Approximating \( \{0, 1, 2\} \)-Survivable Networks with Minimum Number of Steiner Points

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Abstract

We consider low connectivity variants of the Survivable Network with Minimum Number of Steiner Points (SN-MSP) problem: given a finite set \( R \) of terminals in a metric space \((M, d)\), a subset \( B \subseteq R \) of “unstable” terminals, and connectivity requirements \( \{r_{uv} : u, v \in R\} \), find a minimum size set \( S \subseteq M \) of additional points such that the unit-disc graph of \( R \cup S \) contains \( r_{uv} \) pairwise internally edge-disjoint and \( (B \cup S) \)-disjoint \( uv \)-paths for all \( u, v \in R \). The case when \( r_{uv} = 1 \) for all \( u, v \in R \) is the Steiner Tree with Minimum Number of Steiner Points (ST-MSP) problem, and the case \( r_{uv} \in \{0, 1\} \) is the Steiner Forest with Minimum Number of Steiner Points (SF-MSP) problem. Let \( \Delta \) be the maximum number of points in a unit ball such that the distance between any two of them is larger than 1. It is known that \( \Delta = 5 \) in \( \mathbb{R}^2 \). The previous known approximation ratio for ST-MSP was \( \lceil (\Delta + 1)/2 \rceil + 1 + \epsilon \) in an arbitrary normed space [17], and 2.5 + \( \epsilon \) in the Euclidean space \( \mathbb{R}^2 \) [5]. Our approximation ratio for ST-MSP is \( 1 + \ln(\Delta - 1) + \epsilon \) in an arbitrary normed space, which in \( \mathbb{R}^2 \) reduces to \( 1 + \ln 4 + \epsilon < 2.3863 + \epsilon \). For SN-MSP with \( r_{uv} \in \{0, 1, 2\} \), we give a simple \( \Delta \)-approximation algorithm. In particular, for SF-MSP, this improves the previous ratio 2\( \Delta \).
1 Problems considered

A large research effort is focused on developing algorithms for finding a “cheap” network that satisfies a certain property. In wired networks, where connecting any two nodes incurs a cost, many problems can be cast as finding a subgraph of minimum cost that satisfies some prescribed connectivity requirements. Following previous work on min-cost connectivity problems, we use the following generic notion of connectivity.

**Definition 1.1** Let \( G = (V, E) \) be a graph and let \( Q \subseteq V \). The \( Q \)-connectivity \( \lambda_Q^G(u,v) \) of \( u,v \) in \( G \) is the maximum number of pairwise \((E \cup Q \setminus \{u,v\})\)-disjoint \( uv \)-paths in \( G \). Given connectivity requirements \( r = \{ r_{uv} : u,v \in R \subseteq V \} \) on a subset \( R \subseteq V \) of terminals, we denote by \( D_r = \{ u,v \in R : r_{uv} > 0 \} \) the set of “demand edges” of \( r \). We say that \( G \) is \((r,Q)\)-connected, or simply \( r \)-connected if \( Q \) is understood, if \( \lambda_Q^G(u,v) \geq r_{uv} \) for all \( uv \in D_r \).

Note that edge-connectivity is the case \( Q = \emptyset \) and node-connectivity is the case \( Q = V \). The members of \( E \cup Q \) will be called elements, hence \( \lambda_Q^G(u,v) \) is the maximum number of pairwise internally element-disjoint \( uv \)-paths in \( G \). Variants of the following classic problem were extensively studied in the literature.

<table>
<thead>
<tr>
<th>Survivable Network (SN)</th>
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<tbody>
<tr>
<td><strong>Instance:</strong> A graph ( G = (V, E) ) with edge costs, ( Q \subseteq V ), and connectivity requirements ( r = { r_{uv} : uv \in R \subseteq V } ).</td>
</tr>
<tr>
<td><strong>Objective:</strong> Find a minimum-cost ((r,Q))-connected subgraph ( H ) of ( G ).</td>
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</table>

In practical networks the connectivity requirements are rather small, usually \( r_{uv} \in \{0,1,2\} \) – so called \( \{0,1,2\}\)-SN. Particular cases in this setting are Minimum Spanning Tree (MST) \((r_{uv} = 1 \text{ for all } u,v \in V)\), Steiner Tree \((r_{uv} = 1 \text{ for all } u,v \in R)\) and Steiner Forest \((r_{uv} \in \{0,1\} \text{ for all } u,v \in R)\), and 2-Connected Subgraph \((r_{uv} = 2 \text{ for all } u,v \in V)\).

In wireless networks, the range and the location of the transmitters determines the resulting communication network. We consider adding a minimum number of transmitters such that the resulting communication network is \((r,Q)\)-connected. If the range of the transmitters is fixed, our goal is to add a minimum number of transmitters, and we get the following type of problems.

<table>
<thead>
<tr>
<th>Survivable Network with Minimum Number of Steiner Points (SN-MSP)</th>
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<tbody>
<tr>
<td><strong>Instance:</strong> A finite set ( R \subseteq M ) of terminals in a metric space ((M,d)), a set ( B \subseteq R ) of “unstable” terminals, connectivity requirements ( { r_{uv} : uv \in R } ).</td>
</tr>
<tr>
<td><strong>Objective:</strong> Find a minimum size set ( S \subseteq M ) such that the unit-disk graph of ( R \cup S ) is ((r,Q))-connected, where ( Q = B \cup S ).</td>
</tr>
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</table>

As in previous work, we will allow to place several points at the same location, and assume that the maximum distance between terminals is polynomial in the number of terminals.

2 Previous work and our results

On previous work on high connectivity variants of SN problem we refer the reader to a survey in [15] and here only mention some work relevant to this paper. The Steiner Tree problem was studied
Minimum α defines a parameter solution of cost C
instance of cost at most is called a bead solution SN-MSP in the paper of Byrka, Grandoni, Rothvoß, and Sanità [2]. For by Zelikovsky [22], improved in a long series of papers (part of them are [22, 18, 21]), and culminated SN-MSP approximation algorithm for the

Given a finite set Definition 2.1 a bead solution as a solution to the corresponding SN instance obtained as follows. Let |I| denote the optimal solution value of a problem instance at hand. It is easy to see that any solution of cost C to the corresponding SN instance with k = max_{u \in D, r_{u,v}} defines a solution S of size C to the original SN-MSP instance, where every node in S has degree exactly 2; such a solution is called a bead solution. Conversely, any bead solution S can be converted into a solution to the SN instance of cost at most |I| (see [10, 3]). Due to this bijective correspondence, we simply define a bead solution as a solution to the corresponding SN instance, and denote the optimal value of a bead solution to an instance I by \tau = \tau(I). If the SN instance admits a ρ-approximation algorithm, and if for the given SN-MSP instance there exists a bead solution S of size \leq αopt, then we get a ρα-approximation algorithm for the SN-MSP instance. Equivalently, for a class \mathcal{I} of SN-MSP instances, define a parameter α by α = α(\mathcal{I}) = \sup_{I \in \mathcal{I}} \frac{opt(I)}{\tau(I)}. Then approximation ratio ρ for SN instances that correspond to the class \mathcal{I} implies approximation ratio αρ for SN-MSP instances in class \mathcal{I}.

Mándoiu and Zelikovsky [16] showed that for ST-MSP \alpha = \Delta - 1. Since the instance of SN that corresponds to ST-MSP is the MST problem that can be solved in polynomial time, this gives a (\Delta - 1)-approximation algorithm for ST-MSP. A more general method, uses a reduction to the Minimum k-Connected Spanning Subhypergraph problem, see Section ???. This method was initiated by Zelikovsky [22], improved in a long series of papers (part of them are [22, 18, 21]), and culminated in the paper of Byrka, Grandoni, Rothvoß, and Sanità [2]. For ST-MSP in \mathbb{R}^2, Chen and Du [5] applied this method to get the currently best known ratio 2.5 + \epsilon. In arbitrary metric spaces, the ratio \Delta - 1 of [16] was improved to [(\Delta + 1)/2] + 1 + \epsilon in [17], also using the same method. These works assume that ST-MSP instances with a constant number of terminals can be solved in polynomial
time, which holds in $\mathbb{R}^2$ if the maximum distance between terminals is polynomial in the number of terminals, see [4, Lemma 11] and the discussion there. In this paper we apply a variant due to Zelikovsky [23], and obtain the following result.

**Theorem 2.1** ST-MSP with constant $\Delta$ admits an approximation scheme with ratio $1 + \ln(\Delta - 1) + \epsilon$, provided that ST-MSP instances with a constant number of terminals can be solved in polynomial time. In particular, in $\mathbb{R}^2$ the ratio is $1 + \ln 4 + \epsilon < 2.3863 + \epsilon$.

We now discuss SN-MSP problems with $k = \max_{uv \in V} r_{uv} \geq 2$. Bredin, Demaine, Hajiaghayi, and Rus [1] considered a related problem of adding a minimum size $S$ such that the unit disc graph of $R \cup S$ is $k$-node-connected (note that we require $k$-connectivity only between terminals). For this problem in $\mathbb{R}^2$, they gave an $O(k^5)$-approximation algorithm, but essentially they implicitly proved that for this class of problems $\alpha = O(\Delta k^3)$. Recently, it was shown in [17] that $\alpha = \Theta(\Delta k^2)$ for node-connectivity SN-MSP instances in any normed space.

Kashyap, Khuller, and Shayman [11] considered the 2-edge/node-connectivity version of SN-MSP, where $r_{uv} = 2$ for all $u, v \in R$. They used the reduction method described in Definition 2.1, namely, their algorithm constructs an SN instance as in Definition 2.1 and then converts its solution into a bead solution to the SN-MSP instance. Although they analyzed a performance of specific 2-approximation algorithms – the algorithm of Khuller and Vishkin [13] for 2-edge-connectivity and the algorithm of Khuller and Raghavachari [12] for 2-node-connectivity, they essentially proved that $\alpha = \Delta$ in both cases. This implies ratio $2\Delta$ in both cases. The analysis of these specific algorithms was recently improved by Calinescu [3], showing that their tight performance is $\Delta$ for node-connectivity and $2\Delta - 1$ for edge-connectivity. Note that the edge-connectivity version is not included in our model, since in our SN-MSP instances every non-terminal node is in $Q$, namely, the paths are required to be $S$ disjoint.

Let $\tau^* = \tau^*(I)$ denote the optimal value of a fractional bead solution of an SN-MSP instance $I$, namely, $\tau^*$ is the optimum of a standard cut-LP relaxation for the corresponding SN instance (see Section ??). Here we observe, that if the algorithm we use for the corresponding SN instance computes a solution of cost at most $\rho \tau^*$, then the relevant parameter is the following.

**Definition 2.2** For a class $\mathcal{I}$ of SN-MSP instances, let $\alpha^*(\mathcal{I}) = \sup_{I \in \mathcal{I}} \frac{\text{opt}(I)}{\tau^*(I)}$.

**Theorem 2.2** For $Q$-connectivity $\{0,1,2\}$-SN-MSP $\alpha^* = \frac{\Delta}{2}$. Thus if $Q$-connectivity $\{0,1,2\}$-SN admits a polynomial time algorithm that computes a solution of cost at most $\rho \tau^*$, then $Q$-connectivity $\{0,1,2\}$-SN-MSP admits approximation ratio $\rho \cdot \frac{\Delta}{2}$. In particular, for $\rho = 2$ the ratio is $\Delta$, and thus $\{0,1,2\}$-SN-MSP admits a $\Delta$-approximation algorithm.

**References**


