Social choice theory
A brief introduction

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Motivation

Introduction

Aims

- analyze a number of properties of electoral systems
- present a few elements of the classical theory
Motivation

What is Social Choice Theory?

Social Choice Theory

- aim: study decision problems in which a group has to take a decision among several alternatives
- abstract theory
  - nature of the decision
  - size of the group
  - nature of the group
- many (deep) results
  - Economics, Political Science, Applied Mathematics, OR
  - two Nobel Prizes: Kenneth J. Arrow, Amartya Sen

Areas of applications

Applications

- political elections
- other types of elections
  - fewer voters and candidates (e.g., electing a Dean)
- decision with multiple criteria
- artificial intelligence
  - multiple agents
  - multiple rules
Motivation

Problem

Vocabulary: political elections
- group
  - society
- members of the group
  - voters
- alternatives
  - candidates

Problem
- study election problems in which a society has to take a decision among several candidates

Today’s problem

Problem
- choice of one among several candidates
  - French or US presidential elections

Electing several candidates: assembly
- apply same rules in each electoral district
- many specific problems: gerrymandering, technical problems (as sometimes seen in the USA)

Proportional representation
- PR does not solve the decision problem in the Parliament!
  - one bill will adopted on each issue
- PR raises many difficult problems (What is a just PR? How to achieve it? PR and Power indices)
A glimpse at PR

Problem 1: # of seats and power
- Parliament: 100 MPs
- voting rule in the Parliament: simple majority (> 50 %)
- # of votes exactly proportional to # of seats

Example
- party A: 45 % of votes
- party B: 15 % of votes
- party C: 40 % of votes

- all coalitions of 1 party are loosing coalitions
- all coalitions of at least 2 parties are winning coalitions
- entirely symmetric situation
- all parties have the same power

Problem 2: obtaining a fair PR
- in general # of voters $\gg$ # of MPs
- # of MPs must be integer!
- rounding off procedures

Hamilton’s rule
- 2 100 000 voters, 3 parties, 20 MPs
- results
  - party A: 928 000, quota: $r_A = 928 000 / 2 100 000 = 8.84$
  - party B: 635 000, quota: $r_B = 635 000 / 2 100 000 = 6.05$
  - party C: 537 000, quota: $r_C = 537 000 / 2 100 000 = 5.11$
- party $x$ gets at least $\lfloor r_x \rfloor$ seats
- if all seats are allocated: done
- if not: allocate the remaining seats according to the $r_x - \lfloor r_x \rfloor$
### Motivation

**Hamilton’s rule**

- party A: 928,000, quota: \( r_A = 8.84 = 928,000 / 210,000 \)
- party B: 635,000, quota: \( r_B = 6.05 = 635,000 / 210,000 \)
- party C: 537,000, quota: \( r_C = 5.11 = 537,000 / 210,000 \)

### Results

- party A gets 8 seats
- party B gets 6 seats
- party C gets 5 seats
- \( 8 + 6 + 5 = 19 < 20 \)
- party A gets the extra seat because \( 0.84 > 0.11 > 0.05 \)

### Example

20 seats

- party A: \( r_A = 8.84, 8 + 1 = 9 \) seats
- party B: \( r_B = 6.05, 6 \) seats
- party C: \( r_C = 5.11, 5 \) seats

21 seats

- party A: \( r_A = 9.28, 9 \) seats
- party B: \( r_B = 6.35, 6 \) seats
- party C: \( r_C = 5.37, 5 + 1 = 6 \) seats

22 seats: Alabama paradox (1881)

- party A: \( r_A = 9.72, 9 + 1 = 10 \) seats
- party B: \( r_B = 6.65, 6 + 1 = 7 \) seats
- party C: \( r_C = 5.63, 5 \) seats
Motivation

Election of one candidate

Common sense
- the choice of the candidate will affect all members of the society
- the choice of the candidate should take the opinion of all members of society into account

Intuition
Democracy \Rightarrow Elections \Rightarrow Majority

Elections

Philosophical problems
- “general will” and elections
- majority and protection of minorities
- formal vs real freedom

Political problems
- direct or undirect democracy?
- rôle of parties?
- who can vote? (age, sex, nationality, paying taxes, …)
- who can be candidate?
- what type of mandate?
- how to organize the campaign?
- rôle of polls?
Motivation

Technical problems

Majority
When there are only two candidates
- elect the one receiving the more votes

Majority
When there are more than candidates
- many ways to extend this simple idea
- not equivalent
- sometimes leading to unwanted results

Typology of elections

Two main criteria
- type of ballots admitted
  - one name
  - ranking of all candidates
  - other types (acceptable candidates, grading candidates, etc.)
- method for organizing the election and for tallying ballots

Consequences
- many possible types of elections
- many have been proposed
- many have have been used in practice
Two hypotheses

Hypotheses

- all voters are able to rank order the set of all candidates (ties admitted)
  
  \[ a \succ b \succ [d \sim e] \succ c \]

  - each voter has a weak order on the set of all candidates

- voters are sincere
  
  - if I have to vote for one candidate, I vote for \( a \)

Examples

Plurality voting: UK

Rules

- one round of voting
- ballots with one name
- “first past the post”

Remark

- ties are neglected (unlikely)
  
  - one voter has special power (the Queen chooses in case of a tie)
  - one candidate receives special treatment (the older candidate is elected)
  - random tie breaking rule
Plurality voting

Example

- 3 candidates \{a, b, c\}
- 21 voters (or 21 000 000 or 42 000 000...)

<table>
<thead>
<tr>
<th>Voters</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>a ≻ b ≻ c</td>
</tr>
<tr>
<td>6</td>
<td>b ≻ c ≻ a</td>
</tr>
<tr>
<td>5</td>
<td>c ≻ b ≻ a</td>
</tr>
</tbody>
</table>

Results

\[ a : 10 \quad b : 6 \quad c : 5 \]

- a is elected...
- but an absolute majority of voters (11/21) prefer all loosing candidates to the elected one!

\[ a: \text{Tory}, \quad b: \text{Labour}, \quad c: \text{LibDem} \]

Remarks

- problems are expected as soon as there are more than 2 candidates
- a system based on an idea of “majority” may well violate the will of a majority of voters
- sincerity hypothesis is heroic!
Plurality with runoff: France

Rules
- ballots with one name
- first round
  - the candidate with most votes is elected if he receives more than 50% of votes
  - otherwise go to the second round
- second round
  - keep the two candidates having received more votes
  - apply plurality voting

Variants
- rule are slightly different for the “élections législatives”

Examples
Ballots with one name

Plurality with runoff

Previous example
- 3 candidates \{a, b, c\}
- 21 voters

| 10 voters: | a ≻ b ≻ c |
| 6 voters:   | b ≻ c ≻ a |
| 5 voters:   | c ≻ b ≻ a |

Results
- \( a : 10 \quad b : 6 \quad c : 5 \)
- absolute majority: \( \lceil 21/2 \rceil = 11 \) votes
- go to the second round with \( a \) and \( b \)
  - \( a : 10 \quad b : 11 \)
- \( b \) is elected
- no candidate is preferred to \( b \) by a majority of voters
Plurality with runoff

Example

- 4 candidates \( \{a, b, c, d\} \)
- 21 voters (may be also 21 000 000 or 42 000 000)

<table>
<thead>
<tr>
<th>Voters</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>( b \succ a \succ c \succ d )</td>
</tr>
<tr>
<td>6</td>
<td>( c \succ a \succ d \succ b )</td>
</tr>
<tr>
<td>5</td>
<td>( a \succ d \succ b \succ c )</td>
</tr>
</tbody>
</table>

Results: 1st round

- \( a : 5 \)  \( b : 10 \)  \( c : 6 \)  \( d : 0 \)
- absolute majority: \([21/2] = 11\) votes
- go to the second round with \( b \) and \( c \)

Results: 2nd round

- \( b : 15 \)  \( c : 6 \)
- \( b \) is elected (15/21)
- an absolute majority of voters (11/21) prefer \( a \) and \( d \) to \( b \)

Plurality vs plurality with runoff

- the French system does only a little better than the UK one
- preferences used in the above example are not bizarre
  - try replacing \( a, b, c, d \) by MoDem, UMP, PS, PCF, FN, etc.
- sincerity and wasted votes
Examples: Ballots with one name

Plurality with runoff: manipulation

Example

- 4 candidates \{a, b, c, d\}
- 21 voters

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<td>a ≻ d ≻ b ≻ c</td>
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</table>

- b is elected

Non-sincere voting

- the 6 voters for which c ≻ a ≻ d ≻ b vote as if their preferences were a ≻ c ≻ d ≻ b

Results

- a is elected at the first round (11/21)
- profitable to the six manipulating voters (for them a ≻ b)

Manipulable voting rules

Definition

A voting rule is manipulable if it may happen that some voters may have an interest to vote in a non-sincere way.

Problems

- elections are no more a means to reveal preferences
  - manipulations and counter-manipulations
  - equilibrium
- bonus to clever voters
Plurality with runoff: monotonicity

Example: before campaign

- 3 candidates \{a, b, c\}
- 17 voters

<table>
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<th>Voters</th>
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<tr>
<td>6 voters</td>
<td>(a \succ b \succ c)</td>
</tr>
<tr>
<td>5 voters</td>
<td>(c \succ a \succ b)</td>
</tr>
<tr>
<td>4 voters</td>
<td>(b \succ c \succ a)</td>
</tr>
<tr>
<td>2 voters</td>
<td>(b \succ a \succ c)</td>
</tr>
</tbody>
</table>

Results: before campaign

Absolute majority: \([17/2] = 9\)

<table>
<thead>
<tr>
<th>Preference</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>11</td>
</tr>
<tr>
<td>(b)</td>
<td>6</td>
</tr>
</tbody>
</table>

\(a\) is elected!

\(a\) gets more money to campaign against \(b\)

Plurality with runoff

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- 2 voters \(b \succ a \succ c\) change their minds in favor of \(a\)
- New preference: \(a \succ b \succ c\)

Absolute majority: \([17/2] = 9\)

<table>
<thead>
<tr>
<th>Preference</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>8</td>
</tr>
<tr>
<td>(c)</td>
<td>9</td>
</tr>
</tbody>
</table>

\(c\) is elected!

- The good campaign of \(a\) is fatal to him/her
- Non-monotonic method: increasing possibilities of manipulation
Plurality with runoff: participation

Example

- 3 candidates \{a, b, c\}
- 11 voters

\[
\begin{array}{l}
4 \text{ voters: } a \succ b \succ c \\
4 \text{ voters: } c \succ b \succ a \\
3 \text{ voters: } b \succ c \succ a \\
\end{array}
\]

Results

absolute majority: \(\lceil11/2\rceil = 6\)

\[
\begin{align*}
a : 4 & \\
b : 3 & \\
c : 4 & \\
\end{align*}
\]

\[
\begin{align*}
a : 4 & \\
c : 7 & \\
\end{align*}
\]

- c is elected
- this is not a nice outcome for the first 4 voters
- 2 of them go fishing and abstain (at the two rounds)

Before

\[
\begin{array}{l}
4 \text{ voters: } a \succ b \succ c \\
4 \text{ voters: } c \succ b \succ a \\
3 \text{ voters: } b \succ c \succ a \\
\end{array}
\]

\[
\begin{array}{l}
\bullet c \text{ elected} \\
\end{array}
\]

After

\[
\begin{array}{l}
2 \text{ voters: } a \succ b \succ c \\
4 \text{ voters: } c \succ b \succ a \\
3 \text{ voters: } b \succ c \succ a \\
\end{array}
\]

Results

absolute majority: \(\lceil11/2\rceil = 6\)

\[
\begin{align*}
a : 2 & \\
b : 3 & \\
c : 4 & \\
\end{align*}
\]

\[
\begin{align*}
b : 5 & \\
c : 4 & \\
\end{align*}
\]

- b is elected
- the abstention of the two voters who think \(b \succ c\) has been very rational
### Plurality with runoff: separability

**Example**
- 3 candidates \( \{a, b, c\} \)
- 26 voters in two districts \((13 + 13)\)

#### District 1

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>4 voters</td>
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<td>4</td>
</tr>
<tr>
<td>3 voters</td>
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<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( c \succ a \succ b )</td>
<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( c \succ b \succ a )</td>
<td>3</td>
</tr>
</tbody>
</table>

- \( a : 4 \quad b : 3 \quad c : 6 \)
- \( a : 7 \quad c : 6 \)
- \( a \) is elected \((7/13)\)

#### District 2

<table>
<thead>
<tr>
<th>Voters</th>
<th>Preferences</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 voters</td>
<td>( a \succ b \succ c )</td>
<td>4</td>
</tr>
<tr>
<td>3 voters</td>
<td>( c \succ a \succ b )</td>
<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( b \succ c \succ a )</td>
<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( b \succ a \succ c )</td>
<td>3</td>
</tr>
</tbody>
</table>

- \( a : 4 \quad b : 6 \quad c : 3 \)
- \( a : 7 \quad b : 6 \)
- \( a \) is elected \((7/13)\)

**Plurality with runoff**

**Nationwide**

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>4 voters</td>
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<td>3</td>
</tr>
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<td>3 voters</td>
<td>( c \succ a \succ b )</td>
<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( c \succ b \succ a )</td>
<td>3</td>
</tr>
<tr>
<td>4 voters</td>
<td>( a \succ b \succ c )</td>
<td>4</td>
</tr>
<tr>
<td>3 voters</td>
<td>( c \succ a \succ b )</td>
<td>3</td>
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<tr>
<td>3 voters</td>
<td>( b \succ c \succ a )</td>
<td>3</td>
</tr>
<tr>
<td>3 voters</td>
<td>( b \succ a \succ c )</td>
<td>3</td>
</tr>
</tbody>
</table>

- \( a : 8 \quad b : 9 \quad c : 9 \)
- \( a \) looses at the first round
- method is not separable
- decentralization of decisions?
Summary

**French vs UK system**

- the French system does only a little better better than the UK one on the “democratic side”
- it has many other problems
  - manipulable
  - not monotonic
  - no incentive to participate
  - not separable
- are there other (hopefully better!) systems?
- conventional wisdom ("au premier tour on choisit, au deuxième tour on élimine") must be used with great care!

**Amendment procedure**

**Remarks**

- the majority method works well with two candidates
- when there are more than two candidates, organize a series of confrontations between two candidates according to an agenda
- method used in most parliaments
  - a bill is proposed
  - amendments to the bill are proposed
  - compare the amended bill vs the status quo
Amendment procedure

Example

- 4 candidates \{a, b, c, d\}
- agenda: a, b, c, d

majority winner between a and b

- a is a bill
- b, c are amendments
- d is the status quo

- 3 candidates \{a, b, c\}
- 3 voters

| 1 voter: | a > b > c |
| 1 voter: | c > a > b |
| 1 voter: | b > c > a |

- agenda a, b, c: c is elected
- agenda b, c, a: a is elected
- agenda c, a, b: b is elected

- results depending on the (arbitrary) choice of the agenda
  - power given to the agenda-setter
  - candidates not treated equally
    - late-coming candidates are favored
    - method is not neutral
### Amendment procedure

#### Example
- 4 candidates \( \{a, b, c, d\} \)
- 30 voters
- agenda \( a, b, c, d \)

| 10 voters: \( b \succ a \succ d \succ c \) |
| 10 voters: \( c \succ b \succ a \succ d \) |
| 10 voters: \( a \succ d \succ c \succ b \) |

#### Results
- \( b \) beats \( a \)
- \( c \) beats \( b \)
- \( d \) beats \( c \)
- \( d \) is elected…
- 100% of the voters prefer \( a \) to \( d \)!
- method is not unanimous!

### Ballots: ordered lists

#### Ballots with a single name
- poor performances…
- may be due to poor information on preferences
- ask for the full preference on each ballot

#### Remarks
- much richer information
  - practice?
- ballots with one name are a particular case
Examples

Ballots with ordered lists

Condorcet

Principles

- compare all candidates by pair
- declare that $a$ is “socially preferred” to $b$ if (strictly) more voters prefer $a$ to $b$ (social indifference in case of a tie)
- Condorcet’s principle: if one candidate is preferred to all other candidates, it should be elected
- Condorcet Winner (CW: must be unique)

Remarks

- UK and French systems violate Condorcet’s principle
- the UK system may elect a Condorcet looser
- Condorcet’s principle does not solve the “dictature of the majority” difficulty
- a Condorcet winner is not necessarily “ranked high” by voters

Example

- 3 candidates $\{a, b, c\}$
- 21 voters

<table>
<thead>
<tr>
<th></th>
<th>10 voters:</th>
<th>6 voters:</th>
<th>5 voters:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a \succ b \succ c$</td>
<td>$b \succ c \succ a$</td>
<td>$c \succ b \succ a$</td>
</tr>
</tbody>
</table>

- $a$ is the plurality winner
- $a$ is the Condorcet looser
- $b$ is the CW
  - $b$ beats $a$ (11/21)
  - $b$ beats $c$ (16/21)
Examples  Ballots with ordered lists

Condorcet and plurality with runoff

Example

- 4 candidates \{a, b, c, d\}
- 21 voters

<table>
<thead>
<tr>
<th>10 voters:</th>
<th>b ≻ a ≻ c ≻ d</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 voters:</td>
<td>c ≻ a ≻ d ≻ b</td>
</tr>
<tr>
<td>5 voters:</td>
<td>a ≻ d ≻ b ≻ c</td>
</tr>
</tbody>
</table>

- \(b\) is the plurality with runoff winner (beats \(c\) in the second round)
- \(a\) is the CW
  - \(a\) beats \(b\) (11/21)
  - \(a\) beats \(c\) (15/21)
  - \(a\) beats \(d\) (21/21)

Condorcet and ranks

Example

- 5 candidates \{a, b, c, d, e\}
- 50 voters

<table>
<thead>
<tr>
<th>10 voters:</th>
<th>a ≻ b ≻ c ≻ d ≻ e</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 voters:</td>
<td>b ≻ c ≻ e ≻ d ≻ a</td>
</tr>
<tr>
<td>10 voters:</td>
<td>e ≻ a ≻ b ≻ c ≻ d</td>
</tr>
<tr>
<td>10 voters:</td>
<td>a ≻ b ≻ d ≻ e ≻ c</td>
</tr>
<tr>
<td>10 voters:</td>
<td>b ≻ d ≻ c ≻ a ≻ e</td>
</tr>
</tbody>
</table>

- \(a\) is the CW (beats 30/20 all other candidates)

Ranks

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Condorcet and dictatorship of the majority

Example

- 26 candidates \{a, b, c, \ldots, z\}
- 100 voters

\[
\begin{align*}
51 \text{ voters:} & \quad a \succ b \succ c \succ \cdots \succ y \succ z \\
49 \text{ voters:} & \quad z \succ b \succ c \succ \cdots \succ y \succ a
\end{align*}
\]

- a is the CW
- b could be a reasonable choice

---

Condorcet’s paradox

Electing the CW

- attractive …
- but not always effective!

Condorcet’s paradox

- 3 candidates \{a, b, c\}
- 3 voters

\[
\begin{align*}
1 \text{ voter:} & \quad a \succ b \succ c \\
1 \text{ voter:} & \quad c \succ a \succ b \\
1 \text{ voter:} & \quad b \succ c \succ a
\end{align*}
\]
### Condorcet’s paradox

- the social strict preference relation may have circuits
  - prob. $\approx 40\%$ with 7 candidates and a large number of voters (impartial culture)
- McGarvey’s theorem

### Dealing with Condorcet’s paradox

- weaken the principle so as to elect candidates that are not strictly beaten (Weak CW)
  - they may not exist
  - there may be more than one
- find what to do when there is no (weak) Condorcet winner

### Schwartz

### Principle

- build the social preference à la Condorcet
- the strict social preference may not be transitive
  - take its transitive closure
  - take the maximal elements of the resulting weak order
Schwartz

Example

- 4 candidates \( \{a, b, c, d\} \)
- 30 voters

\[
\begin{align*}
\text{10 voters:} & \quad a \succ b \succ c \succ d \\
\text{10 voters:} & \quad d \succ a \succ b \succ c \\
\text{10 voters:} & \quad c \succ d \succ a \succ b
\end{align*}
\]

\[
\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (1,-1) {c};
\node (d) at (0,-1) {d};
\draw (a) -- (b);
\draw (a) -- (c);
\draw (a) -- (d);
\draw (b) -- (c);
\draw (b) -- (d);
\draw (c) -- (d);
\end{tikzpicture}
\]

- taking the transitive closure gives a clique
- all candidates are declared socially indifferent
- but 100% of voters prefer \( a \) to \( b \)!

Copeland

Principles

- build the social preference à la Condorcet
- count the number of candidates that are beaten by one candidate minus the number of candidates that beat him (Copeland score)
- elect the candidate with the highest score
- sports league
  - +2 for a victory, +1 for a tie, 0 for a defeat
  - equivalent to Copeland’s rule
## Copeland

### Example
- 5 candidates \{x, a, b, c, d\}
- 40 voters

<table>
<thead>
<tr>
<th>Voters</th>
<th>Ballots</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>x ≻ a ≻ d ≻ c ≻ b</td>
</tr>
<tr>
<td>10</td>
<td>x ≻ a ≻ b ≻ c ≻ d</td>
</tr>
<tr>
<td>10</td>
<td>a ≻ d ≻ c ≻ b ≻ x</td>
</tr>
<tr>
<td>10</td>
<td>b ≻ c ≻ d ≻ x ≻ a</td>
</tr>
</tbody>
</table>

\[ \begin{array}{ccccc} x & a & b & c & d \\ 1 & 2 & -2 & -1 & 0 \end{array} \]

- \( a \) is elected!
- \( x \) is the unique weak CW

## Borda

### Principles
- each ballot is an ordered list of candidates (exclude ties for simplicity)
- on each ballot compute the rank of the candidates in the list
- rank order the candidates according to the decreasing sum of their ranks

### Remarks
- simple
- efficient: always lead to a result
- separable, monotonic, participation incentive
Borda and Condorcet principle

Example
- 4 candidates \{a, b, c, d\}
- 3 voters
  - 2 voters: \(b \succ a \succ c \succ d\)
  - 1 voter: \(a \succ c \succ d \succ b\)

Borda scores
\[
\begin{array}{cccc}
a & b & c & d \\
5 & 6 & 8 & 11 \\
\end{array}
\]

Results
- \(a\) is elected
- \(b\) is the obvious CW

Borda and withdrawals

Example
- 4 candidates \{a, b, c, d\}
- 3 voters
  - 2 voters: \(b \succ a \succ c \succ d\)
  - 1 voter: \(a \succ c \succ d \succ b\)

Borda scores
\[
\begin{array}{cccc}
a & b & c & d \\
5 & 6 & 8 & 11 \\
\end{array}
\]

- \(a\) is elected

Example
- \(c\) and \(d\) are withdrawing
- 2 candidates \{a, b\}
- 3 voters
  - 2 voters: \(b \succ a\)
  - 1 voter: \(a \succ b\)

Borda scores
\[
\begin{array}{cc}
a & b \\
5 & 4 \\
\end{array}
\]

- \(b\) is elected!
- door wide open for manipulations
- introduce dummy candidates
Examples

Ballots with ordered lists

Summary

Example

- 4 candidates \( \{a, b, c, d\} \)
- 27 voters

<table>
<thead>
<tr>
<th>Voters</th>
<th>Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( a \succ b \succ c \succ d )</td>
</tr>
<tr>
<td>4</td>
<td>( a \succ c \succ b \succ d )</td>
</tr>
<tr>
<td>2</td>
<td>( d \succ b \succ a \succ c )</td>
</tr>
<tr>
<td>6</td>
<td>( d \succ b \succ c \succ a )</td>
</tr>
<tr>
<td>8</td>
<td>( c \succ b \succ a \succ d )</td>
</tr>
<tr>
<td>2</td>
<td>( d \succ c \succ b \succ a )</td>
</tr>
</tbody>
</table>

Results

- \( a \) is the plurality with runoff winner
- \( d \) is the plurality winner
- \( b \) is the Borda winner
- \( c \) is the CW

Democratic method

- always giving a result like Borda
- always electing the Condorcet winner
- consistent w.r.t. withdrawals
- monotonic, separable, incentive to participate, not manipulable
- etc.
### Framework
- $n \geq 3$ candidates (otherwise use plurality)
- $m$ voters ($m \geq 2$ and finite)
- ballots: ordered list of candidates

### Problem
- find all electoral methods respecting a small number of “desirable” principles

### Principles
- universality
  - the method should be able to deal with any configuration of ordered lists
- transitivity
  - the result of the method should be an ordered list of candidates
- unanimity
  - the method should respect a unanimous preference of the voters
- absence of dictator
  - the method should not allow for dictators
- independence
  - the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters
**Arrow’s theorem (1951)**

**Theorem**
There is no method respecting the five principles

**Borda**
- universal, transitive, unanimous with no dictator
- cannot be independent

**Condorcet**
- universal, independent, unanimous with no dictator
- cannot be transitive

**Sketch of proof**

**Decisive coalitions**

\[ V \subseteq N \text{ is decisive for } (a, b) \text{ if } \]
\[ a \succ_i b \text{ for all } i \in V \Rightarrow a \succ b \]

**Almost decisive coalitions**

\[ V \subseteq N \text{ is almost decisive for } (a, b) \text{ if } \]
\[ \begin{align*}
   a \succ_i b \text{ for all } i \in V \\
   b \succ_j a \text{ for all } j \notin V
\end{align*} \Rightarrow a \succ b \]
Lemma 1

If $V$ is almost decisive over some ordered pair $(a, b)$, it is decisive over all ordered pairs.

Sketch of proof

Take \{a, b, x, y\} and use universality to obtain:

\[ V : x \succ a \succ b \succ y \]
\[ N \setminus V : x \succ a, b \succ y, b \succ a \]

The relative position of $x$ and $y$ for $N \setminus V$ is not specified.
Unanimity implies $x \succ a$ and $b \succ y$.
Almost decisiveness of $V$ for $(a, b)$ implies $a \succ b$.
Transitivity implies $x \succ y$.
Independence implies that this does not depends on the position of $a$ and $b$.
Hence $V$ is decisive for $(x, y)$.

Lemma 2

If $V$ is decisive and $|V| > 1$, some proper subset of $V$ is decisive.

Sketch of proof

Partition $V$ into $V_1$ and $V_2$.
Take \{x, y, z\} and use universality to obtain:

\[ V_1 : x \succ y \succ z \]
\[ V_2 : y \succ z \succ x \]
\[ N \setminus V : z \succ x \succ y \]

Decisiveness of $V$ implies $y \succ z$.
If $x \succ z$ then $V_1$ is almost decisive for $(x, z)$ and use Lemma 1 to conclude.
Otherwise, we have $z \preceq x$, so that $y \succ x$. This implies that $V_2$ is almost decisive for $(y, x)$ and use Lemma 1 to conclude.
Proof

- unanimity implies that $N$ is decisive
- since $N$ is finite, the iterated use of Lemma 2 leads to the existence of a dictator

Analysis of principles

Principles

- Unanimity: no apparent problem
- Absence of dictator: minimal requirement of democracy!
- Universality: a group adopting functioning rules that would not function in “difficult situations” could be in big trouble!
Unimodal preferences

**Ideal point**

left \[\uparrow\] right

**Consequences**
- if the preferences of all voters are unimodal with the same underlying axis
- Condorcet’s paradox cannot occur

**Problem**
- not true if more than one axis!

Independence

**Interpretation**
- no intensity of preference considerations
  - I “intensely” or “barely” prefer \(a\) to \(b\)
  - practice: manipulation, interpersonal comparisons?
- no consideration of a third alternative to rank order \(a\) and \(b\)
Borda and independence

Example
- 4 candidates \( \{a, b, c, d\} \)
- 3 voters
  - 2 voters: \( c \succ a \succ b \succ d \)
  - 1 voter: \( a \succ b \succ d \succ c \)

Borda scores
\[
\begin{array}{cccc}
a & b & c & d \\
5 & 6 & 8 & 11 \\
\end{array}
\]
- \( a \) is elected

Example
- 4 candidates \( \{a, b, c, d\} \)
- 3 voters
  - 2 voters: \( c \succ a \succ b \succ d \)
  - 1 voter: \( a \succ c \succ b \succ d \)

Borda scores
\[
\begin{array}{cccc}
a & b & c & d \\
5 & 9 & 4 & 12 \\
\end{array}
\]
- \( c \) is elected

Transitivity

Remarks
- maybe too demanding if the only problem is to elect a candidate
  - absence of circuit is sufficient
  - but... guarantees consistency

in \( \{a, c\} \), the maximal elements are \( a \) and \( c \)
in \( \{a, b, c\} \), the maximal element is \( a \)
Relaxing transitivity

From weak orders to...

- semi-orders and interval orders
  - no change (if more than 4 candidates)
- transitivity of strict preference
  - oligarchy: group $O$ of voters st
    
    \[ a \succ_i b, \forall i \in O \Rightarrow a \succ b, \exists i \in O : a \succ_i b \Rightarrow \text{Not}[b \succ a] \]

- absence of circuits
  - some voter has a veto power
    
    \[ a \succ_i b \Rightarrow \text{Not}[b \succ a] \]

Underlying message

Naive conclusion

- despair

But...

- the existence of an “ideal” method would be dull!
  - analyze the pros and cons of each method
  - beware of “method-sellers”
- a group is “more complex” than an individual
Extensions

Impossibility results

- logical tension between conditions
- Arrow
- Gibbard-Satterthwaite
  - all “reasonable methods” may be manipulated (more or less easily or frequently)
- Moulin
  - no separable method can be Condorcet
  - no Condorcet method can give an incentive to participate
- Sen
  - tensions between unanimity and individual freedom

Paretian Liberal Paradox

Remarks

- obvious tensions between the majority principle and the respect of individual rights
- tensions between the respect of individual rights and the unanimity principle

Theorem (Sen, 1970)

The combination of unanimity, universality and respect of individual rights implies problems
Sen: Paretian liberal paradox

Example
- 2 (male) individuals on a desert island
  - $x$ the Puritan
  - $y$ the Liberal
- a pornographic brochure
- 3 social states
  - $a$: $x$ reads
  - $b$: $y$ reads
  - $c$: nobody reads
- preferences
  - $x$: $c \succ a \succ b$
  - $y$: $a \succ b \succ c$

Characterization results
- find a list of properties that a method is the only one to satisfy simultaneously
  - Borda
  - Copeland
  - Plurality

Example of result
- neutral, anonymous and separable method are of Borda-type (Young, 1975)
Analysis results

- find a list of desirable properties
- not an easy task!
- fill up the methods / properties table

Ideally

- characterization results will use intuitive axioms
- analysis results will lead to characterization and/or impossibility results

Other aspects

- institutional setting
- welfare judgments
  - voting on taxes
- direct vs indirect democracy
- electoral platforms
- paradox of voting (why vote?)
Why vote?

Voting has a cost
- I have to go to the polling station
- I had rather go fishing

Analysis
- the probability that my vote will change the results is nil
- why should I bother?

Models
- economic explanations
- sociological explanations
  - not fully convincing on their own

Ostrogorski’s Paradox

Representative democracy
- you vote for a party that has a position on several issues (economic, social, international, etc.)
- no party can be expected to represent your opinion on every issue
- why vote for parties instead of issues?
Ostrogorski’s Paradox

Example

- 5 voters, 2 parties (X and Y), 3 issues

<table>
<thead>
<tr>
<th></th>
<th>issue 1</th>
<th>issue 2</th>
<th>issue 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>voter 1</td>
<td>X</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>voter 2</td>
<td>Y</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>voter 3</td>
<td>Y</td>
<td>Y</td>
<td>X</td>
</tr>
<tr>
<td>voter 4</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>voter 5</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

- on issue 1, voter 1 agrees with party X

- if each voter votes for the party with which he agrees on a majority of issues, Y wins
- the loosing party X agrees with a majority of voters on each issue!

Anscombe’s paradox

Example

- 5 voters, 2 parties (X and Y), 3 issues

<table>
<thead>
<tr>
<th></th>
<th>issue 1</th>
<th>issue 2</th>
<th>issue 3</th>
<th>minority</th>
<th>majority</th>
</tr>
</thead>
<tbody>
<tr>
<td>voter 1</td>
<td>X</td>
<td>X</td>
<td>Y</td>
<td>minority</td>
<td></td>
</tr>
<tr>
<td>voter 2</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>minority</td>
<td></td>
</tr>
<tr>
<td>voter 3</td>
<td>Y</td>
<td>X</td>
<td>X</td>
<td>minority</td>
<td></td>
</tr>
<tr>
<td>voter 4</td>
<td>X</td>
<td>Y</td>
<td>X</td>
<td>majority</td>
<td></td>
</tr>
<tr>
<td>voter 5</td>
<td>X</td>
<td>Y</td>
<td>X</td>
<td>majority</td>
<td></td>
</tr>
<tr>
<td>result</td>
<td>X</td>
<td>Y</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- on issue 1, voter 1 agrees with party X

Analysis

- vote on issues
- a majority of voters can be frustrated on a majority of issues!
Direct and undirect democracy

Referendum paradox
- direct democracy: referendum
- indirect (representative) democracy: parliament

Paradox
- these two methods can lead to different results...
- even if each MP votes according to the opinion of the majority of his electors

<table>
<thead>
<tr>
<th></th>
<th>MP1</th>
<th>...</th>
<th>MP167</th>
<th>MP168</th>
<th>...</th>
<th>MP200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>7000</td>
<td></td>
<td>7000</td>
<td>15000</td>
<td></td>
<td>15000</td>
</tr>
<tr>
<td>No</td>
<td>8000</td>
<td></td>
<td>8000</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

“No” wins in assembly (167/200 = 83%)
“Yes” wins in referendum (55%)