Strengthened Hardness for Approximating Minimum Unique Game and Small Set Expansion

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Abstract. In this paper, the author strengthens the previous known hardness for approximating Minimum Unique Game Problem, $5/4 - \epsilon$, by proving that Min 2Lin-2 is NP-hard to be approximated within $2 - \epsilon$ and strengthens the previous known hardness for approximating Small Set Expansion Problem, $4/3 - \epsilon$, by proving that Min Bisection is NP-hard to be approximated within $3 - \epsilon$, assuming a variation of Feige’s Hypothesis, which claims that refuting Unbalanced Biased Max 3XOR is NP-hard on average on a natural distribution.

In Unique Game Problem (UGP), we are given a graph $G = (V, E)$, and a set of labels, $[k]$. Each edge $e = (u, v)$ in the graph is equipped with a permutation $\pi_e : [k] \to [k]$. The solution of the problem is a labeling $f : V \to [k]$ that assigns a label to each vertex of $G$. An edge $e = (u, v)$ is said to be satisfied under $f$ if $\pi_e(f(u)) = f(v)$. The goal of the problem is to find a labeling such that the number of the satisfied edges under this labeling is maximized. The value of the instance $Val(I)$ is defined as the maximum fraction of the satisfied edges under all labeling. In the same situation of UGP, the goal of Minimum Unique Game Problem (Min UGP) is to find a labeling such that the number of the unsatisfied edges under this labeling is minimized. The value of the instance $Val(I)$ is defined as the minimum fraction of the unsatisfied edges under all labelling.

In Max 2Lin-2, we are given a set of linear equations over $GF_2$. Each equation contains exactly two variables. The goal of the problem is to seek an assignment of the variables such that the number of satisfied equations is maximized. In the same situation of Max 2Lin-2, the goal of Min 2Lin-2 is to seek an assignment of the variables such that the number of unsatisfied equations is minimized. Note that Max 2Lin-2 is a special case of UGP, and Min 2Lin-2 is special case of Min UGP.

The Unique Game Conjecture (UGC)[1] states: for every $\zeta, \delta > 0$, there is a $k = k(\zeta, \delta)$ such that given an instance $I$ of UGP with $k$ labels it is NP-hard to distinguish whether $Val(I) > 1 - \zeta$ or $Val(I) < \delta$. The $(c, s)$-approximation NP-hardness of UGP is defined as: for some fixed $0 < s < c < 1$, there is a $k$ such that given an instance $I$ of UGP with $k$ labels it is NP-hard to distinguish whether $Val(I) \geq c$ or $Val(I) < s + \epsilon$ for any $\epsilon > 0$. The $(c', s')$-approximation NP-hardness of Min UGP is defined as: for some fixed $0 < c' < s' < 1$, there is a $k$ such that given an instance $I$ of Min UGP with $k$ labels it is NP-hard to distinguish whether $Val(I) > s' - \epsilon$ or $Val(I) < c' + \epsilon$ for any $\epsilon > 0$. 
In Small Set Expansion Problem (SSE), we are given a graph $G = (V, E)$ and a constant $0 < \delta \leq 1/2$. The goal of the problem is to find a subset $S \subseteq V$ satisfying $|S|/|V| = \delta$ such that $\Phi(S)$, the edge expansion of $S$ is minimized. The edge expansion $\Phi(S)$ of a subset $S \subseteq V$ is defined as: $\Phi(S) = \frac{|E(S, V \setminus S)|}{|S|}$. The expansion profile is defined as: $\Phi_{G}(\delta) = \min_{|S|=\delta} \Phi(S)$, where $0 < \delta \leq 1/2$. The Small Set Expansion Hypothesis (SSEH) states: for every $\eta > 0$, there is a $\delta$ such that it is NP-hard to distinguish whether $\Phi_{G}(\delta) > 1 - \eta$ or $\Phi_{G}(\delta) < \eta$. As a special case of Small Set Expansion Problem, Min Bisection is defined as: given a graph $G$ with $n$ vertices, where $n$ is even, find a set $S$ of $n/2$ vertices (a bisection) such that the number of edges connecting $S$ and $V \setminus S$ (the bisection width) is minimized.

In the paper[4], the author shows a new point of $(1/2, 3/8)$-approximation NP-hardness of UGP, which is an improvement of previously known $(3/4, 11/16)$-approximation NP-hardness of UGP[2]. Their result determines a two-dimensional region of for $(c, s)$-approximation NP-hardness of UGP, namely, the triangle with the three vertices $(0, 0)$, $(1/2, 3/8)$, and $(1, 1)$. All known points of $(c, s)$-approximation NP-hardness of UGP are in the triangle, plus an inferior bump area near the origin by [3]. However, the best known hardness for approximating Min UGP is still $5/4 - \epsilon$ despite all the efforts. It is known that Min Bisection (and SSE) is NP-hard to be approximated within $4/3 - \epsilon$ by [5]. It would be interesting to answer the question whether we can further enlarge the hardness gap of Min UGP and SSE.

In this paper, the author strengthens the previous known hardness for approximating Minimum Unique Game Problem, $5/4 - \epsilon$, by proving that Min 2Lin-2 is NP-hard to be approximated within $2 - \epsilon$ and strengthens the previous known hardness for approximating Small Set Expansion Problem, $4/3 - \epsilon$, by proving that Min Bisection is NP-hard to be approximated within $3 - \epsilon$, assuming a variation of Feige’s Hypothesis, which claims that refuting Unbalanced Biased Max 3XOR is NP-hard on average on a natural distribution.

In Fig. 1, the red area is the known region of $(c, s)$-approximation NP-hardness of UGP, the yellow area is the region of $(c, s)$-approximation NP-hardness of UGP assuming the variation of Feige’s Hypothesis.

In Max 3XOR, we are given a set of XOR clauses, each clause contains exactly three literals. The goal of the problem is to seek an assignment of the Boolean variables such that the number of satisfied clauses is maximized. In Max 3AND, we are given a set of AND clauses, each clause contains exactly three literals. The goal of the problem is to seek an assignment of the Boolean variables such that the number of satisfied clauses is maximized.

In Unbalanced Biased Max 3XOR, we are given a set of XOR clauses, each clause contains exactly three literals. The fraction of negative occurrence of each variable is exactly $\beta < 1/2$, which is called balance of the instance. In addition, the domain of the variables is restricted such that the fraction of variables assigned to 0 is no more than $\gamma < \beta$, which is called bias of the instance.

In random Unbalanced Biased Max 3XOR, we assume that formulas are generated by the following random process. Given parameters $n$ and $m$, each clause is generated independently at random by selecting the three variables in it independently at random and inserting the negative literal of the variable into the clause with probability $\beta < 1/2$.
and inserting the positive literal of the variable into the clause with probability $1 - \beta$. $\beta$ is called balance of the instance. In addition, the domain of the variables is restricted such that the fraction of variables assigned to 0 is no more than $\gamma < \beta$, which is called bias of the instance. In random Unbalanced Biased Max 3AND, formulas are generated similarly, and we can define the notations, balance and bias, similarly. Throughout this paper, let $\alpha = \beta + \gamma - 2\beta\gamma$, and $\epsilon$ generally denotes a negligible quantity.

In this paper, the author considers the average complexity of random Unbalanced Biased Max 3XOR, and put forward a variation of Feige’s Hypothesis[5, 6].

**Conjecture 1.** For every fixed $\epsilon > 0$, for $\Delta$ a sufficiently large constant independent of $n$, assuming NP ≠ P, there is no polynomial time algorithm that refutes most $\beta$-balanced $\gamma$-biased Max 3XOR formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - \epsilon$ satisfiable formula.

The following conjecture states that Unbalanced Biased Max 3XOR is approximation resistant. It is easy to observe that Conjecture 1 implies Conjecture 2.

**Conjecture 2.** For every fixed $\epsilon > 0$, for $\Delta$ a sufficiently large constant independent of $n$, assuming NP ≠ P, given a $\beta$-balanced $\gamma$-biased Max 3XOR formula with $n$ variables and $m = \Delta n$ clauses, there is no polynomial time algorithm that distinguishes whether the formula is $1 - \epsilon$ satisfiable or at most $(1 - \beta)^3 + 3\beta^2(1 - \beta) + \epsilon$ satisfiable.
Conjecture 1 implies Conjecture 3.

We use the three-dimensional cube gadget that is similar to the gadgets used by Conjecture 2 holds for $\gamma$-biased Max 3AND formulas with $n$ variables and $m = \Delta n$ clauses, but never refutes a $1 - \frac{1}{2}\alpha - \epsilon$ satisfiable formula.

Theorem 1. Conjecture 1 implies Conjecture 3.

Proof. We rewrite a formula of $\beta$-balanced $\gamma$-biased Max 3XOR to a formula of $\beta$-balanced $\gamma$-biased Max 3AND. If the formula of Max 3XOR is random, then the formula of Max 3AND is also random. The formula of Max 3XOR $\phi$ is $1 - \epsilon$ satisfiable, we show in the following that at least $1 - \frac{1}{2}\alpha - \epsilon$ fraction of clauses in $\phi$ have all the three literals satisfied.

On average, each positive literal has $3(1 - \beta)\Delta$ appearance in $\phi$, and each negative literal has $3\beta\Delta$ appearance in $\phi$. When $\Delta$ is large enough, standard bounds on large deviations show that with high probability, all but an $\epsilon$ fraction of the occurrences of positive literals correspond to positive literals that appear between $(3(1 - \beta)\pm \epsilon)\Delta$ times in $\phi$, and all but an $\epsilon$ fraction of the occurrences of negative literals correspond to negative literals that appear between $(3\beta \pm \epsilon)\Delta$ times in $\phi$.

If this does hold, observe that every $\gamma$-biased assignment $\psi$ does not satisfy on average at most $3(\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon$ variables per clauses in $\phi$. It then follows that at most $3\beta(1 - \gamma) + \gamma(1 - \beta)) + \epsilon$ clauses have exactly one literal satisfied by $\psi$.

Theorem 2. Conjecture 2 holds for $0 < \gamma < \beta < 1/2$ implies $(c', s')$-approximation NP-hardness of Min UGP for $c' = \frac{3}{4}\alpha$ and $s' = \frac{3}{4}\beta - \frac{1}{2}\beta^2 + \beta^3$.

Proof. We use the three-dimensional cube gadget that is similar to the gadgets used by authors of [2, 7].

Let $l_1 \oplus l_2 \oplus l_3$ be a clause in the formula of Max 3XOR, where $l_i$ is either a variable $x_i$ or its negation $\bar{x}_i$, for $i = 1, 2, 3$. The set of equations we construct have variables at the corners of a three-dimensional cube, which take value 1 or $-1$. For each $\alpha \in \{0, 1\}^3$, we have a variable $v_\alpha$. The variable $v_{000}$ is replaced by $w$ taking value 1. We let $u_i$ take the place of $v_{011}$, $u_2$ the place of $v_{101}$, and $u_3$ the place of $v_{110}$, where $u_i = 1$ if $x_i = 1$, and $u_i = -1$ if $x_i = 0$. For each edge $(w, \alpha)$ of the cube, we have the equation $wv_\alpha = -1$. For each edge $(u_i, \alpha)$ of the cube, we have the equation $u_iv_\alpha = -1$ if $l_i$ is positive, and the equation $u_iv_\alpha = 1$ if $l_i$ is negative, for all $i = 1, 2, 3$.

If all $l_i$ are satisfied in the clause, we assign $v_\alpha$ the value $(-1)^{x_1 + x_2 + x_3}$. All the twelve edge equation are satisfied and left no equation unsatisfied. Otherwise, an enumeration establishes that is only possible to satisfy nine equations and left three equations unsatisfied.

Given a $\beta$-balanced $\gamma$-biased Max 3XOR formula $\phi$ with $1 - \epsilon$ satisfiable clauses, every $\gamma$-biased assignment $\psi$ does not satisfy at most $3(\beta(1 - \gamma) + \gamma(1 - \beta))$ variables per clauses in $\phi$. It then follows that at most $\frac{3}{4}(\beta(1 - \gamma) + \gamma(1 - \beta))$ clauses have exactly one literal satisfied by $\psi$. In another word, at least $1 - \frac{1}{2}\alpha - \epsilon$ fraction of clauses in $\phi$ have all the three literals satisfied.

The author also considers the average complexity of random Unbalanced Biased Max 3AND, and put forward the following conjecture.
Given a $\beta$-balanced $\gamma$-biased Max 3XOR formulas $\phi$ that is at most $(1-\beta)^3 + 3\beta^2(1-\beta) + \epsilon$ satisfiable, at least $3\beta(1-\beta)^2 + \beta^3 - \epsilon$ clauses in $\phi$ are unsatisfiable.

Now we reduce a formula of $\beta$-balanced $\gamma$-biased Max 3XOR to an instance of Min 2-Lin-2 using the gadget introduced above. If the formula is $1 - \epsilon$ satisfiable, then the value of the instance of Min 2-Lin-2 is at most $\frac{1}{2}\alpha + \epsilon$. If the formula is at most $(1-\beta)^3 + 3\beta^2(1-\beta) + \epsilon$ satisfiable, then the value of the instance of Min 2-Lin-2 is at least $\frac{1}{2}\beta - \frac{3}{2}\beta^2 + \beta^3 - \epsilon$.

**Corollary 1.** Conjecture 1 holds for arbitrarily small $\beta$ and $\gamma$ implies Min UGP is NP-hard to be approximated within $2 - \epsilon$.

**Lemma 1.** For every $\epsilon > 0$, there is some $\Delta, \Delta_\epsilon > 0$ such that for every $\Delta > \Delta_\epsilon$, $n$ large enough, and $0 < \gamma < \beta < 1/2$, with high probability the following holds. Every set of $((1-\alpha)^3 + \epsilon)m$ clauses in a random $\beta$-balanced $\gamma$-biased Max 3AND formula with $m = \Delta n$ clauses contains at least $\gamma n + 1$ different negative literals or $(1-\gamma)n + 1$ different positive literals.

**Proof.** Fix a set $S$ of $n$ literals with exactly $\gamma$ fraction of positive literals to be avoided. The probability that a random clause with three literals avoids these literals is $(1-\alpha)^3$. For large enough $\Delta$, standard bounds on large deviations implies that with probability greater than $1 - (1-\alpha)^3n$, less than $((1-\alpha)^3 + \epsilon)m$ random clauses avoid the set $S$. As there are roughly $2^{\gamma m}$ ways of choosing the set $S$, the union bound implies that on one of them is avoided by a set of $((1-\alpha)^3 + \epsilon)m$ clauses.

**Theorem 3.** Conjecture 3 holds for $0 < \gamma < \beta < 1/2$ implies Small Set Expansion is NP-hard to be approximated within $\frac{2(1-\gamma\alpha)^3}{2\alpha} - 1 - \epsilon$.

**Proof.** We reduce $\beta$-balanced $\gamma$-biased Max 3AND to Min Bisection. Given a Max 3AND formula with $n'$ variables and $m' = \Delta n'$ clauses in which we want to distinguish between the case at most $((1-\beta)^3 + \epsilon)m'$ clauses are satisfiable and the case that at least $(1-\frac{3}{2}\alpha + \epsilon)m'$ clauses are satisfiable, construct the following graph.

The left hand side (LHS) contains $2n$ vertices, one for each literal. The right hand side (RHS) contains $m'$ clusters, one for each clause, where each cluster is a clique of size $4m'$. In addition, the graph contains a clique of size $m'' = 4(1-3\alpha + \epsilon)m't^2$. In each cluster there is a unique vertex that is a "connecting vertex". Place an edge between a vertex that corresponds a literal and the connecting vertex of a cluster if the literal is in the clause that corresponds the cluster. These are called the "bipartite" edges.

In this graph, find a minimum bisection, which contains exactly $n'$ LHS vertices, and $(1 - \frac{3}{2}\alpha - \epsilon)m'$ clusters. It suffices to consider only the connecting vertices from each of the $m'$ clusters, and we need to find a cut of minimum width that contains $n'$ vertices from the LHS, and $(1 - \frac{3}{2}\alpha - \epsilon)m'$ connecting vertices.

When the 3AND formula has $(1 - \frac{3}{2}\alpha - \epsilon)m'$ satisfiable clauses, we pick the set $S$ to contain the clauses corresponding to these clauses and the $n'$ literals corresponding to the assignments consistent with these clauses. The only edges cut by this bisection connect the satisfying literals to unsatisfied clauses. The number of bipartite edges within the set $S$ is $3(1-\frac{3}{2}\alpha - \epsilon)m'$. The sum of degrees of the satisfied literals is $3(1-\alpha)m'$. Hence the width of the bisection is $\frac{3}{2}\alpha + \epsilon$. 
In a random 3AND formula, we still need one side of the cut to contain \( n' \) vertices and \((1 - \frac{3}{2}\alpha - \epsilon)m'\) clusters. This set of \( n' \) literals has at most \(((1 - \alpha)^3 + \epsilon)m'\) of these clauses 3-connected to it (by Lemma 1) and the other \((1 - \frac{3}{2}\alpha - (1 - \alpha)^3 - 2\epsilon)m'\) clauses are 2-connected to it. Hence the width of the cut is at least

\[
3(1 - \alpha)m' - 3((1 - \alpha)^3 + \epsilon)m' - (1 - \frac{3}{2}\alpha - (1 - \alpha)^3 - 2\epsilon)m' = (2(1 - (1 - \alpha)^3) - \frac{3}{2}\alpha - \epsilon)m'
\]

**Corollary 2.** Conjecture 1 holds for and arbitrarily small \( \gamma \) and \( \beta \) implies SSE is NP-hard to be approximated within \( 3 - \epsilon \).

**References**