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Comparing two territory partitions in districting problems: Indices and practical issues

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Abstract

Planning as part of public sector decision-making situations is an activity of critical importance, with direct relevance for urban planners. The ramifications of such decisions generally have significant effect on peoples' lives. The current paper deals with the comparison between territorial maps in the context of districting problems with a strong socio-economic component. The theoretical problem involves the comparison of two partitions in a connected, undirected, and planar graph. In considering this problem, we introduce three new indices to compare territory partitions: *compatibility*, *inclusion*, and *distance*, all of which have importance for real-world planning situations. Numerical experiments of these indices were carried out for the communes network in (*Île de France*), France.

Keywords: Decision-making; Districting problems; Comparison indices; Combinatorics

1. Introduction

Planning as part of public sector decision-making situations that include a socio-economic dimension is of critical importance, with direct relevance for urban planners. Frequently, the ramifications of such decisions have significant effects on the lives of selected populations. A vast array of applications that include socio-economic elements, the wide panoply of techniques, methods, and methodologies available, and the complex nature of real decision-making situations all have implications for the way a decision study should be designed and conducted. The current paper deals with districting problems that involve such circumstances.

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1.1. On districting problems and some practical concerns

In the last three decades, many researchers, academics, and practitioners from a variety of fields (not necessarily urban planners) have developed models, built algorithms, and implemented solutions for *districting problems*. Such problems can be viewed as a grouping process of *elementary units* or *atoms* of a given *territory* into larger pieces of land or *zones*, thus giving rise to a *partition*, also called a *district map*.

There are many practical questions/applications related to districting problems, including:

- defining the electoral districts of a country [1–4];
- establishing different work or delivery zones for a traveling salesperson team [5–9];
- defining areas within metropolitan internet networks in order to install hubs [10];
- defining a public transportation network pricing system [11];
- designing a school districting plan [12]; electrical power zones [13]; and a police districting map [14];
- constructing a districting map for salt spreading operations [15,16]; and
- defining a district-based health information system [17].

These are but a few of the many public sector partition problems that have been discussed in the literature.

1.2. Comparing two partitions

Quantitative analysts have largely concentrated their attention on the question of how to form political districts. Among the most frequent criteria employed, one deserves particular attention: comparing and evaluating the "differences" between an alternative/proposed partition and an existing one. In general, current or existing partitions were conceived some time in the past, and, for historical reasons, gain some level of popularity/acceptance over time.

As for the *political districting problem*, the importance of implementing a new partition depends on the need to regain or create voting power balance across a set of districts that constitutes the relevant political district map. In such a situation, the natural objective is to construct a new partition that "minimizes" resulting changes with respect to the existing structure [1].

The comparison of partitions may occur in a variety of fields as, for example, in the case of defining school zones. A school map determines to which school a student should be assigned according to his/her place of residence. Moreover, some public transport pricing systems are grounded on a "transport map", the price being equal for any journey within a given zone (see [11]). In such a case, it is not desirable that the school and transport maps differ significantly. Thus, a comparison between the school map and the partition transportation pricing system map becomes important.

1.3. Measuring the difference between two partitions

When analyzing the many districting problems found in the literature (see the selected list above), the comparison of two partitions can be motivated by different objectives. In the current paper, we propose three transparent measures of how two partitions of a territory can be compared with one another. Before discussing the measures, it is important to note that the literature considers the comparison of two territory partitions to be an "obvious" problem (see [18]). In contrast, we show that such comparisons can be made in (at least) three well-defined ways. Each of the proposed indices thus represents a different perspective or "take" on the problem.

Importantly, these indices can be used to (a) identify possible discrepancies between partitions, and/or (b) design new partitions, e.g., for inclusion in algorithms as criteria to be optimized, or as constraints to be fulfilled. Section 7 provides illustrations of such uses.

The proposed measures, listed in no particular order of importance, are as follows:

(1) *Compatibility*. This index is relevant when the comparison consists of checking whether a zone of the first partition is equal to a group of contiguous zones of the second; or, together with other zones of the first partition, it defines a single zone of the second.

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(2) *Inclusion*. This index is relevant when the objective is to evaluate the differences between two partitions based on the notion of "fineness", i.e., when the first partition is composed only of zones that result from a "splitting out" operation, or a division of the zones of the second partition.
(3) *Distance*. This index seeks to model the differences between two partitions. Note that it does not focus on the possible inclusions/compatibilities between the zones of two partitions, but only on how the partitions differ.

1.4. Scope and purpose of the paper

The theoretical problem of interest here is how to compare two partitions in a connected, undirected, and planar graph according to a specified attribute/factor. As suggested above, to the best of our knowledge, this problem has not been previously addressed in the literature. A few related studies were identified, but they deal with rather different situations. In particular, two earlier efforts involve attempts to measure the degree of similarity between the partitioning of political districts [1,19].

The question of how to compare two partitions led us to propose the three new concepts/indices noted above. These indices represent the current paper's contribution to the partitioning/districting literature. They are strongly dependent on real-world decision-making situations and, as we will show, they can be used with great effect in districting optimization algorithms (e.g. [20]).

1.5. Outline of the paper

Section 2 presents the key concepts and notation of our proposed framework. Sections 3, 4, and 5 are devoted to the three indices, compatibility, inclusion, and distance, respectively. Section 6 presents computational experiments and results, while Section 7 considers use of the indices in managerial problems. Finally, Section 8 presents the main conclusions, and avenues for future research.

2. Concepts, definitions, and notation

Consider the following notation:

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_n\}$ denotes a *territory*, in which each element a_i represents an indivisible *elementary unit*;
- $y = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_i, \ldots, \hat{a}_L\}$ denotes a set of contiguous elementary units, called a *zone*;
- $Y = \{y_1, y_2, \ldots, y_u, \ldots, y_K\}$ denotes a *partition* or *district map* of territory A;
- $\mathscr{Y} = \{Y', Y'', \ldots, Y^{(m)}, \ldots, Y^{(M)}\}$ denotes the set of all *feasible partitions* of territory A.

For each elementary unit a_i there is one and only one zone $y_u \in Y$ such that a_i belongs to y_u.

For the sake of simplicity, an elementary unit, a_i, is also represented by its index i. Fig. 1(a) represents a territory composed of 16 elementary units, divided into four zones, $Y = \{y_1, y_2, y_3, y_4\}$.

Given a territory $A = \{1, 2, \ldots, i, \ldots, n\}$, a *contiguity graph* is associated with A as an *undirected*, *connected*, and *planar* graph $G = (V, E)$, where $V = \{1, 2, \ldots, i, \ldots, n\}$ denotes the set of *vertices* representing elementary territorial units and $E = \{e_1, e_2, \ldots, e_k, \ldots, e_m\} \subset V \times V$ denotes the set of *edges*, where $e_k = \{i, j\}$ represents a border between two adjacent elementary units i and j. Fig. 1(b) shows the contiguity graph G corresponding to the territory of Fig. 1(a). In the current paper, territory A and its representation through the set of vertices V are considered indifferently.

Definition 2.1 (*Attribute*). Consider a contiguity graph $G = (V, E)$. An *attribute* P in V is a real-valued function defined in V such that, for each $i \in V$, $P(i) \equiv p_i \in \mathrm{R}^+$. The value p_i is thus a non-negative value.

For any subset of elementary units, $\bar{y} \subseteq V$, $P_{\bar{y}} = \sum_{i \in \bar{y}} p_i$ is the overall value of the attribute P in $\bar{y}$, and $\mathscr{P} = \sum_{i \in V} p_i$ represents the overall value of attribute P, for graph G (assume that $P_{\bar{y}} = 0$ when $\bar{y} = \emptyset$).

Let $Y = \{y_1, y_2, \ldots, y_u, \ldots, y_K\}$ and $Y' = \{y'_1, y'_2, \ldots, y'_v, \ldots, y'_{K'}\}$ denote two partitions, where $|Y| = K$ and $|Y'| = K'$.

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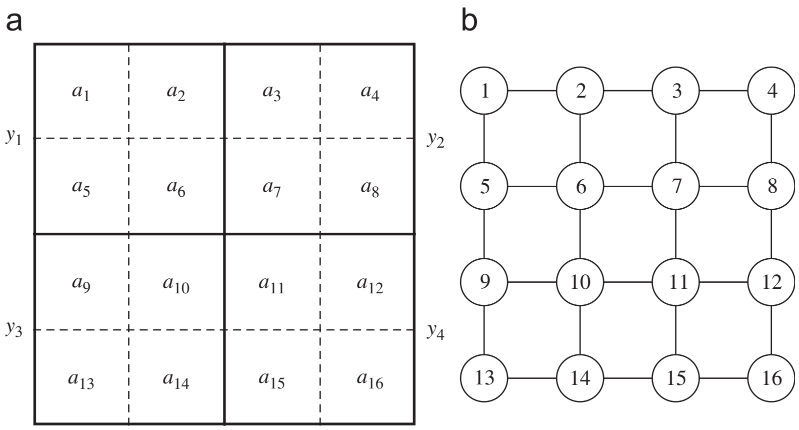

Fig. 1. A territory and the associated contiguity graph.

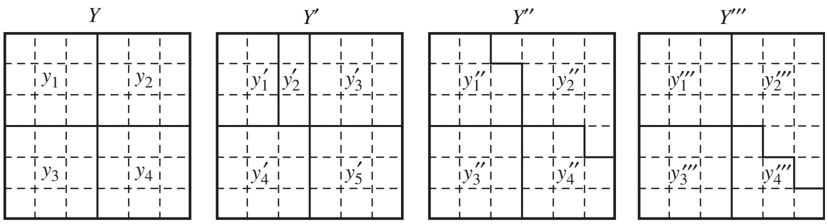

Fig. 2. Four different partitions, Y, Y', Y'', and Y'''.

Definition 2.2 (*Inclusion between two zones*). Consider two zones $y \in Y$ and $y' \in Y'$. Zone y is included into y', according to P (denoted $y \subseteq_P y'$), if, for all $i \in y$ such that $i \notin y'$ $(i \in y \backslash y')$, $p_i = 0$, i.e., $P_{y \backslash y'} = 0$.

Definition 2.3 (*Equality between two zones*). A zone $y \in Y$ is equal to $y' \in Y'$, according to attribute P (denoted $y =_P y'$), if $y \subseteq_P y'$ and $y' \subseteq_P y$.

Definition 2.4 (*Equality between two partitions*). Two partitions $Y, Y' \in \mathscr{Y}$ are equal, according to attribute P (denoted $Y =_P Y'$), if for all pairs of zones $\{y, y'\} \in Y \times Y'$, $y =_P y'$ or $P_{y \cap y'} = 0$.

Definition 2.5 (*Reference zone*). Consider two partitions Y, $Y' \in \mathscr{Y}$. The function $R_{Y'}$ is called a *reference zone function*:

$$\begin{aligned} R_{Y'}: \quad & Y \longrightarrow Y', \\ & y \longmapsto R_{Y'}(y), \end{aligned}$$

where $R_{Y'}(y) \in Y'$ maximizes $P_{y \cap y'_v}$, $v = 1, \ldots, K'$. Zone $R_{Y'}(y)$ is called the *reference zone* of y in Y'.

In other words, $R_{Y'}(y)$ is the zone belonging to Y' that contains the largest quantity of attribute P common to y.

Remark 2.1. When $P_{y \cap y'_v}$ is maximal for several $y'_v \in Y'$, $R_{Y'}(y)$ is defined arbitrarily, as the zone whose index is minimal among all the zones for which $P_{y \cap y'_v}$ is maximal.

Figs. 2 and 3 illustrate the previous definitions. Zone y'_1 is included in y''_1, for whatever attribute. When considering attribute P^1, y''_1 is also included in y'_1, i.e., $y''_1 \subseteq_{P^1} y'_1$. In this case, the equality between y'_1 and y''_1, according to P^1, is verified. If P^2 is considered, the reference zone of y''_2 in Y' is $R_{Y'}(y''_2) = y'_3$. But, the same

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P^1

1	1	0	1	1	1
1	1	0	1	1	1
1	1	0	1	1	1
1	1	1	1	1	10
1	1	1	1	1	1
1	1	1	1	1	1

P^2

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

P^3

1	1	0	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	0	0
1	1	1	1	1	0
1	1	1	1	1	1

Fig. 3. Three different attributes, P^1, P^2, and P^3.

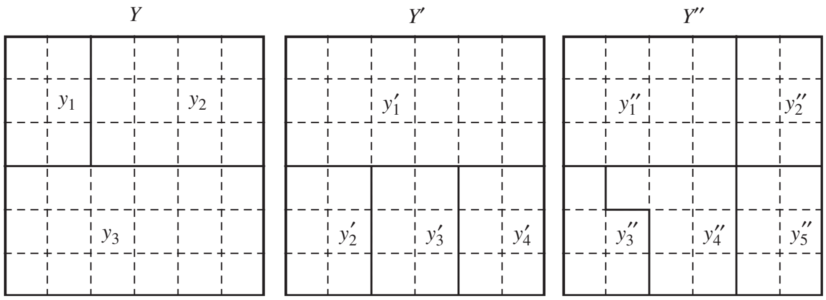

Fig. 4. Compatibility.

does not occur when taking into account P^1; in such a case, $R_{Y'}(y''_2) = y'_5$, because $P^1_{y''_2 \cap y'_5} = 10 = \max_{1 \leqslant u \leqslant 5}\{P_{y''_2 \cap y'_u}\}$.

3. Compatibility index

In this section, we consider an index that evaluates the degree of compatibility between two partitions. Two partitions Y and Y' are said to be totally compatible if, for any pair of zones $y \in Y$ and $y' \in Y'$ that overlap, either y is included in y' or y' is included in y.

3.1. Definition and structural properties

Definition 3.1 (*Total compatibility between two partitions*). Two partitions Y and Y' are *totally compatible*, *according to the attribute P* (denoted $Y \equiv_P Y'$), if for any pair of zones $\{y, y'\} \in Y \times Y'$ such that $P_{y \cap y'} > 0$, one of the following inclusions occurs, $y \subseteq_P y'$ or $y' \subseteq_P y$.

In Fig. 4, Y and Y' are totally compatible, whatever the attribute P. However, the total compatibility between Y and Y'' will depend on attribute P.

Definition 3.2 (*Overlapping pairs*). A pair of zones $\{y, y'\} \in Y \times Y'$ for which $P_{y \cap y'} > 0$ and such that $y \nsubseteq_P y'$ and $y' \nsubseteq_P y$ is called an *overlapping pair*.

Remark 3.1. Two factors should contribute to the degradation of the index if compatibility is at its best value (which occurs when there is total compatibility):

- The number of overlapping pairs: The degradation of the compatibility index increases when the number of overlapping pairs increases.
- The way the pairs of zones overlap: In Fig. 5(a), y' is "almost" included into y. Hence this pair of zones should not significantly contribute to the compatibility index. The same holds in Fig. 5(c) since the

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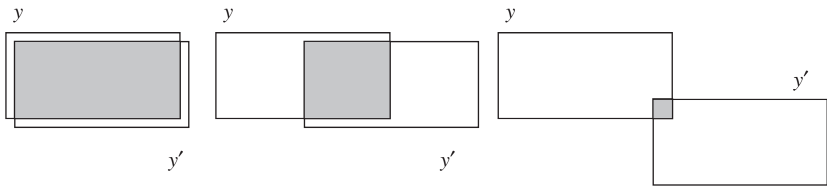

Fig. 5. Overlapping pairs of zones.

intersection between y and y' is "almost" empty. On the contrary, Fig. 5(b) depicts two overlapping zones such that $y \cap y'$, $y \backslash y'$and $y' \backslash y$ contain "almost" the same quantity of attribute P.

Remark 3.2. Moreover, the definition of a compatibility index $C_P(Y, Y')$ that satisfies the following properties is suitable:

(1) *Total compatibility*: $Y \equiv_P Y'$ iff $C_P(Y, Y') = 1$;
(2) *Idempotence*: $\forall Y \in \mathcal{Y}$, $C_P(Y, Y) = 1$; and
(3) *Symmetry*: $C_P(Y, Y') = C_P(Y', Y)$.

The first property means that the compatibility index, $C_P(Y, Y')$, has a maximum value when Y and Y' are totally compatible and it occurs only in this case. The second property states that any partition Y is compatible with itself, while the last (3) requires that the index be symmetric.

3.2. Implementation

The proposed implementation of the compatibility index is defined by taking into account the minimum value between the three elements, $P_{y \backslash y'}$, $P_{y \cap y'}$, and $P_{y' \backslash y}$ as given in

$$C_P(Y, Y') = 1 - \frac{1}{\mathcal{P}} \sum_{y \in Y} \sum_{y' \in Y'} \min\{P_{y \backslash y'}, P_{y \cap y'}, P_{y' \backslash y}\}. \tag{1}$$

Consider attribute P^2 and the examples of Fig. 2. Two overlapping pairs of zones, $\{y_1, y_2''\}$, and $\{y_4, y_2''\}$ are considered to determine the compatibility index between Y and Y''. The value of the index is $C_{P^2}(Y, Y') = 1 - \frac{1}{36}(\min\{8,1,9\} + \min\{8,1,9\}) = 1 - \frac{2}{36} = \frac{34}{36}$. Considering partitions Y' and Y''', there is only one overlapping pair, $\{y_5', y_2'''\}$; $C_{P^2}(Y', Y''') = 1 - \frac{1}{36}\min\{6,3,9\} = \frac{33}{36}$. When $P = P^3$, Y' and Y''' are totally compatible since $P^3_{y_5' \cap y_2'''} = 0$.

It is obvious that the compatibility index can be computed in $\mathcal{O}(n^2)$ elementary operations.

3.3. Analysis of the index

Consider territory A as a set, where each elementary unit a_i is an element of the set. It is obvious that a partition of territory A is also a partition of set A, in terms of *Set Theory*. It is well-known that the set $\Pi = \{y \cap y' \neq \emptyset : y \in Y, \ y' \in Y'\}$ constitutes a partition of set A, in which Y and Y' represent two partitions of A. Then

$$\sum_{\{y,y'\} \in Y \times Y'} P_{y \cap y'} = \mathcal{P}.$$

The following proposition shows that $C_P(Y, Y')$ is bounded from above and below.

Proposition 3.1. *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, Y, $Y' \in \mathcal{Y}$. The index $C_P(Y, Y') \in [0, 1]$ and the minimum and maximum values for C_P are* 0 *and* 1, *respectively.*

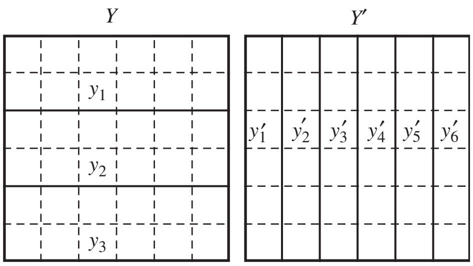

Fig. 6. Totally “incompatible” partitions, Y and Y'.

Proof. Since for any pair $\{y, y'\} \in Y \times Y'$, $\min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\} \leqslant P_{y\cap y'}$, and the summation $(1/\mathscr{P})\sum_{y\in Y}\sum_{y'\in Y'}\min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\} \leqslant 1$. Thus, $C_P(Y, Y') \geqslant 0$. Because attribute P is non-negative, the summation in $C_P(Y, Y')$ is also non-negative. Therefore, $C_P(Y, Y') \leqslant 1$.

Now we must prove that 0 and 1 are also the minimum and maximum values of $C_P(Y, Y')$. Suppose that Y and Y' are totally compatible. Then, when $P_{y\cap y'} > 0$, one of the inclusions, $y \subseteq_P y'$, $y' \subseteq_P y$, is verified, i.e., when $P_{y\cap y'} > 0$, either $P_{y\backslash y'} = 0$ or $P_{y'\backslash y} = 0$. Thus, $\sum_{y\in Y}\sum_{y'\in Y'}\min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\} = 0$, and, consequently, $C_P(Y, Y') = 1$. Now, suppose that Y, Y' are two partitions such that, for any pair $\{y, y'\} \in Y \times Y'$, $P_{y\cap y'} = \min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\}$. See Fig. 6 and consider attribute P^2 in Fig. 3; it represents the worst case. Then

$$C_P(Y, Y') = 1 - \frac{1}{\mathscr{P}}\sum_{y\in Y}\sum_{y'\in Y'} P_{y\cap y'} = 1 - \frac{1}{\mathscr{P}}\mathscr{P} = 0. \qquad \square$$

Proposition 3.2 connects the key properties discussed in Section 3.1.

Proposition 3.2. *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathscr{Y}$. The index C_P verifies the following properties*:

(1) *Total compatibility*: $Y \equiv_P Y'$ *iff* $C_P(Y, Y') = 1$;
(2) *Idempotence*: $C_P(Y, Y) = 1$; *and*
(3) *Symmetry*: $C_P(Y, Y') = C_P(Y', Y)$.

Proof.

(1) $Y \equiv_p Y' \Rightarrow C_P(Y, Y') = 1$. The proof was provided in Proposition 3.1.
$C_P(Y, Y') = 1 \Rightarrow Y \equiv_p Y'$. If $C_P(Y, Y') = 1$ then $\sum_{y\in Y}\sum_{y'\in Y'}\min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\} = 0$. Since all of its elements are non-negative, then for any pair $\{y, y'\} \in Y \times Y'$, $\min\{P_{y\backslash y'}, P_{y\cap y'}, P_{y'\backslash y}\} = 0$. Therefore, if $P_{y\cap y'} > 0$ then $P_{y\backslash y'} = 0$ or $P_{y'\backslash y} = 0$, i.e., $y \subseteq_P y'$ or $y' \subseteq_P y$. Thus $Y \equiv_p Y'$.
(2) Since for any two zones $y, z \in Y$, when $P_{y\cap z} > 0$ then $y \subseteq_p z$, $z \subseteq_p y$. Therefore $C_P(Y, Y) = 1$.
(3) The sum, the intersection, and the min operators are commutative thus $C_P(Y', Y) = C_P(Y, Y')$. $\square$

4. Inclusion index

This section defines an inclusion index, $I_P(Y, Y')$, that measures the extent to which the zones of partition Y are included into the zones of Y'. The aim is thus to evaluate the degree of “inclusion” of any zone $y \in Y$ into Y'. This degree is based, for each zone $y \in Y$, on the extent to which y is included into its reference zone in Y'. The concept of inclusion between two partitions is close to the notion of compatibility, but differs in that it is asymmetric (compatibility being symmetric).

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4.1. Definition and structural properties

Definition 4.1 (*Total inclusion between two partitions*). The partition Y is totally included into Y', according to P (denoted, $Y \subseteq_P Y'$), if $\forall y \in Y$, $\exists y' \in Y'$ such that $y \subseteq_P y'$.

In other words, $Y \subseteq_P Y'$ if each zone of Y is totally included into a zone of Y'.

In Fig. 2, the partition Y' is totally included in Y, whatever the attribute P. The reverse inclusion $Y \subseteq_P Y'$, however, is not verified when attribute P^2 is taken into account (see Fig. 3). But, it holds when considering P^1. It should also be noted that for P^3 the two partitions Y'' and Y''' are totally included into Y.

Remark 4.1. It is desirable that the proposed inclusion index verifies the following properties:

(1) *Total inclusion*: $Y \subseteq_P Y'$ iff $I_P(Y, Y') = 1$;
(2) *Idempotence*: $\forall Y \in \mathscr{Y}$, $I_P(Y, Y) = 1$;
(3) *Asymmetry*: If $I_P(Y, Y') = 1$ and $Y \neq_P Y'$ then $I_P(Y', Y) < 1$;
(4) *Transitivity*: If $I_P(Y, Y') = 1$ and $I_P(Y', Y'') = 1$ then $I_P(Y, Y'') = 1$.

Property (1) establishes that $I_P(Y, Y')$ has a maximal value when Y is totally included into Y' and only in this case. Property (2) states that any partition Y is contained in itself. Property (3) establishes that, when total inclusion between Y and Y' is verified, then the total reverse inclusion is false. Property (4) states that when there is total inclusion between Y and Y' and also between Y', and Y'' then Y is completely included into Y''.

4.2. Implementation

The proposed index measures the inclusion of Y into Y' and is modeled as follows:

$$I_P(Y, Y') = \frac{1}{\mathscr{P}} \sum_{y \in Y} P_{y \cap R_{Y'}(y)}.$$

Thus $I_P(Y, Y')$ represents the proportion of the sum, for each zone y, of the quantity P_y that belongs to its reference zone. It is obvious that its upper bound is equal to 1.

Remark 4.2. The value of $I_P(Y, Y')$ is independent of the choice of the reference zone, when for one $y \in Y$, there is more than one zone in Y' with the same maximum value of attribute P common to y.

Consider again Figs. 2 and 3. The index, $I_{P^2}(Y'', Y) = \frac{8+9+9+8}{36} = \frac{34}{36}$, for P^2. Note that only y''_2 is not totally included in some zone of Y; y_2 is its reference zone in Y. When P^1 is considered, the reference zone of y''_2 will change. The overall quantity $P^1_{y''_2}$ is 19. Its largest part, 10 units, belongs to y_4. Obviously, as y''_1, y''_3, and y''_4 are included into y_1, y_3, and y_4, respectively, $I_{P^1}(Y'', Y) = \frac{6+10+9+8}{42} = \frac{33}{42}$. The same value is obtained for $I_{P^1}(Y'', Y')$. It should be noted that $y''_1 \subseteq_{P_1} y'_1$.

It is obvious that this index can be computed in $\mathcal{O}(n)$ elementary operations.

4.3. Analysis of the index

The following proposition states that $I_P(Y, Y')$ is bounded from above and below.

Proposition 4.1. *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathscr{Y}$. The index $I_P(Y, Y') \in [1/n, 1]$ while the minimum and maximum values for $I_P(Y, Y')$ are $1/n$ and 1, respectively.*

Proof. Since $P_{y \cap R_{Y'}(y)} \leqslant P_y$ for all $y \in Y$, $\sum_{y \in Y} P_{y \cap R_{Y'}(y)} \leqslant \sum_{y \in Y} P_y$. Then $I_P(Y, Y') = (1/\mathscr{P}) \sum_{y \in Y} P_{y \cap R_{Y'}(y)} \leqslant (1/\mathscr{P}) \sum_{y \in Y} P_y = 1$. It is also the maximum value of I_P because, when $Y \subseteq_P Y'$, $y \subseteq_P R_{Y'}(y)$, for all $y \in Y$, and, therefore, $P_{y \cap R_{Y'}(y)} = P_y$. Thus, $I_P(Y, Y') = 1$.

For each $y \in Y$, $P_{y\cap R_{Y'}(y)}$ is minimal when the overall amount of its attribute is equally distributed by all the zones of Y'. Thus, for any $y \in Y$, $P_{y\cap y'} = P_y/K'$ for all $y' \in Y'$. Consequently, $P_{y\cap R'_Y(y)} = P_y/K'$. In this case, $I_P(Y, Y') = (1/\mathscr{P})\sum_{y\in Y}(P_y/K') = (1/K')(1/\mathscr{P})\mathscr{P} = 1/K'$. Therefore, the greater the number of zones in Y', the more the degradation of $I_P(Y, Y')$.

The condition, "*the overall amount of attribute P of each $y \in Y$ is equally distributed among all the zones of Y'*", imposes an upper bound on the number of zones, K' in Y'. The maximum value for K' is n/K if n is a multiple of K and $p_i = \mathscr{P}/n$, i.e., any elementary unit has the same amount of attribute. Consequently, the maximum value that K' can take is reached when K is minimum, i.e., when Y contains only one zone ($K = 1$). Finally, assuming that each elementary unit has the same quantity of attribute $K = 1$ and $K' = n$, the index $I_P(Y, Y')$ reaches its minimal value, $1/n$. □

The following proposition establishes the properties of Section 4.1.

Proposition 4.2. *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathscr{Y}$. The index $I_P(Y, Y')$ fulfills the following properties*:

(1) *Total inclusion*: $Y \subseteq_P Y'$ iff $I_P(Y, Y') = 1$;
(2) *Idempotence*: $I_P(Y, Y) = 1$;
(3) *Asymmetry*: if $I_P(Y, Y') = 1$ and $Y \neq_P Y'$ then $I_P(Y', Y) < 1$; and
(4) *Transitivity:* if $I_P(Y, Y') = 1$ and $I_P(Y', Y'') = 1$ then $I_P(Y, Y'') = 1$.

Proof.

(1) $Y \subseteq_P Y' \Rightarrow I_P(Y, Y') = 1$ was proved in Proposition 4.1. Let us now consider the reverse implication. If there is a $y_{u_0} \in Y$ such that $y_{u_0} \nsubseteq_P y'$ for any $y' \in Y'$, i.e., $Y \nsubseteq_P Y'$, then $P_{y_{u_0}\cap R_{Y'}(y_{u_0})} < P_{y_{u_0}}$. Therefore, since $P_{y\cap R_{Y'}(y)} \leqslant P_y$, $\sum_{y\in Y} P_{y\cap R_{Y'}(y)} < \sum_{y\in Y} P_y = \mathscr{P}$. That is, $I_P(Y, Y') < 1$. Thus, if $I_P(Y, Y') = 1$ then $Y \subseteq_P Y'$.
(2) For all $Y \in \mathscr{Y}$, $Y \subseteq_P Y$, then $I_P(Y, Y) = 1$.
(3) If $I_P(Y, Y') = 1$ and $Y \neq_P Y'$ then there is at least one $y'_{v_0} \in Y'$ for which there are $y_{u_1}, \ldots, y_{u_{K_{v_0}}} \in Y$ such that $y'_{v_0} =_P \bigcup_{u=1}^{K_{v_0}} y_{u_i}$, and $P_{y'_{v_0}\cap y_{u_i}} > 0$, for more than one y_{u_i}. Consequently, $P_{y'_{v_0}\cap R_Y(y'_{v_0})} < P_{y'_{v_0}}$. Then $I_P(Y', Y) < 1$.
(4) If $I_P(Y, Y') = 1$ and $I_P(Y', Y'') = 1$, then $Y \subseteq_P Y'$ and $Y' \subseteq_P Y''$. Therefore, for each $y \in Y$, there exists a $y' \in Y'$ such that $y \subseteq_P y'$ and for which there is $y'' \in Y''$ that verifies $y' \subseteq_P y''$. We may now verify that $y \subseteq_P y''$. Since $y \subseteq y'$ then $P_{y\cap y'} > 0$, i.e., $\exists\, i \in y \cap y'$ such that $p_i > 0$. Since $y' \subseteq y''$ then $i \in y''$ as well. Consequently $P_{y\cap y''} > 0$. Suppose that there is an elementary unit $i \in y$ such that $i \notin y''$. By *reductio ad absurdum*, suppose that $p_i \neq 0$. Therefore, $i \in y'$ because $y \subseteq_P y'$. Consequently, $y' \subseteq_P y''$, $p_i = 0$. Contradiction! This means that $p_i = 0$. Then $y \subseteq_P y''$ and, thus, $Y \subseteq_P Y''$, i.e., $I_P(Y, Y'') = 1$. □

5. Distance index

This section defines a distance index, $D_P(Y, Y')$, that evaluates "how different" two partitions can be. The attribute P is considered here as a strictly positive function, i.e., $p_i > 0$, for all $i \in V$.

5.1. Definition and structural properties

Construction of this index is based on the notion of equality between partitions, according to an attribute P, presented in Definition 2.4.

Remark 5.1. The equality between partitions is associated with the attribute P. For example, in Fig. 2 the partitions Y and Y' are equal for P^1 (see also Fig. 3), while partitions Y and Y''' are equal for P^3.

Remark 5.2. The aim is to define a distance $D_P(Y, Y')$ in $\mathcal{Y}$ that fulfills the metric properties. Consider the following three partitions $Y, Y', Y'' \in \mathcal{Y}$:

(1) $D_P(Y, Y') = 0$ iff $Y =_P Y'$;
(2) *Symmetry*: $D_P(Y, Y') = D_P(Y', Y)$; and
(3) *Triangular inequality*: $D_P(Y, Y'') \leqslant D_P(Y, Y') + D_P(Y', Y'')$.

5.2. Implementation

The proposed implementation for the distance index takes into account all edges $\{i,j\} \in E$, corresponding to the border zones in one and only one of the partitions.

Consider the following additional notation:

- $I_Y = \{\{i,j\} \in E : \exists y \in Y, i,j \in y\}$; and
- $B_Y = \{\{i,j\} \in E : \forall y \in Y, i,j \notin y\}$;

where, for any $Y \in \mathcal{Y}$, I_Y represents the set of edges that are included into some zone, while B_Y represents the set of all edges corresponding to border zones. It is obvious that $I_Y \cup B_Y = E$ and $I_Y \cap B_Y = \emptyset$.

Consider, now, $IB_{YY'} \subseteq E$, defined as follows:

$$IB_{YY'} = (B_Y \cap I_{Y'}) \cup (B_{Y'} \cap I_Y),$$

where $Y, Y' \in \mathcal{Y}$. In other words, $IB_{YY'}$, represents the set of edges whose adjacent vertices belong to the same zone in one of the partitions and pertain to different zones in the other partition.

A distance D_P between Y and Y', according to an attribute P, can be defined as follows:

$$D_P(Y, Y') = \frac{1}{\Delta} \sum_{e \in IB_{YY'}} \delta_e, \tag{2}$$

where, for each edge $e = \{i,j\} \in E$, $\delta_e = \min\{p_i, p_j\}$ and $\Delta = \sum_{e \in E} \delta_e$.

The distance $D_{P^1}(Y, Y')$ between Y and Y' (according to P^1), represented in Fig. 2, is equal to zero. Note that the set $IB_{YY'}$ has three edges corresponding to the border between y'_1 and y'_2. However, the values δ_e are equal to zero thus, $P^1_{y'_2} = 0$. Consider, now, Y and Y'' and also P^1, where the set $IB_{YY''}$ has six edges. For three of them $\delta_e = 0$ while the value for the others is equal to one. Therefore, $D_{P^1}(Y, Y'') = \frac{1}{51}(0 + 0 + 0 + 1 + 1 + 1) = \frac{3}{51}$.

5.3. Analysis of the index

Proposition 5.1. *Consider a territory A, composed of n elementary units, an attribute P defined on A, and $Y, Y' \in \mathcal{Y}$. The index $D_P(Y, Y') \in [0, 1]$ and the minimum and maximum values for D_P, are 0 and 1, respectively.*

Proof. Since $p_i \geqslant 0$ $\forall i \in V$, then $\delta_e \geqslant 0$. Consequently, $D_P(Y, Y') \geqslant 0$. Since $IB_{YY'} \subseteq E$, then $\sum_{e \in IB_{YY'}} \delta_e \leqslant \Delta$. Therefore, $D_P(Y, Y') \leqslant 1$.

Now it is needed to prove that 0 and 1 are also the minimum and maximum value of $D_P(Y, Y')$, respectively. Obviously, $IB_{YY} = \emptyset$, then $D_P(Y, Y) = 0$. Suppose now that Y has only one zone and Y' has n zones (see Fig. 7). It is obvious that $IB_{YY'} = E$. Therefore, $D_P(Y, Y') = 1$. □

Let us now prove the properties presented in Section 5.1.

Proposition 5.2. *The operator D_P, defined in* (2) *is a distance, i.e., D_P verifies the properties*:

(1) $D_P(Y, Y') = 0$ iff $Y =_P Y'$;
(2) *Symmetry*: $D_P(Y, Y') = D_P(Y', Y)$; and
(3) *Triangular inequality*: $D_P(Y, Y'') \leqslant D_P(Y, Y') + D_P(Y', Y'')$.

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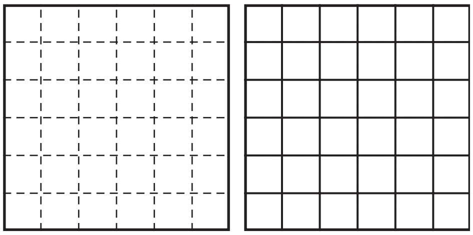

Fig. 7. Trivial partitions.

Proof.

(1) ($\Rightarrow$) Suppose $Y \neq_P Y'$, i.e., there is a couple $\{y, y'\} \in Y \times Y'$ such that $y \neq_P y'$ and $P_{y \cap y'} > 0$. Without loss of generality, suppose $y \nsubseteq_P y'$, i.e., $\sum_{i \in y \backslash y'} p_i > 0$. Therefore, because each zone is contiguous, there is at least an edge $e = \{i, j\}$, such that $i \in y \backslash y'$ and $j \in y \cap y'$, i.e., $e \in IB_{YY'}$. Since the attribute is strictly positive then $D_P(Y, Y') > 0$.
($\Leftarrow$) If $D_P(Y, Y') > 0$ then there is an $e = \{i, j\} \in IB_{YY'}$ such that $\min\{p_i, p_j\} > 0$. Assume $e \in B_Y \cap I_{Y'}$. Then $\exists y' \in Y' : i, j \in y'$ and $\exists y \in Y, i \in y$ and $j \notin y$ or $j \in y$ and $i \notin y$. Thus $y' \nsubseteq_P y$, i.e., $y' \neq_P y$. Since $P_{y \cap y'} > 0$, $Y \neq_P Y'$.
(2) Since $IB_{YY'} = IB_{Y'Y}$, then $D_P(Y, Y') = D_P(Y', Y)$.
(3) It is obvious that if $IB_{YY''} \subseteq IB_{YY'} \cup IB_{Y'Y''}$ then $D_P(Y, Y'') \leqslant D_P(Y, Y') + D_P(Y', Y'')$. Therefore, it is necessary to prove that $IB_{YY''} \subseteq IB_{YY'} \cup IB_{Y'Y''}$. Consider $e \in IB_{YY''}$. From the definition of $IB_{YY''}$, either

$$e \in B_Y \cap I_{Y''} \tag{3}$$

or

$$e \in B_{Y''} \cap I_Y. \tag{4}$$

Concerning the partition Y', either $e \in I_{Y'}$ or $e \in B_{Y'}$. In the first case, if (3) is true then $e \in IB_{YY'}$. Otherwise, from (4), $e \in IB_{Y'Y''}$. Similarly, in the second case ($e \in B_{Y'}$), if (3) is true then $e \in IB_{Y'Y''}$. Otherwise, from (4), $e \in IB_{YY'}$. Therefore, in all possible cases, $e \in IB_{YY'} \cup IB_{Y'Y''}$. □

In this section, it was necessary to consider P as being a strictly positive function. Without this constraint Property 1 of Proposition 5.2 does not hold. This restriction does not represent a significant loss of applicability in real-world problems. Therefore, in a large number of cases, the attribute of a territory is represented by a strictly positive number. If the implementation of distance suggested in (2) seems inadequate, it can be modified without changing the properties proven in this section. It is then necessary to redefine I_Y and B_Y as follows:

- $I'_Y = \{\{i, j\} \in V \times V : \exists y \in Y, i, j \in y\}$; and
- $B'_Y = \{\{i, j\} \in V \times V : \forall y \in Y, i, j \notin y\}$.

That is, I'_Y and B'_Y are now subsets of pairs of vertices. The set $IB_{YY'}$ keeps the same definition. This increases the difficulty of computing the distance index. The number of elementary operations is bounded by $\mathcal{O}(n^2)$. Given the previous definition, calculation of $D_P(Y, Y')$ is done in $\mathcal{O}(n)$ elementary operations. This is true because the number of edges in a planar graph is $(3n - 6)$.

6. Numerical experiments and behavior of the indices

In this section, we present numerical experiments that investigate the behavior of our three proposed indices. We consider data from Ile de France (Paris region), which is composed of 1300 elementary territorial units (towns). In Fig. 8, each town is represented by a vertex, and each vertex corresponds to a pair of towns

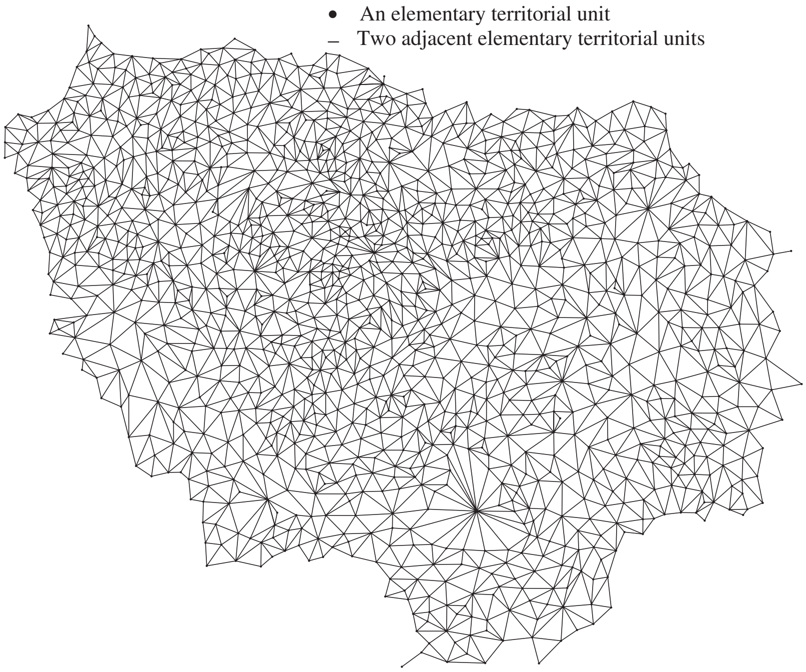

Fig. 8. The Paris region contiguity graph.

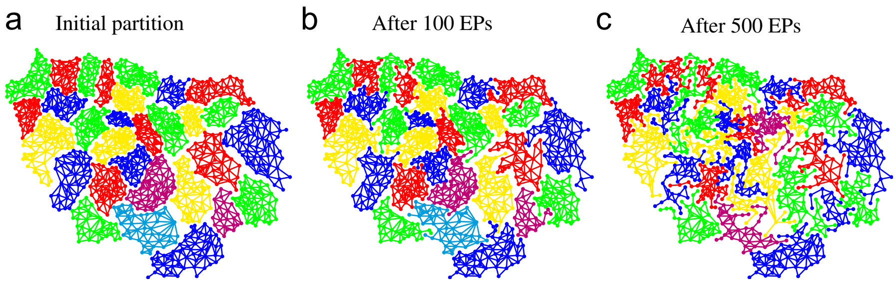

Fig. 9. Successive elementary perturbations (EPs). (a) Initial partition. (b) After 100 EPs. (c) After 500 EPs.

having a common boundary (the size of this contiguity graph $G = (V, E)$ is such that $|V| = 1300$ and $|E| = 3719$). The attribute P, considered in the experiments is the "working population", i.e., $P =$working population.

In this analysis, the following two partitions were considered: a *root* partition Y^R and a *current* one, Y. In the experiments, Y^R remains the same, while Y is progressively modified through successive elementary perturbations (EPs). An EP consists in moving an elementary unit between two neighboring zones. Fig. 9 shows how successive EPs modify an initial partition formed by 30 zones (a), then, after 100 EPs (b), and, finally, after 500 EPs (c).

6.1. Compatibility index

Consider the initial pair of partitions (Y^R, Y) which fulfill total compatibility ($Y^R \equiv_P Y$ hence $C_P(Y^R, Y) = 1$). Applying α successive EPs to Y^R, it leads to Y^α. The compatibility index between Y^R and Y^α was computed for 100 randomly generated instances of Y^α.

Results are provided in Fig. 10. Note how the value of $C_P(Y^R, Y^\alpha)$ evolves when $\alpha = 100, 200, \ldots, 1000$. As expected, $C_P(Y^R, Y^\alpha)$ decreases as α increases.

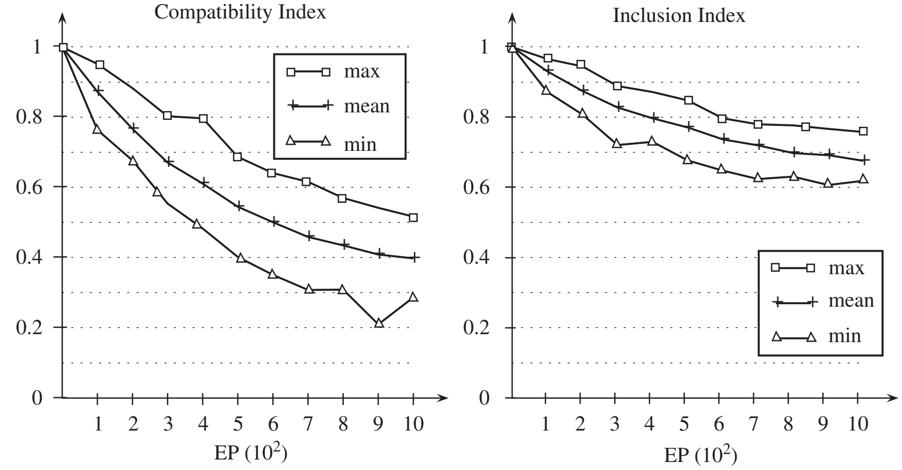

Fig. 10. Behavior of the compatibility and inclusion indices.

6.2. Inclusion index

Now, consider partitions (Y^R, Y) which fulfill total inclusion ($Y^R \subseteq_P Y$, hence $I_P(Y^R, Y) = 1$). Applying α successive EPs to Y, it leads to Y^α. The inclusion index between Y^R and Y^α was computed for 100 randomly generated instances of Y^α.

Results are again provided in Fig. 10. Note how the value of $I_P(Y^R, Y^\alpha)$ evolves when $\alpha = 100, 200, \ldots, 1000$. As expected, $I_P(Y^R, Y^\alpha)$ decreases when α increases. Moreover, the observed minimal value for $I_P(Y^R, Y^\alpha)$ was 0.619, which is far from its smallest possible value. Nevertheless, it should be remarked that the minimum corresponds to a very particular case: when Y is composed of a single zone and when the zones of Y' are the elementary units (the comparison of these two partitions is indeed a very singular case).

6.3. Distance index

Consider partitions (Y^R, Y^R) which obviously verify $D_P(Y^R, Y^R) = 0$. Applying α successive EPs to Y^R leads to Y^{R^α}. As in the previous cases, the distance index between Y^R and Y^{R^α} was computed for 100 randomly generated instances of Y^{R^α}.

These results are given in Fig. 11. Note how the value of $D_P(Y^R, Y^{R^\alpha})$ evolves when $\alpha = 100, 200, \ldots, 1000$. As expected, $D_P(Y^R, Y^{R^\alpha})$ increases when α increases.

6.4. Conclusion

In comparing the computational behavior of the three indices, some similarities can be found. It is clear that the compatibility index is more sensitive to an increase in EPs than are the inclusion and the distance indices. From the proof of Proposition 3.1 it is easy to conclude that the lower the number of zones in Y and the greater the number of zones in Y', the lower the value of the inclusion index. As for the distance index, the greater the difference between the overall number in Y and in Y', the greater the distance between Y and Y'. Since the number of zones remains constant when the number of EPs increases, one has to impose a restriction on the variation that occurs in these two indices. Up to 1000 EPs, the observed value for variation is approximately 0.4. The inclusion index decreased from 1 to 0.67, while the distance index increased from 0 to 0.45. This pattern can be explained as follows. When applying an EP to a given partition from a pair that verifies equality between partitions, the number of edges added to $IB_{YY'}$ is equal to the number of incident edges of the vertices that were moved from one zone to another one. The proportion of such a number with respect to m is approximately $1/n$. It is also the same as the variation that occurs in the inclusion index when

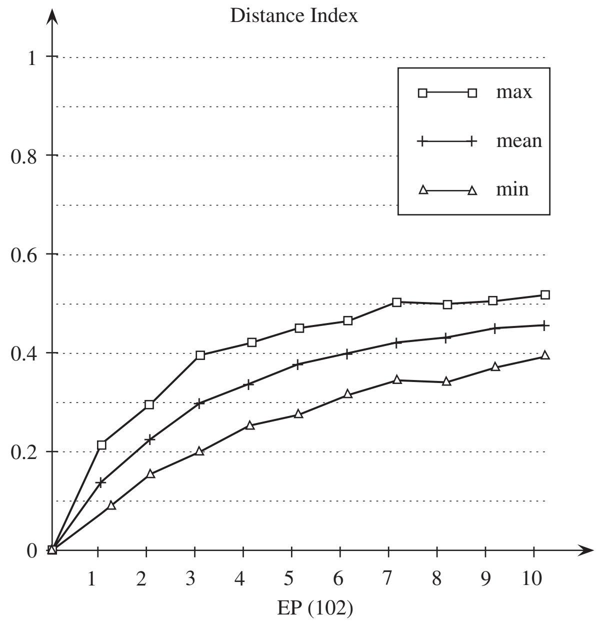

Fig. 11. Behavior of the distance index.

an EP is applied to a given partition from a pair that verifies total inclusion. (It was assumed that the attribute is equal to 1 in each elementary unit.)

Considering the compatibility index, it was expected that its variation was greater than the variation of the other two indices. It should be noted that it is possible to reach the worst case of compatibility between partitions with the same number of zones. Application of an EP in one of the partitions of a totally compatible pair can lead to set two pairs of overlapping zones and, as a consequence, a decrease of $2/n$.

7. Implications for management and policy making

In this section, we provide selected applications of our proposed indices to management problems involving territorial partitions. These examples illustrate implications of the proposed indices for management purposes and for setting public sector policy.

7.1. Compatibility index

Territory partitions are frequently used in the field of sales management wherein a commercial zone is assigned to each salesperson (see, for example [5,9,21]). Consider a company that commercializes products grouped into several ranges, with salespersons specialized in such ranges. Hence, there exists a sales territory partition for each range of product. Obviously, clients can buy products from different ranges. As demand is not geographically homogenous among product ranges, the sales territory partitions do not usually match.

Consider two ranges of products A and B, and the corresponding partitions Y^A and Y^B. In order to optimize customer relations, Y^A and Y^B should be defined in such a way that vendors specialized in range A share clients with a limited number of vendors specialized in range B. More precisely, any vendor of range A (of range B, respectively) should share clients:

- either with only one vendor of range B (of range A, respectively),
- or with several vendors of range B (of range A, respectively), knowing that these vendors share clients only with him/her.

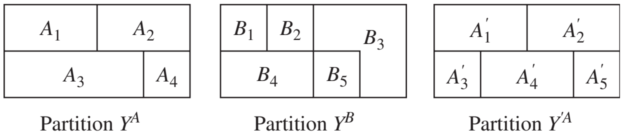

Fig. 12. Sales management partitions.

Such a property is verified for the pair of partitions (Y^A, Y^B) depicted in Fig. 12, but does not hold for the pair of partitions (Y'^A, Y^B), as vendor A_4' shares clients with vendors B_4 and B_5, while vendor B_4 shares clients both with A_3' and A_4'. Such a property is in line with the concept of total compatibility between two partitions (see Section 3) when the attribute considered is the number of clients in each territorial unit. Hence, the compatibility index $C_P(Y^A, Y^B)$ appears to be useful in such a context when evaluating the extent to which partitions Y^A and Y^B optimize customer relations management between ranges A and B.

Although the above example involves sales management in a private firm, similar problems surely occur in the public sector, e.g., when considering health care services (cancer, pediatrics, psychiatry, etc.). For each medical specialty, a patient is assigned to a hospital according to his/her home residence. If we consider any two specialties, A and B, the partitions Y'_A and Y'_B of the territory that assign patients to a hospital for specialties A and B should be as compatible as possible, in the same way Y^A and Y^B should match in the sales territory scenario.

7.2. Inclusion index

Let us illustrate here the interest of the inclusion index within the context of pricing of public transportation where pricing zones exist. A reform of this pricing system has been undertaken by the STIF: the Paris regional transportation authority (see [11]). The current partition, consisting of concentric rings, is considered unsatisfactory as it no longer corresponds to travel patterns or customer needs. The STIF thus wishes to define a new partition in which the zones are autonomous with respect to the transportation function. Such a partition is needed to establish the pricing system.

In defining its new pricing partition, the STIF considered an existing partition: the "school map". In the French educational system, the "school map" (see [22]) defines that high school to which a student should be assigned, i.e., to the school associated to his/her residential zone. Such a map thus allows to plan in which schools to open/close teaching positions in response to their zone's demographic evolution. The size of each zone in the map thus corresponds to size of its high school and its population density.

An important quality of a new transport pricing partition is its ability to account for the "school map". As noted above, ideally, each student should be able to go to the school located within the same pricing zone. This requires that the school partition, denoted Y^{sch}, should be included in the pricing partition. Hence, in order to evaluate and compare alternative pricing partitions $Y^1, Y^2, \ldots,$ it would be relevant to consider the inclusion index $I_P(Y^{\text{sch}}, Y^i)$, $i = 1, 2, \ldots$. In this way, we should seek the partition that "minimizes" the student journeys between different pricing zones.

7.3. Distance index

As noted earlier, much research dealing with territorial partitions has been devoted to political districting (e.g. [1–4]). In modern democracies, parliamentary members represent voters attached to an electoral district, hence defining an electoral partition of the territory. Basic democratic principles impose, among other things, that each district should contain approximately the same number of voters (or, that the number of representatives account for differences in voter populations).

Moreover, the demographic evolution of populations requires ongoing revisions of political districts. Obviously, such revisions can become a highly sensitive issue. For example, if drastically different districts were to result, political manipulation would surely be suspected. Moreover, candidates are, by design, involved in the political life of their respective district. A radical change in the geography or other dimension(s) of a district would run counter to local political debate.

Therefore, when revising an electoral partition, the resulting entity should be as close as possible to the previous one in terms of selected characteristics. A key criterion to be minimized can thus be the "distance" with respect to the number of voters (as defined in Section 5), and/or other factors of choice.

8. Conclusions

In the last few years, there has been tremendous growth in the use of models and software for partitioning a territory into "homogeneous" zones. The examples provided in the previous section are representative of areas covered in this arena. A key dimension of such problems is the need to compare two different partitions of the same territory while quantifying that comparison.

The current research represents an initial attempt to identify and detail selected comparison indices in districting problems. Specifically, we proposed three objective classes of measures: compatibility, inclusion, and distance. We strongly believe that this classification can account for most metrics of importance in the districting of physical areas.

Thus, compatibility incorporates those measures that evaluate the following scenario: if each zone of a first partition results from a group of zones of the second one, or if each zone along with the remaining zones of the same partition define a single zone of the second partition. The inclusion class covers those measures devoted to evaluating the degree of fineness of a given partition with respect to a second one. Finally, the distance class consists of those measures that evaluate any differences between two partitions.

For each of the three classes, we provided a set of elementary properties, as well as typical forms of implementation. The concepts and measures introduced in this paper do not consider a territory by itself, but as attributes associated with each region according to the problem of interest. In this way, we developed the abstract form of each measure, as well as suggestions for the universe of its applicability. We also implemented the measures on a real-world network. The experiments dealt with the progressive degradation in the similarity between two partitions. We subsequently showed the value of utilizing our measures in such an evaluation.

Future work should seek to apply our three measures in actual managerial decision problems. They are particularly relevant in such areas as telecommunications districting and public services management. Incorporating the measures in Geographical Information Systems with user friendly interfaces would be important in facilitating decision making within both public and private districting problems. At a theoretical level, extensions of the proposed indices to account for more than a single attribute should be investigated.

Acknowledgments

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