The simplest possible model with criteria interactions

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UMONS

Sorting Boolean vectors in 2 categories

- Alternatives or objects are Boolean vectors $x = (x_1, \ldots, x_n)$
- They are sorted, monotonically, in two categories $\mathcal{G} =$ "good" and $\mathcal{B} =$ "bad"

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- a "positive" Boolean function f

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- "positive" means "non-decreasing"

$$x \geq y \quad \Rightarrow \quad f(x) \geq f(y)$$

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- $f(x) = 1$ means that x is a sufficient (or winning) coalition (SC) of criteria

Threshold functions

Some assignment rules f can be represented by additive weights and a threshold

Threshold Boolean functions

$$\begin{aligned} f(x) = 1 \quad &\Leftrightarrow \quad \sum_{i=1}^{n} w_i x_i \geq \lambda \\ &\Leftrightarrow \quad \sum_{i \in A} w_i \geq \lambda \quad \text{for } x = 1_A \end{aligned}$$

- The true points of f ($f = 1$) can be separated from the false points ($f = 0$) by an affine function

Examples

Example 1

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- $x = (1110) \quad \rightarrow \quad \sum_{i=1}^{4} = 3/4 \geq \lambda$

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$$\begin{cases} w_1 + w_3 & \geq & \lambda \\ w_1 + w_4 & \geq & \lambda \\ w_2 + w_3 & \geq & \lambda \\ w_2 + w_4 & \geq & \lambda \\ w_1 + w_2 & < & \lambda \\ w_3 + w_4 & < & \lambda \end{cases}$$

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- summing up the first four inequalities, we get that $\lambda \leq 1/2 \sum_{i=1}^{4} w_i$;

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- summing up the first four inequalities, we get that $\lambda \leq 1/2 \sum_{i=1}^{4} w_i$;
- summing up the last two yields $\lambda > 1/2 \sum_{i=1}^{4} w_i$.

k-additive positive Boolean functions

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$$p(x) = \sum_{A:|A|\leq k} c(A) \prod_{i\in A} x_i$$

and

$$f(x) = 1 \quad \Leftrightarrow \quad p(x) \geq 0$$

Cases for $p(x) = \sum_{A:|A|\leq k} c(A) \prod_{i\in A} x_i \geq 0$

$k = 1$: threshold functions

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Example 2 : Sufficient Coalitions $= \{13, 14, 23, 24\}$

$\mathcal{G}$ and $\mathcal{B}$ can be separated by a multilinear polynomial of degree 2

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Interpretation

There is a negative synergy between 1,2 and between 3,4

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Another interpretation is possible

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There are positive synergies between $\{1, 3\}$; $\{1, 4\}$; $\{2, 3\}$ and between $\{2, 4\}$

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- For example 2, some non-null interaction is needed
- For example 1, the separating function can be linear ($k = 1$)

Enumerating and categorizing positive Boolean functions up to $k = 6$

n	$\mathcal{C}_1$		$\mathcal{C}_2$		$\mathcal{C}_3$		$R(n)$
0	2	(100.0 %)	0	(00.00 %)	0	(00.00 %)	2
1	3	(100.0 %)	0	(00.00 %)	0	(00.00 %)	3
2	5	(100.0 %)	0	(00.00 %)	0	(00.00 %)	5
3	10	(100.0 %)	0	(00.00 %)	0	(00.00 %)	10
4	27	(90.00 %)	3	(10.00 %)	0	(00.00 %)	30
5	119	(56.67 %)	91	(43.33 %)	0	(00.00 %)	210
6	1113	(06.81 %)	14902	(91.13 %)	338	(02.07 %)	16 353

Table: Number and proportion of inequivalent families of SCs that are representable by a 1-, 2- or 3-additive capacity

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- Up to $n = 5$, all positive Boolean functions are $k = 2$-additive
- In other words, it can always be avoided to use 3-interactions up to $n = 5$

Examples requiring 3-interactions for $n = 6$

Families of Minimal Sufficient Coalitions

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Reference : Enumerating and categorizing positive Boolean functions separable by a *k*-additive capacity. E. Ersek Uyanık, O.Sobrie, V. Mousseau, M. Pirlot, DAM9907 Discrete Applied Mathematics, to appear.