Approval Voting for Committees:

Threshold Approaches

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Abstract

When electing a committee from the pool of individual candidates, it is not sufficient to elicit voters preferences among individual candidates and one should also take into account voters' opinions about synergetic effects of candidate interactions if they are jointly elected into a committee. We propose an approval voting method in which each voter selects a set of candidates indicating his or her approval of any committee that has sufficiently many candidates from the selected set. The committee approved by most voters is elected.

1 Introduction

Selecting a set of candidates (committee, group, team) is as commonplace as selecting a single candidate. Shareholders of a company need to elect and approve its corporate directors, consulting firms assign teams of consultants to projects, legislative bodies select committees and subcommittees, coaches select starting players on a sports team, etc. In some cases, the selection is made by a single decision-maker, but in many cases the selection procedure has to aggregate preferences (opinions, votes) from a large number of stakeholders (e.g., shareholders). In this paper, we will focus on the aggregation the latter problem, i.e., we will investigate voting procedures for committee selection.

There is a large social choice theory literature devoted exclusively to methods of aggregating preferences into a consensus choice. The central results of the social choice theory are impossibility results showing that no aggregation procedure can have some rather reasonable properties (e.g., Arrow's Theorem [1], Gibbard-Satterhwaite Theorem [4, 6]) and some of the research activities in this field revolve around analyzing what restrictions allow avoiding these impossibility results (e.g., restrictions on preference structures, [7]). Despite these negative results, there is a continual research interest in the analysis of desirable and less desirable properties of and comparison of standard voting procedures that are used in practice, such as Borda-like methods (e.g. [5]), approval voting [2], etc.

All these results and methods could be applied to aggregate voters' preferences over committees, but one does not observe any of such methods in practice. One problem is that the number of possible committees that can be constructed from even a modest pool of candidates could be prohibitively large and unmanageable. For example, there are more than two million different seven-member committees that could be constructed from a pool of 30 candidates. Identifying several most preferred candidates from a pool of 30 is doable, but even considering a small portion of more than two million possible seven member committees might be too much to ask from a voter.

Instead, a typical committee selection aggregates voters' preferences over candidates and then constructs a committee consisting of the top individual candidates. For example, if a committee of seven is to be selected, seven candidates receiving the most votes are chosen into a committee. A slightly more involved method, that ensures representativeness of different types of candidates, is to divide candidates according to some criterion and to form a committee from the top candidates in each category. For example, five starters in an all-star NBA team consist of the top center and top two forwards and top two guards, according to the submitted votes for individual players.

While methods of selecting a committee based on the aggregation of preferences over individual candidates are simple and intuitive, such aggregation by its design does not take into account any interdependencies among candidates. Consider selecting a two-member committee from the pool of three candidates 1,2,3 and suppose that every voter agrees that candidates 1 and 2 are better than candidate 3, and that having both 1 and 2 on the same committee is the worst possible choice (i.e., every voter prefers committees {1,3,} and {2,3} to the committee {1,2}). Clearly, every sensible procedure of ranking individual candidates ranks 1 and 2 over 3, and cannot be a basis for selecting any other committee but {1,2}, which is the least preferred by all voters.

The purpose of this paper is to propose procedures for committee selection that consider interdependencies, but that remain simple and avoid the managability issue due to the large number of potential committees to choose from. In particular, we adopt the approval voting procedure and modify its aggregation rule to reflect an "approval" of a committee. Informally, every voter selects a set of candidates and the procedure assumes that the voter approves of every committee that has a sufficiently large intersection with the selected set of candidates. (As will be discussed, the definition of "sufficiently large" intersection could depend on the size of the committee; all this is predefined and publicly known prior to the selection of the set of candidates.) Note that a voter can approve more than one committee, but our procedure does not allow for different intensities of approvals (i.e., every committee is either approved or not by each voter). The selected (winning) committee is that committee that is approved by the most voters. The procedure will be defined in the next section.

2 Definitions

We assume throughout that there are n voters ($i = 1, 2, \ldots, n$) and m candidates ($j = 1, 2, \ldots, m$). The set of all subsets of $[m] = \{1, 2, \ldots, m\}$ is denoted by $2^{[m]}$, and the set of all potential committees with one or more candidates is $2^{[m]} \setminus \{\emptyset\}$.

We denote by ξ the set of admissible committees for a particular application and assume that ξ is a nonempty subset of $2^{[m]} \setminus \{\emptyset\}$. The set of all k-member committees is denoted by ξ_k. Thus, when a committee of k is to be elected and all k-member committees are admissible, we set $\xi = \xi_k$. When all nonempty committees are admissible, $\xi = 2^{[m]} \setminus \{\emptyset\}$.

Let V_i be the set of candidates approved of by voter i. We allow V_i to be any subset of $[m]$, including the empty set. An approval voting ballot profile, or profile for short, is an n-tuple

$$V = (V_1, V_2, \ldots, V_n)$$

We will often abuse the notation and denote sets of candidates approved by a voter, without using standard set notation and by simply listing the approved candidates. For example, instead of writing $V_i = \{3, 5, 6\}$, we will sometimes write 356.

The set of all possible profiles is $\mathcal{V}=\left(2^{[m]}\right)^n$. We define V_iS as the number of candidates in committee S approved of by voter i so that

$$V_iS = |V_i \cap S|.$$

Then n-tuple $VS = (V_1S, V_2S, \ldots, V_nS)$, tells how many candidates in S each voter approves of.

Example 2.1 A committee of three is to be chosen from eight candidates (1,2,. . . ,8). There are nine voters (1,2,. . . ,9) with the following approval voting ballot profile:

Voter	1	2	3	4	5	6	7	8	9
AV Ballot	2	12	12	13	37	45	46	47	48

In other words, we have n=9, m=8, $\xi=\xi_3$, and the profile is $V = (2, 12, 12, 13, 37, 45, 46, 47, 48)$. To illustrate notation VS, consider a committee $S = \{1, 3, 4\}$. Then $VS = (0, 1, 1, 2, 1, 0, 0, 0, 0)$.

∎

A threshold function is a map t from ξ into $\Re^+$, where $\Re^+ = \{x \in \Re : x > 0\}$. It will become clear that the codomain of t could be taken as $\{1, 2, 3, \ldots\}$, but we will not presume this. Let $\mathcal{T}$ be the set of threshold functions.

Given ξ, let C be a map from $\mathcal{V}$ to the set of nonempty subsets of ξ. We refer to C as a choice function and to $S \in \xi \cap C(V)$ as a choice for profile V. Given ξ and $t \in \mathcal{T}$, the choice function for threshold function t is defined by

$$S \in C_t(V) \text{ if } [S \in \xi \text{ and } |\{i : V_iS \geq t(S)\}| \geq |\{i : V_iT \geq t(T)\}| \text{ for all } T \in \xi\ ].$$

Thus $S \in \xi$ is a choice for profile V and threshold function t if as many voters approve of S (according to t) as any other admissible committee. We refer to C_t for $t \in \mathcal{T}$ as an approval voting threshold choice function, or simply TCF for short.

Remark 2.2 When TCF is used, technically, one can allow for all committees to be admissible extending t to all of $2^{[m]} \setminus \{\emptyset\}$ by setting $t(S) = |S| + 1$ for all $S \notin \xi$. Obviously, with such t only $S \in \xi$ can be approved. ▮

Remark 2.3 Note that TCF does not account for intensity of voter's approval of a committee. It simply records whether $V_i S$ is above the threshold $t(S)$ (i.e., whether voter i approves of committee S under given threshold function t) and not the actual value of $V_i S$. Note that the method based on approval voting ballots in which voter i assigns $V_i S$ votes to a committee S is equivalent to choosing a committee of candidates that maximizes the sum of individual approval voting scores of candidates from a committee:

$$\sum_{i=1}^{n} V_i S = \sum_{i=1}^{n} \sum_{j \in S} V_i\{j\} = \sum_{j \in S} \sum_{i=1}^{n} V_i\{j\} = \sum_{j \in S} |\{i : j \in V_i\}|.$$

In other words, aggregating approval voting ballots in this way is nothing else but the method of first aggregating voters preferences for individual candidates and then constructing a committee based on this aggregated ranking. As mentioned in the introduction, in this paper we explore methods that allow for aggregating voters preferences that go beyond individual preferences, so we do not study this method (or similar ones that can be constructed from any method that aggregates preferences over individual candidates by utilizing only the aggregated ranking of individuals to select a committee). ▮

Example 2.4 *Example 2.1 revisited.*

Before illustrating threshold functions, note that under ordinary approval voting for individual candidates (where each voter i approves of candidate j if and only if $j \in V_i$, and the ranking of candidates is determined by the number of voters who approve of each of them), the three candidates with the most approval votes are 1,2, and 4 (4 being approved by four voters, 1 and 2 each approved by three voters; any other candidate is approved by at most two voters). Thus, if one were to use the results of the approval voting for individual candidates to construct a winning three member committee, the elected committee would be {1,2,4}. As mentioned in Remark 2.3, this outcome could be reached by assigning $V_1S + \ldots + V_9S$ votes to committee S. Committee $\{1, 2, 4\}$ gets 10 votes, while any other three member committee gets at most nine votes.

Next consider two threshold rules.

t ≡ 1. Voter i "approves" of a committee S if and only if there is at least one of her/his "approved" candidates is among the three members of the committee S, i.e., if and only if $V_iS \geq 1$. The only three member committee approved of by all voters is {2,3,4}. Thus $C_t(V) = \{\{2, 3, 4\}\}$.

t ≡ 2. Voter i "approves" of a three-member committee S if and only if the majority (i.e., at least two) of the committee members are approved by her/him, i.e., if and only if $V_iS \geq 2$. Note that under this rule voters 2, 3 and 4 approve of committee {1,2,3}, and no other committee of three has more than two approving voters. Thus $C_t(V) = \{\{1, 2, 3, \}\}$.

∎

Before moving on to study properties of TCF's, let us note that computing TCF, and selecting the committee based on some numerical score in general, could be time-consuming. Essentially, one is unlikely to avoid having to compute the number of votes (or a score) for (almost) every admissible committee $S \in \xi$. If ξ is large, this computational issue could become important. More formally, TCF can be computed in time that is polynomial in $nm + |\xi|$ (the first term corresponds to the number of bits needed to represent the approval voting ballot profile and the second is the number of admissible committees) by simply calculating $|\{i : V_i S \geq t(S)\}|$, but computing TCF is NP-complete if the input is only nm (e.g., if ξ is predefined).

Example 2.5 Consider electing a k-member committee, i.e., let $\xi = \xi_k$. Let $t \equiv 1$, and let $|V_i| = 2$ for all i. To see this, note that there is a one-to-one correspondence between all such voter profiles and graphs whose vertex set is $[m]$ and whose edges are $V_1, V_2, \ldots, V_n$. Thus determining whether there exists $S \in \xi$ approved by all voters , i.e., such that $V_i S > 0$ for all i, is equivalent to determining whether S is a vertex cover for the corresponding graph. Determining whether a graph has a vertex cover of size k is one of the fundamental NP-complete problems [3]. Therefore, determining whether there exists $S \in \xi_k$ approved by all voters is also NP-complete. ∎

The fact that computing TCF requires comparing any admissible committee S with every voter profile V_i is not the source of NP-completeness. Even if one has to select a committee that maximizes the sum of scores of individuals in the committee (as is the case with any method that first aggregates votes for individuals and then, based on these votes chooses an admissible committee) is computationally hard: given the list of individual votes $v(j)$, $j = 1, ..., m$, finding $S \in \xi$ that maximizes $\sum_{j \in S} v(j)$ is also NP-complete [3].

3 Properties of TCF's

We note three properties shared by all approval voting threshold choice functions before we consider specialized aspects of threshold functions and sets of admissible committees. The first property merely reiterates features of the definition of C_t and mentions t explicitly.

Property 1: Consistency. For all $S, T \in \xi$ and all $V \in \mathcal{V}$, if $VS = VT$ and $t(S) = t(T)$ then $S \in C_t(V) \Longleftrightarrow T \in C_t(V)$.

The other two properties do not involve t explicitly. The first is a voter anonymity condition.

Property 2: Anonymity. Suppose choice function C is a TCF. For all $S \in \xi$ and all $U, V \in \mathcal{V}$ if US is a permutation of VS then $S \in C(V) \Longleftrightarrow S \in C(U)$.

Property 1 can be strengthened to reflect anonymity. Given V and S, let $s(V, S) = (s_0(V, S), s_1(V, S), \ldots, s_m(V, S)) = (s_0, s_1, \ldots, s_m)$, where

$$s_h = s_h(V, S) = |\{i : V_i S = h\}|,$$

the number of voters whose approval sets contain exactly h members of S. We refer to $s(V, S)$ as the <u>score sequence</u> for V and S. Obviously,

$$\sum_{h=0}^{m} s_h = n, \qquad s_h = 0 \text{ for all } h > |S|.$$

Now, consistency and anonymity could be combined into a single property:

Property 1*: Consistency and Anonymity. For all $S, T \in \xi$ and all $U, V \in \mathcal{V}$, if $s(V, S) = s(U, T)$ and $t(S) = t(T)$ then $S \in C_t(V) \Longleftrightarrow T \in C_t(U)$.

Next we define a partition-consistency condition. The partition aspect refers to a division of the n voters into two disjoint groups with p and $n-p$ voters. Given $V=(V_1,\dots V_n)$ let

$$V_p^- = (V_1, \dots V_p, \emptyset, \dots, \emptyset), V_p^+ = (\emptyset, \dots, \emptyset, V_{p+1}, \dots V_n),$$

with $1 \le p < n$

Property 3: Partition-consistency. Suppose $S \in C(V_p^-)$ and $S \in C(V_p^+)$, then $S \in C(V)$.

This says, in effect, that if $S \in \xi$ is a choice for each of two disjoint groups of voters under the same type of TCF, then S is also a choice for the combined group under the same type of TCF.

If we remove the TCF restriction from Properties 2 and 3, then the resulting conditions are necessary for a choice function C to be a TCF. It is then natural to consider additional conditions on C to yield a set of conditions that are necessary and sufficient to imply that C is a TCF, i.e., that there exists some $t \in \mathcal{T}$ for which $C = C_t$.

4 Threshold Functions

We use TF henceforth as abbreviation of threshold function. We refer to $t \in \mathcal{T}$ as a cardinal TF if, for every $S \in \xi$, $t(S)$ depends only on $|S|$, and as a constant TF is $t(S) = t(T)$ for all $S, T \in \xi$. A constant TF is clearly a cardinal TF, and if $\xi \subseteq \xi_k$ for some k, then a cardinal TF is also a constant TF. (Example 2.4 provides an example of a threshold function.) For larger admissible sets, for example $\xi = 2^{[m]} \setminus \{\emptyset\}$, a cardinal TF is nondecreasing if, for all

$S, T \in \xi$, $|S| < |T|$ implies $t(S) \leq t(T)$. Similarly, a cardinal TF is nonincreasing if, for all $S, T \in \xi$, $|S| < |T|$ implies $t(S) \geq t(T)$. Apart from constant TF's, we will not pay much attention to nonincreasing TF's because it seems odd to have $t(S) > t(T)$ when $|S| < |T|$. The following lemma provides a connection to choice functions.

Lemma 4.1 *If t is a nonincreasing cardinal TF then, for all $S, T \in \xi$ for which $S \subset T, S \in C_t(V) \Rightarrow T \in C_t(V)$.*

Proof. When t is nonincreasing and $S \subset T$, we have $V_i S \leq V_i T$ and $t(S) \geq t(T)$, so if a voter approves of S, i.e., if $V_i S \geq t(S)$, then that voter also approves of T, since $V_i T \geq V_i S \geq t(S) \geq t(T)$. It follows that if $S \in C_t(V)$ then $T \in C_t(V)$. ▮

Because Lemma 4.1 applies to constant TF's, such functions favor larger committees as choices when ξ includes committees of various sizes. For example, if $t(S) = 1$ for all $S \in \xi$, and $\xi = 2^{[m]} \setminus \{\emptyset\}$, then $[m]$, the committee of the whole, is a choice for every $V \in \mathcal{V}$. We shall therefore consider constant functions primarily in restricted settings such as $\xi = \xi_k$.

Note that cardinal TFs satisfy neutrality: for every permutation π of [m] and every $S = \{j_1, \ldots, j_k\}$, $t(S) = t(\{\pi(j_1), \ldots, \pi(j_k)\})$, i.e., they treat candidates equally. This feature may make cardinal TFs inappropriate for some contexts. For example, suppose a committee of five is to be chosen from 18 candidates who divide naturally into three distinct groups. If it is desirable but not mandatory to have at least one candidate from each group on the chosen committee, then t could be biased in favor of representativeness. With ξ_5^r the set of committees is ξ_5 that represent precisely r of the three groups, one might take $t(S) = 2$ for $S \in \xi_5^3$, $t(S) = 3$ for $S \in \xi_5^2$, and t(S)=5 for $S \in \xi_5^1$. However, if only the committees in S_5^3 are admissible, we could let $\xi = \xi_5^3$ and take t constant there.

Other restrictions on t could be considered for larger admissible sets such as, for all $S, T \in \xi$, (i) $1 \leq t(S) \leq |S|$, (ii) $S \subset T \Rightarrow t(S) + 1 \leq t(T)$, and/or (c) $S \cap T = \emptyset \Rightarrow t(S \cup T) \leq t(S) + t(T)$, etc. For example, when $\xi = 2^{[m]} \setminus \{\emptyset\}$, (i) and (ii) imply that $t(S) = |S|$ for all $S \in \xi$.

We focus henceforth on cardinal TF's. (Thus, when $\xi \subseteq \xi_k$, cardinal TF is a constant TF.)

The following lemma for this case provides a counterpart to Lemma 1.

Lemma 4.2 *If $\xi = 2^{[m]} \setminus \{\emptyset\}$ and $t(S) = |S|$ for all $S \in \xi$ then, for all $S, T \in \xi$ for which $S \subset T$, $T \in C_t(V) \Rightarrow S \in C_t(V)$.*

Proof. Under the lemma's hypotheses, including $S \subset T$, every voter who approves of T also approves of S, and it follows, that if $T \in C_t(V)$ then $S \in C_t(V)$. ∎

Lemma 4.2 shows that some TF's in the general case could favor smaller committees as choices from ξ. If necessary, this could be counteracted by adopting a secondary choice that only larger members of $C_t(V)$ are "acceptable". It could also be counteracted by adopting an entirely different threshold function that is biased towards committees of a certain size.

A related idea that might tend to elect moderate-sized committees concerns majority threshold functions. We define the majority TF by $t(S) = |S|/2$ for all $S \in \xi$, and the strict majority TF by $t(S) = (|S|+1)/2$ for all $S \in \xi$. The definitions apply to all admissible sets, including ξ_k, where the majority and strict majority TF's are equivalent for C when k is odd. In Example 2.4 we have already seen the (strict) majority threshold function ($t \equiv 2$) when $\xi = \xi_3$. The following example illustrates the strict majority TF with $\xi = 2^{[m]} \setminus \{\emptyset\}$.

Example 4.3 Suppose n=12, m=8, and

$$V = (123, 15, 16, 278, 23, 24, 2578, 34, 347, 46, 567, 568).$$

Under the strict majority TF, the maximum approvals for a 1-member committee is 4, for 2-member committee it is 2 (34,23,56,57,58,78), for a 3-member committee it is 5 (only for 234), for a 4-member committee it is 3 (5678), and for a 5-member committee it is 4 (15678). Hence {2,3,4} is the only member of $C_t(V)$. ∎

5 Top individuals might not be top team players

In this section we demonstrate that TCF procedure aggregates voters preferences over teams; i.e., that top individuals might not be selected into a winning committee.

Under ordinary approval voting, the candidates that appear in the most $V_i's$ constitute the "best" committees. This is not generally true for the majority and strict majority TF's, as already demonstrated by Example 2.4.

Proposition 5.1 *When* $\xi = 2^{[m]} \setminus \{\emptyset\}$ *and t is the majority or the strict majority TF, there exist* $n, m,$ *and* $V \in \mathcal{V}$ *such that candidate 1 is in a majority of the* V_i *and in more* V_i*'s than any of other candidates, but candidate 1 is in no* $S \in C_t(V)$.

Proof. First consider the majority TF. Suppose n=6, m=5, and $V = (123, 124, 135, 145, 25, 34)$. Note that $C_t(V) = \{2345\}$ since every voter approves of the 4-member committee 2345, and no other committee has unanimous approval. However, candidate 1 is in the most V_i's (four), while every other candidate is in three V_i's.

Next consider the strict majority TF. Suppose $n = 10$, $m = 6$, and

$$V = (1234, 1236, 1245, 1256, 1345, 1346, 1456, 235, 246, 356).$$

Because every voter approves of three of the five candidates in {2,3,4,5,6}, this 5-member committee has unanimous approval. No other committee has unanimous approval (the closest being 12356 with 9 approvals), so $C_t(V) = \{23456\}$. But candidate 1 is in seven V_i's, while the others are in six V_i's. ▮

The examples from the preceding proof can be modified by adding a few more voters and many more candidates without changing the conclusion of the Proposition 5.1. For example, if we add three voters to the latter example and take $V_{11} = V_{12} = V_{13} = \{7, 8, \ldots 100\}$ then $C_t(V)$ is still {2,3,4,5,6} and candidate 1 is in more V_i's (a majority) than any other candidate.

Note that the proposition also holds for almost any threshold function t:

Proposition 5.2 *Let t be a threshold function such that $t(S) \geq 2$ for every $S \in \xi$ for which $1 \in S$. Suppose that there exist $T \in \xi$ such that $1 \notin T$, $t(T) \leq |T|$ and such that $AT < t(A)$ for every $A \neq T$. Then, for every $n \geq 3$, there exists a $V \in \mathcal{V}$ such that candidate 1 is in a majority of the V_i's and in more V_i's than any of other candidates, but candidate 1 is in no $S \in C_t(V)$.*

Proof. Take $V_n = T$ and $V_1 = V_2 = \ldots = V_{n-1} = \{1\}$. Then, $C_t(V) = \{T\}$ since T is the only committee approved by voter n, and none of the other voters approves of any admissible committee. On the other hand, 1 is approved by $n - 1$ voters, while any other candidate is approved by at most one voter. ▮

The proof assumes that there are voters who choose not to approve of any committee by approving candidate 1 only. In the next proposition this is not the case and the same result still holds.

Proposition 5.3 *Let* $S', S", T \in \xi$, $\{1\} = S' \bigcap S''$, $(S' \cup S'') \cap T = \emptyset$. *Let* $t(S') \leq |S'|, t(S'') \leq |S''|, t(T) \leq |T|$. *Suppose that for every* A *such that* $1 \in A$, *(i)* $AT < t(A)$ *and (ii) either* $S'A < t(A)$ *or* $S''A < t(A)$. *Then, for every* $n \geq 9$, *there exists a* $V \in \mathcal{V}$ *such that candidate 1 is in a majority of the* V_i *and in more* V_i*'s than any of other candidates, but candidate 1 is in no* $S \in C_t(V)$.

Proof. First observe that for every $n \geq 9$ (and for $n = 7$), there exists positive integers p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = n$, $p_1 + p_2 > p_3$ and $p_3 > max\{p_1, p_2\}$. Let

$$V_i = \begin{cases} S' & \text{if } 1 \leq i \leq p_1 \\ S" & \text{if } p_1 < i \leq p_1 + p_2 \\ T & \text{if } p_1 + p_2 < i \leq n \end{cases}$$

By assumptions made, in approval voting for individual candidates 1 is approved by $p_1 + p_2$ voters, while no other candidate is approved by more than $p_3 < p_1 + p_2$ voters. Note that any set A, such that $1 \in A$ is approved by at most $\max\{p_1, p_2\}$ candidates and thus no such set is in $C_t(V)$ because T is approved by $p_3 > \max\{p_1, p_2\}$ voters. ▮

There can be even more candidates that are top individual choices and that do not belong to any winning committee. For example, when $t \equiv 1$, $k = 3$ and (n,m)=(3,4), profile $V = (123, 123, 4)$ has $C_1(V) = \{124, 134, 234\}$. In this case no $S \in C_1(V)$ contains all three of the most popular candidates. The following theorem notes a stronger result.

Theorem 5.4 *Suppose* $1 \leq t \equiv \alpha < k$. *Then there are* n, m *and a profile* V *with* $C_\alpha(V) = \{S\}$ *such that at least* k *candidates not in* S *each appears in more* V_i*'s than any candidates in* S.

Proof. Suppose $t \equiv 1 < k$. When $k = 2$, profile

$$V = \{1, 2, 134, 134, 234, 234\}$$

has $C_1(V) = \{12\}$, and each of 3 and 4 is in more V_i's than either of 1 or 2. When $k \geq 3$, let $n = m = 2k$ with

$$V = (1, 2, \ldots, k, \{1, k+1, k+2, \ldots 2k\}, \ldots, \{k, k+1, k+2, \ldots, 2k\}).$$

Then $C_1(V) = \{1, 2, \ldots, k\}$, Each $j \leq k$ is in two V_i's and each $j > k$ is in $k > 2$ V_i's.

Suppose $2 \leq \alpha < k$. Let L be a list of the $\binom{k}{\alpha}$ subsets of $\{1, 2, \ldots, k\}$ that have α members. Let $m = 2k$, and let L' be a list of the $\binom{k}{\alpha}$ subsets of $\{1, 2, \ldots, k\}$ with α members in union with $\{k+1, k+2, \ldots, 2k\}$. For example, when $\alpha = 2$ and $k = 3$, $L = (12, 13, 23)$ and $L' = (12456, 13456, 23456)$. Let r be a positive integer greater than $\alpha/(k - \alpha)$, let $n = (r+1)\binom{k}{\alpha}$, and let

$$V = (L, rL'),$$

where rL' denotes r repetitions of L'. Then $\{1, 2, \ldots, k\}$ has unanimous approval under

$t \equiv \alpha$ and is the only member of ξ_k with this property, for any other member of ξ_k lacks the approval of at least one of the first $\binom{k}{\alpha}$ voters. Hence $C_\alpha(V) = \{1, 2, \ldots, k\}$. The number of voters with candidate $j > k$ in their approval sets is $\binom{k}{\alpha}$, and the number with candidate $j \leq k$ in their approval sets is

$$(r+1)\left[\binom{k}{\alpha} - \binom{k-1}{\alpha}\right] = (r+1)\binom{k}{\alpha}\frac{\alpha}{k}.$$

Because our choice of r ensures that $r > (r+1)\frac{\alpha}{k}$, each candidate in $\{k+1, \ldots, 2k\}$ is in more approval sets than each candidate in $\{1, 2, \ldots, k\}$. ▮

Remark 5.5 An interesting combinatorial adjunct of Theorem 5.7 is to determine the minimum n for each $k \geq 2$ which admits a V that satisfies the conclusion of the theorem. The minimum n for $k = 2$ is easily seen to be $n = 4$ with $V = (1, 2, 134, 234)$. Then $C_1(V) = \{\{1, 2\}\}$ and $C_2(V) = \{\{3, 4\}\}$. The minimum n for $k = 3$ is $n = 6$ with $V = (1, 3, 256, 346, 2789, 145789)$, in which case $C_1(V) = \{123\}$, $C_2(V) = \{456\}$ and $C_3(V) = \{789\}$. We are not certain of the minimum n for $k = 4$ but know that it is no greater than 10 because the 10-voter profile

$$V = (1, 2, 3, 4, \{1, 5, 8, 9, 11, 12\}, \{2, 6, 7, 10, 11, 12\}, \{3, 5, 7, 9, 10, 11\},$$

$$\{4, 6, 8, 9, 10, 12\}, \{1, 5, 6, 13, 14, 15, 16\}, \{2, 7, 8, 13, 14, 15, 16\})$$

has $C_1(V) = \{1234\}$, $C_2(V) = \{5678\}$, $C_3(V) = \{\{9, 10, 11, 12\}\}$ and $C_4(V) = \{\{13, 14, 15, 16\}\}$.

▮

It should be noted that the behavior exhibited by the preceding propositions and the theorem shows only what is possible and not necessarily what is probable. We would ordinarily expect that the most popular candidates will be contained in some of the committees chosen by the majority and and strict majority TF'. However, one can envision situations in which top individual performers are not considered suitable for committee work.

The following proposition shows that, when all committees are acceptable, there is only one cardinality TF ensures that every candidate in any selected committee (i.e., every member of C_t) is approved by at least one voter.

Proposition 5.6 *Suppose t is a cardinal TF with $1 \leq t(S) \leq |S|$ for every $S \in \xi = 2^{[m]} \setminus \{\emptyset\}$, where $m \geq 2$. Then $t(S) = |S|$ for all $S \in \xi$ if and only if, for every $V \in \mathcal{V}$ that has $|V_i| > 0$ for at least one i, every $S \in C_t(V)$ contains only candidates with one or more votes.*

Proof. If $t(S) = |S|$ for all $S \in \xi$ and V is not $(\emptyset, \ldots \emptyset)$, then every $S \in C_t(V)$ is a subset of some V_i and therefore has positive support for every candidate therein.

Suppose t is not equivalent to the preceding TF. Then there is a k between 2 and m and an integer α such that $1 \leq t(S) = \alpha \leq k - 1$ for all $S \in \xi_k$. With $K = \{1, 2, \ldots \alpha\}$, the constant profile $V = (K, K, \ldots K)$ has unanimous approval for every $S \in \xi_k$ that includes K and $k - \alpha$ other candidates from $[m] \setminus K$ with no votes. ▮

As the preceding proposition shows, when cardinality TFs are used, the only way when there is a guaranteed link between approval of individual candidates and committees is that approval of all members of the committee is needed before a committee is approved.

Finally, we note that the choice of an appropriate threshold has important consequences on the committee that will be chosen. More precisely, Given ξ_k with $k \geq 2$, a voter's approval set may depend on the constant TF used to determine choice, i.e., on the value of α. However, if V is fixed and we vary α, $C_\alpha(V)$ might vary widely for the different α's. An extreme possibility is noted in the next theorem.

Theorem 5.7 *Given $k \geq 2$, there are n, m and a corresponding V such that $C_\alpha(V) = \{S_\alpha\}$ for $\alpha = 1, 2, \ldots k$ with the S_α mutually disjoint.*

Proof. Given $k \geq 2$, let S_α be a k-element set for each α in $\{1, 2, \ldots, k\}$ with the S_α mutually disjoint. Let $\bigcup S_\alpha$ be the candidate set, so $m = k^2$. With $\lambda_1, \lambda_2, \ldots, \lambda_k$ as-yet-unspecified positive integers, set

$$n = \sum_{\alpha=1}^{k} \lambda_\alpha \binom{k}{\alpha}$$

and for each α let $V \in \mathcal{V}$ have λ_α copies of each α-member subset of S_α. We will choose the λ_α so that $C_\alpha(V) = \{S_\alpha\}$.

Set $\lambda_k = 1$. Then $C_k(V) = \{S_k\}$ because S_k is the only V_i that contains at least k candidates.

Set $\lambda_{k-1} = 2$. Note that the only V_i's with $|V_i| \geq k-1$ are S_k and the two copies of each $(k-1)$-member subset of S_{k-1}. This ensures that $C_{k-1} = \{S_{k-1}\}$. (For example, when $k = 2$, $V = (1, 1, 2, 2, 34)$ with $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$.)

When $k \geq 3$, let $\lambda_{k-2}, \ldots, \lambda_1$ be increasingly larger integers so that $C_\alpha(V) = \{S_\alpha\}$ for $\alpha = k-2, \ldots, 1$. For any such α, the number of V_i which are α-member subsets of S_α equals $\lambda_\alpha \binom{k}{\alpha}$, and a suitably large λ_α will make this greater than the number of V_i that contain α or more of any given k-member subset of $S_{\alpha+1} \cup \ldots \cup S_k$. ∎

6 Concluding Remarks

The main motivation of this paper is to point out that procedures for electing a team could and should take into account voters' assessments of the team quality, and not just voters' assessments of the candidates as individuals. We suggested a method that just modifies the way approval voting ballots are counted so that a voter approves of a team if and only if the team includes a sufficient number of candidates he or she voted for. The ballots are counted using threshold functions which have natural interpretations. Constant threshold functions simply allow voters to approve of any committee that has at least some fixed number of members that are approved by the voter. Similarly, cardinal threshold functions (such as the majority threshold functions,) allow voters to approve of any committee that has, e.g., some percentage (e.g., majority) of members that are approved by the voter.

The simplicity of the proposed procedure is important since overly complicated procedures that demand too much information from the voters have a little chance of being implemented. By having to select only the set of candidates they approve of, voters avoid a potentially tremendous communicational burden of having to report their preference ranking over all possible committees. Furthermore, requiring voters to select a set of candidates instead of requiring that they rank individual candidates, is aligned with the purpose of the aggregation procedure (selecting a committee, i.e., a set of candidates). Thus, approval voting, in contrast to other widespread voting methods for selecting individual candidates, seems to be an obvious starting point for developing an aggregation method for committee selection.

While social choice theory impossibility theorems eliminate the possibility of the existence of the single best voting procedure for the committee selection, it would nevertheless be interesting to propose new committee selection methods that take into account interdependencies among individuals within aa committee and then to identify strengths and weaknesses of these methods. Such task goes beyond the scope of this paper.

It would also be interesting to analyze strategic behavior of the voters in the proposed (as well any other) committee selection procedure. Even though voters submit the ballot in the same form as in the approval voting procedure for selecting an individual winner, they could strategize in a different way when a committee is being selected. For example, depending on a threshold rule, a voter could approve of a small set of candidates that he or she would like to see jointly on the committee, thereby expressing an opinion about the interdependencies among those candidates. Of course, strategic considerations with respect to voter's assessments of the opinions and preferences of other voters also need to be analyzed.

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