*: A reflexive relation $R = (P, I)$ on a finite set A is a semi-order iff there exists a real function g, defined on A, a non negative constant q and a positive constant ε such that $\forall a, b \in A$

$$\begin{cases} aPb & \text{iff} \quad g(a) > g(b) + q + \varepsilon \\ aIb & \text{iff} \quad |g(a) - g(b)| \leq q \end{cases}$$

Such a triple (g, q, ε) is called an $\varepsilon - representation$ of (P, I). Any representation (g, q), as in the definition of semi-order given in 5.1, yields an ε-representation where

$$\varepsilon = min_{(a,b) \in P}(g(a) - g(b) - q)$$

Let (A, R) be an associated to the semi-order $R = (P, I)$, we denote $G(q, \varepsilon)$ the valued graph obtained by giving the value ($q + \varepsilon$) to the arcs P and ($-q$) to the arcs I.

Theorem 6.10. *: If R=(P, I) is a semi-order on the finite set A, there exists an ε-representation with threshold q iff:*

$$\frac{q}{\varepsilon} \geq \alpha = max_C\{\frac{|C \cap P|}{|C \cap I| - |C \cap P|}, \text{ C circuit of (A, R)}\}$$

where $|C \cap P|$ (resp. $|C \cap I|$), represents the number of arcs P (resp. I) in the circuit C of the graph (A, R).

An algorithm to find a numerical representation of a semi-order is as follows:

- Choose any value for εk, e.g. ε= 1;
- Choose a large enough value of $\frac{q}{\varepsilon}$(e.g. $\frac{q}{\varepsilon} = |P|$;
- Solve the maximal value path problem in the graph $G(q, \varepsilon)$ (e.g. by using the Bellman algorithm, see [120]).

Denote by $g_{q,\varepsilon}$, the solution of the maximal path problem in $G(q, \varepsilon)$; we have:

$$g_{q,\varepsilon} \leq g(a) \forall a \in A$$

Example 6.1. *We consider the example given by Pirlot and Vincke ([159]):*
Let $S = (P, I)$ be a semiorder on $A = \{a, b, c\}$ defined by $P = \{(a, c)\}$.

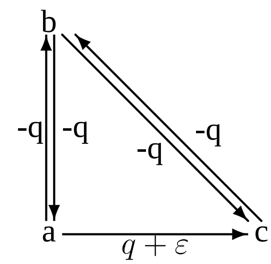

Figure 3: Graphical representation of the semiorder

		a	b	c
q=1	g_1=1	2	1	0
q=1	g_2=1	9.5	8.5	7.5
q=2.5	g_3=1	3.5	1	0
q=2.5	g_4=1	10.5	8.5	7
q=2.5	g_5=1	3.5	2.5	0

Table 2: Various ε-representations with ε=1

The first inequality of 6.2.2 gives the following equations:

$$
\begin{aligned}
g(a) &\geq g(c) + q + \varepsilon \\
g(a) &\geq g(b) - q \\
g(b) &\geq g(a) - q \\
g(b) &\geq g(c) - q \\
g(c) &\geq g(b) - q
\end{aligned}
$$

Figure 3 shows the graphical representation of this semiorder.

As the non-trivial circuit $C = \{(a, c), (c, b), (b, a)\}$ is $-q+\varepsilon$ ($-q+\varepsilon = (q+\varepsilon)+(-q)+(-q)$), necessary and sufficient conditions for the existence of an ε-representation is $q \geq \varepsilon$.
The table2 provides an example of possible numerical representation of this semiorder:

Definition 6.1. *A representation (g^*, q^*, ε) is minimal in the set of all non-negative ε-representations (g, q, ε) of a semiorder iff $\forall a \in A$ $g^*(a) \leq g(a)$.*

Theorem 6.11. *The representation ($g_{q^*,\varepsilon}, q^*, \varepsilon$) is minimal in the set of all ε-representations of a semiorder R.*

6.2.3 Interval Order

An interval can be represented by two real functions l and r on the finite set A which satisfy:

$$(\forall a \in A, l(a) \leq r(a))^{10}$$

Definition 6.2. *A reflexive relation R = $(P \cup I)$ on a finite set A is an interval order iff there exists a pair of functions l, r: A $\longrightarrow R^{+}$ and a positive constant ε such that $\forall a, b \in A$*

$$\begin{cases} aPb & \textit{iff} \quad l(a) > r(b) + q + \varepsilon \\ aIb & \textit{iff} \quad l(a) \geq r(b) \quad \textit{and} \quad l(b) \geq r(a) \end{cases}$$

Such a triplet (l, r, ε) is called an ε-representation of the interval order $P \cup I$.

Definition 6.3. *The ε-representation (l^*, r^*, ε) of the interval order $P \cup I$ is minimal iff for any other ε-representation (l, r, ε) we have, $\forall a \in A$,*

$$l^*(a) \leq l(a)$$
$$r^*(a) \leq r(a)$$

Theorem 6.12. *For any interval order $P \cup I$, there exists a minimal ε-representation (l^*, r^*, ε); the values of l^* and r^* are integral multiples of ε.*

7 Logic of Preferences

The increasing importance of preference modelling immediately interested people from other disciplines, particularly logicians and philosophers. The strict relation with deontic logic (see [1]) raised some questions such as:
- does a general logic exist where any preferences can be represented and used?
- if yes, what is the language and what are the axioms?
- is it possible, via this formalisation, to give a definition of bad or good as absolute values?

It is clear that this attempt had a clear positivist and normative objective: to define the one well-formed logic that people should follow when expressing preferences. The first work on the subject is the one by [94], but it is Von Wright's book ([207]) that tries to give the first axiomatisation of a logic of preferences. Inspired by this work some important contributions have been made ([102], [42], [41], [161], [100], [99]). Influence of this idea can also be found in [108] and [162], but in related fields (statistics and value theory respectively). The discussion apparently was concluded by [208], but [103], [104] continued on later on [95] and [210], [211] also worked on this.

The general idea can be presented as follows. At least two questions should be clarified: preferences among what? How should preferences be understood? Von Wright ([207]) argues that

[10] One can imagine that $l(a)$ represents the evaluation of the alternative a ($g(a)$) which is the left limit of the interval and $r(a)$ represents the value of ($g(a) + q(a)$) which is the right limit of the interval. One can remark that a semi-order is an interval order with a constant length.

preferences can be distinguished as extrinsic and intrinsic. The first ones are derived as *a reason from a specific purpose,* while the second ones are *self-referential* to an actor expressing the preferences. In this sense intrinsic preferences are the expression of the actor's system of values of the actor. Moreover, preferences can be expressed for different things, the most general being (following Von Wright) *"states of affairs"*. That is, the expression "a is preferred to b" should be understood as the preference of a state (a world) where a occurs (whatever a represents: sentences, objects, relations etc.) over a state where b occurs. On the basis of Von Wright expressed a theory based on five axioms:

A^{W}1. $\forall x, y \;\; p(x, y) \;\rightarrow\; \neg p(y, x)$
A^{W}2. $\forall x, y, z \;\; p(x, y) \;\wedge\; p(y, z) \;\rightarrow\; p(x, z)$
A^{W}3. $p(a, b) \;\equiv\; p(a \wedge \neg b, \neg a \wedge b)$
A^{W}4. $p(a \vee b, c) \;\equiv\; p(a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c) \wedge p(a \wedge \neg b \wedge \neg c, \neg a \wedge \neg b \wedge c) \wedge p(\neg a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c)$
A^{W}5. $p(a, b) \;\equiv\; p(a \wedge c, b \wedge c) \;\wedge\; p(a \wedge \neg c, b \wedge \neg c)$

The first two axioms are asymmetry and transitivity of the preference relation, while the following three axioms face the problem of combinations of states of affairs. The use of specific elements instead of the variables and quantifiers reflects the fact that von Wright considered the axioms not as logical ones, but as "reasoning principles". This distinction has important consequences on the calculus level. In the first two axioms, preference is considered as a binary relation (therefore the use of a predicate), in the three "principles", preference is a proposition. Von Wright does not make this distinction directly, considering the expression aPb ($p(a, b)$ in our notation) as a well-formed formulation of his logic. However, this does not change the problem since the first two axioms are referred to the binary relation and the others are not. The difference appears if one tries to introduce quantifications; in this case the three principles appear to be weak. The problem with this axiomatisation is that empirical observation of human behavior provides counterexamples of these axioms. Moreover, from a philosophical point of view (following the normative objective that this approach assumed), a logic of intrinsic preferences about general states of affairs should allow to define what is good (the always preferred?) and what is bad (the always not preferred?). But this axiomatization fails to enable such a definition.

Chisholm and Sosa ([42]) rejected axioms A^{W}3 to A^{W}5 and built an alternative axiomatization based on the concepts of *"good"* and *"intrinsically better"*. Their idea is to postulate the concept of good and to axiomatize preferences consequently. So a *good* state of affairs is one that is always preferred to its negation ($p(a, \neg a)$); Chisholm and Sosa, use this definition only for its operational potential as they argue that it does not capture the whole concept of "good"). In this case we have:

A^{S}1. $\forall x, y \;\; p(x, y) \;\rightarrow\; \neg p(y, x)$
A^{S}2. $\forall x, y, z \;\; \neg p(x, y) \;\wedge\; \neg p(y, z) \;\rightarrow\; \neg p(x, z)$
A^{S}3. $\forall x, y \;\; \neg p(x, \neg x) \;\wedge\; \neg p(\neg x, x) \;\wedge\; \neg p(y, \neg y) \;\wedge\; \neg p(\neg y, y) \;\rightarrow\; \neg p(y, x) \;\wedge\; \neg p(x, y)$
A^{S}4. $\forall x, y \;\; p(x, y) \;\wedge\; \neg p(y, \neg y) \;\wedge\; \neg p(\neg y, y) \;\rightarrow\; p(x, \neg x)$
A^{S}5. $\forall x, y \;\; p(y, \neg x) \;\wedge\; \neg p(y, \neg y) \;\wedge\; \neg p(\neg y, y) \;\rightarrow\; p(x, \neg x)$

Again in this axiomatisation there are counterexamples of the axioms. The assumption of the concept of good can be argued as it allows circularities in the definitions of preferences between combinations of states of affairs. This criticism leaded ([100]) to consider only two fundamental, universally recognised axioms:

A^{H}1. $\forall x, y, z \;\; s(x, y) \;\wedge\; s(y, z) \;\rightarrow\; s(x, z)$
A^{H}2. $\forall x, y \;\; s(x, y) \;\vee\; s(y, x)$

where s is a "large preference relation" and two specific preference relations are defined, p (strict preference) and i (indifference):

D^{H}1. $\forall x, y \;\; p(x, y) \;\equiv\; s(x, y) \;\wedge\; \neg s(y, x)$
D^{H}2. $\forall x, y \;\; i(x, y) \;\equiv\; s(x, y) \;\wedge\; s(y, x)$

He also introduces two more axioms, although he recognises their controversial nature:

A^{H}3. $\forall x, y, z \;\; s(x, y) \;\wedge\; s(x, z) \;\rightarrow\; s(x, y \vee z)$
A^{H}4. $\forall x, y, z \;\; s(x, z) \;\wedge\; s(y, z) \;\rightarrow\; s(x \vee y, z)$

Von Wright in his reply ([208]), trying to argue for his theory, introduced a more general frame to define intrinsic *"holistic"* preferences or as he called them *"ceteris paribus"* preferences. In this approach he considers a set S of states where the elements are the ones of A (n elements) and all the 2^n combinations of these elements. Given two states s and t (elementary or combinations of m states of S) you have i ($i = 2^{n-m}$) combinations C_i of the other states. You call an s-world any state that holds when s holds. A combination C_i of states is also a state so you can define it in the same way a C_i-world. Von Wright gives two definitions (strong and weak) of preference:

1. (strong): s is preferred to t under the circumstances C_i **iff** every C_i-world that is also an s-world and not a t-world is preferred to every C_i-world that is also t-world and not s-world.
2. (weak): s is preferred to t under the circumstances C_i **iff** some C_i-world that is an s-world is preferred to a C_i-world that is a t-world, but a C_i-world that is a t-world that is preferred to a C_i-world that is an s-world does not exist.

Now s is *"ceteris paribus"* preferred to t **iff** it is preferred under all C_i. We leave the discussion to the interested reader, but we point out that, with these definitions, it is difficult to axiomatize both transitivity and complete comparability unless they are assumed as necessary truths for "coherence" and "rationality" (see [208]).

It can be concluded that the philosophical discussion about preferences failed the objective to give a unifying frame of generalized preference relations that could hold for any kind of states, based on a well-defined axiomatization (for an interesting discussion see [135]). It is still difficult (if not impossible) to give a definition of good or bad in absolute terms based on reasoning about preferences and the properties of these relations are not unanimously accepted as axioms of preference modelling. For more recent advances in deontic logic see [143].

More recently, Von Wright's ideas and the discussion about "logical representation of preferences" attracted attention again. This is due to problems found in the field of Artificial Intelligence field due to essentially two reasons:
- the necessity to introduce some "preferential reasoning" (see [26], [27], [34], [62], [63], [64], [118], [121], [178]);
- the large dimension of the sets to which such a reasoning might apply, thus demanding a compact representation of preferences (see [21], [19], [20], [61], [119])

8 Conclusion

We hope that this paper on preference modelling, gave the non-specialist reader a general idea of the field by providing a list of the most important references of a very vast and technical literature. In this paper, we have tried to present the necessary technical support for the reader to understand most of the relevant literature. One can note that our survey does not interpret all the questions related to preference modelling. Let us mention some of them:
- How to get and validate preference information ([206]), [12]
- Relation between preference modelling and the problem of signifiance in measurement theory ([163])
- Statistical analysis of preferential data ([44], [93])
- Interrogations on the relations between preferences and the value system, and the nature of these values([37], [46], [192], [207]).

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