Accepted Manuscript

Ordinal regression revisited: multiple criteria ranking using a set of additive value functions

Salvatore Greco, Vincent Mousseau, Roman Słowiński

PII: S0377-2217(07)00875-2
DOI: 10.1016/j.ejor.2007.08.013
Reference: EOR 8586

To appear in: *European Journal of Operational Research*

Received Date: 3 October 2005
Revised Date: 14 July 2007
Accepted Date: 2 August 2007

Please cite this article as: Greco, S., Mousseau, V., Słowiński, R., Ordinal regression revisited: multiple criteria ranking using a set of additive value functions, *European Journal of Operational Research* (2007), doi: 10.1016/j.ejor.2007.08.013

Ordinal regression revisited: multiple criteria ranking using a set of additive value functions

Salvatore Greco[*], Vincent Mousseau[†], Roman Słowiński[‡]

July 14, 2007

[*]Faculty of Economics, University of Catania, Corso Italia, 55, 95129 Catania, Italy, e-mail: salgreco@mbox.unict.it

[†]LAMSADE, Université Paris Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France, e-mail: mousseau@lamsade.dauphine.fr

[‡]Institute of Computing Science, Poznań University of Technology, 60-965 Poznań, and Systems Research Institute, Polish Academy of Sciences, 01-447 Warsaw, Poland, e-mail: roman.slowinski@cs.put.poznan.pl

Abstract

We present a new method, called UTA$^{\text{GMS}}$, for multiple criteria ranking of alternatives from set A using a set of additive value functions which result from an ordinal regression. The preference information provided by the decision maker is a set of pairwise comparisons on a subset of alternatives $A^R \subseteq A$, called reference alternatives. The preference model built via ordinal regression is the set of all additive value functions compatible with the preference information. Using this model, one can define two relations in the set A: the necessary weak preference relation which holds for any two alternatives a, b from set A if and only if for all compatible value functions a is preferred to b, and the possible weak preference relation which holds for this pair if and only if for at least one compatible value function a is preferred to b. These relations establish a necessary and a possible ranking of alternatives from A, being, respectively, a partial preorder and a strongly complete relation. The UTA$^{\text{GMS}}$ method is intended to be used interactively, with an increasing subset A^R and a progressive statement of pairwise comparisons. When no preference information is provided, the necessary weak preference relation is a weak dominance relation, and the possible weak preference relation is a complete relation. Every new pairwise comparison of reference alternatives, for which the dominance relation does not hold, is enriching the necessary relation and it is impoverishing the possible relation, so that they converge with the growth of the preference information. Distinguishing necessary and possible consequences of preference information on the complete set of actions, UTA$^{\text{GMS}}$ answers questions of robustness analysis. Moreover, the method can support the decision maker when his/her preference statements cannot be represented in terms of an additive value function. The method is illustrated by an example solved using the UTA$^{\text{GMS}}$ software. Some extensions of the method are also presented.

Keywords: *Multiple criteria ranking, Ordinal regression approach, Additive value function.*

1 Introduction

We are considering a decision situation in which a finite and non-empty set of alternatives (actions) $A = \{a, b, c, d, ...\}$ is evaluated on a family of n criteria $g_1, \ldots, g_i, \ldots, g_n$, with $g_i : A \mapsto \mathbb{R}$ for all $i \in G = \{1, \ldots, n\}$. We assume, without loss of generality, that the greater $g_i(a)$, $a \in A$, the better alternative a on criterion g_i, for all $i \in G$. A decision maker (DM) is willing to rank the alternatives in A from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information supplied by the DM and on the way of using this information. The family of criteria G is supposed to satisfy the following consistency conditions (see [25]):

- exhaustivity - any two alternatives having the same evaluations on all criteria from G should be considered indifferent,
- monotonicity - when comparing two alternatives, an improvement of one of them on at least one criterion from G should not deteriorate its comparison to the other alternative,
- non-redundancy - deletion of any criterion from G will contradict one of the two above conditions.

Such a decision problem is called multiple criteria ranking problem. It is known that the only information coming out from the formulation of this problem is the weak dominance relation. Let us recall that the weak dominance relation is a partial preorder, i.e. it is a reflexive and transitive binary relation. According to the weak dominance relation, alternative $a \in A$ is preferred to alternative $b \in A$ if and only if $g_i(a) \geq g_i(b)$ for all $i \in G$, with at least one strict inequality; moreover, a is indifferent to b if and only if $g_i(a) = g_i(b)$ for all $i \in G$; finally, a is incomparable with b otherwise, i.e. if $g_i(a) > g_i(b)$ for at least one criterion $i \in G$ and $g_j(a) < g_j(b)$ for at least another criterion $j \in G$. Since incomparability is very often the most frequent situation, the weak dominance relation is usually very poor.

In order to enrich the weak dominance relation, multiple criteria decision aiding (MCDA) helps in construction of an aggregation model on the base of preference information provided by the DM. Such an aggregation model is called preference model - it induces a preference structure in set A whose proper exploitation permits to work out a ranking proposed to the DM.

The preference information may be either direct or indirect, depending if it specifies directly values of some parameters used in the preference model (e.g. trade-off weights, aspiration levels, discrimination thresholds, etc.), or if it specifies some examples of holistic judgments from which compatible values of the preference model parameters are induced. Direct preference information is used in the traditional aggregation paradigm, according to which the aggregation model is first constructed and then applied on set A to rank the alternatives.

Indirect preference information is used in the disaggregation (or regression) paradigm, according to which the holistic preferences on a subset of alternatives $A^R \subseteq A$ are known first, and then a consistent aggregation model is inferred from this information to be applied on set A in order to rank the alternatives.

Presently, MCDA methods based on indirect preference information and the disaggregation paradigm are of increasing interest for they require relatively less cognitive effort from the DM. Indeed, the disaggregation paradigm is consistent with the "posterior rationality" postulated by

March [19] and with the inductive learning used in artificial intelligence approaches (see [20]). Typical applications of this paradigm in MCDA are presented in [30], [23], [10], [13], [1], [22], [7], [8], [9].

Let X_i denote the evaluation scale of criterion g_i, $i \in G$. Consequently, $X = \prod_{i=1}^{n} X_i$ is the evaluation space, and $x = [x_1, ..., x_n] \in X$ denotes a profile in the evaluation space.

From a pragmatic point of view, it is reasonable to assume that $X_i = [\alpha_i, \beta_i]$, i.e. the evaluation scale on each criterion g_i is bounded, such that $\alpha_i < \beta_i$ are the worst and the best (finite) evaluations, respectively. Thus, $g_i : \ A \mapsto X_i, \ i \in G$, therefore, each alternative $a \in A$ is associated with the profile $[g_1(a), ..., g_n(a)]$ in the evaluation space X. In consequence, A is obviously associated with a finite subset of X.

In this paper, we are considering the aggregation model in form of an additive value function $U : X \mapsto \mathbb{R}$, such that, for each $x \in X$,

$$U(x) = \sum_{i=1}^{n} u_i(x_i), \tag{1}$$

where u_i are non-decreasing marginal value functions, $u_i : X_i \mapsto \mathbb{R}$, $i = 1, \ldots, n$.

To simplify notation, when considering any alternative $a \in A$, we shall write $U(a)$ instead of $U(g_1(a), ..., g_n(a))$, and $u_i(a)$ instead of $u_i(g_i(a))$, even if $U : X \mapsto \mathbb{R}$ and $g_i : \ A \mapsto X_i, \ i \in G$.

While the additive value function involves compensation between criteria and requires a rather strong assumption about their independence in the sense of preference [11], it is often used for its intuitive interpretation and relatively easy computation. The weighted-sum aggregation model, which is a particular case of the additive value function, is used even more frequently, in spite of its simplistic form (see e.g. [1], [26]).

We are using the additive aggregation model in the settings of the disaggregation paradigm, as it has been proposed in the UTA method (see [10]). In fact, our method generalizes the UTA method in three aspects:

- it takes into account all additive value functions (1) compatible with indirect preference information, while UTA is using only one such function,
- the marginal value functions of (1) are general non-decreasing functions, and not piecewise linear, as in UTA,
- the DM's ranking of reference alternatives does not need to be complete.

The preference information used by our method is provided in the form of a set of pairwise comparisons of some alternatives from a subset $A^R \subseteq A$, called reference alternatives. The method is producing two rankings in the set of alternatives A, such that for any pair of alternatives $a, b \in A$:

- in the *necessary* ranking, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for <u>all</u> value functions compatible with the preference information,
- in the *possible* ranking, a is ranked at least as good as b if and only if, $U(a) \geq U(b)$ for <u>at least one</u> value function compatible with the preference information.

The necessary ranking can be considered as robust with respect to the indirect preference information. Such robustness of the necessary ranking refers to the fact that any pair of alternatives compares in the same way whatever the additive value function compatible with the indirect preference information. Indeed, when no indirect preference information is given, the necessary ranking boils down to the weak dominance relation, and the possible ranking is a complete relation. Every new pairwise comparison of reference alternatives, for which the dominance relation does not hold, is enriching the necessary ranking and it is impoverishing the possible ranking, so that they converge with the growth of the preference information.

Another appeal of such an approach stems from the fact that it gives space for interactivity with the DM. Presentation of the necessary ranking, resulting from an indirect preference information provided by the DM, is a good support for generating reactions from the DM. Namely, (s)he could wish to enrich the ranking or to contradict a part of it. This reaction can be integrated in the indirect preference information in the next iteration.

The organization of the paper is the following. In the next section, we will outline the principle of the ordinal regression via linear programming, as proposed in the original UTA method (see [10]). In section 3, we give a brief overview of existing approaches to multiple criteria ranking using a set of additive value functions, and we provide motivations for our approach. The new UTA$^{\text{GMS}}$ method is presented in section 4. Some extensions are considered in section 5. Section 6 provides an illustrative example showing how the method can be applied in practice. The last section includes conclusions.

2 Ordinal regression via linear programming - principle of the UTA method

In the following, we recall the principle of the UTA method as presented recently in [29]. The indirect preference information is given in the form of a complete preorder $\succsim$ on a subset of reference alternatives $A^R \subseteq A$, called reference preorder, such that, for all $a, b \in A^R$:

$$a \succsim b \;\Leftrightarrow\; \text{“}a \text{ is at least as good as } b\text{”}.$$

This weak preference relation can be decomposed into its asymmetric and symmetric parts, as follows:

- $a \succ b \;\Leftrightarrow\; [a \succsim b \text{ and not}(b \succsim a)] \;\Leftrightarrow\;$ “a is preferred to b”,
- $a \sim b \;\Leftrightarrow\; [a \succsim b \text{ and } b \succsim a] \;\Leftrightarrow\;$ “a is indifferent to b”.

The reference alternatives are usually those alternatives in set A for which the DM is ready to express holistic preferences. Let the set of reference alternatives $A^R = \{a_1, ..., a_m\}$ be rearranged such that $a_k \succsim a_{k+1},\ k = 1, ..., m-1$, where $m = |A^R|$. The disaggregation paradigm consists here in inferring an additive value function (1) ranking the reference alternatives in exactly the same way as it was done by the DM. Such a value function U is called *compatible*. A compatible value function U is supposed to represent DM's preferences on the whole evaluation space X. In other words, for all profiles $x, y \in X$, x is considered at least as good as y according to compatible value function U, if $U(x) \geq U(y)$.

Let us remark that the transition from the preorder $\succsim$ provided by the DM on A^R to the compatible marginal value functions exploits the ordinal character of the criterion scale X_i. Note, however, that for the considered additive representation of preferences on X, the scale of the marginal value functions is a conjoint interval scale (see e.g. Theorem 13 in Chapter 6 of [16] or Theorem III.4.1 in [31]). More precisely, the admissible transformations on the marginal value functions $u_i(x_i)$ have the form $u_i^*(x_i) = k \times u_i(x_i) + h_i,\ h_i \in \mathbb{R},\ i = 1, ..., n,\ k > 0$, such that for all $[x_1, ..., x_n], [y_1, ..., y_n] \in \prod_{i=1}^{n} X_i$

$$\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \Leftrightarrow \sum_{i=1}^{n} u_i^*(x_i) \geq \sum_{i=1}^{n} u_i^*(y_i).$$

An alternative way of representing the same preference model is:

$$U(a) = \sum_{i=1}^{n} w_i \hat{u}_i(a), \quad \text{where } \hat{u}(\alpha_i) = 0,\ \hat{u}(\beta_i) = 1,\ w_i \geq 0 \quad \text{for all } i \in G,\ \text{and} \sum_{i=1}^{n} w_i = 1. \tag{2}$$

Note that the correspondence between (2) and (1) is such that $w_i = u_i(\beta_i)$, for all $i \in G$. Due to the cardinal character of the marginal value function scale, the parameters w_i can be interpreted as tradeoff weights among marginal value functions $\hat{u}_i(a)$. We will use, however, the preference model (1) with normalization constraints bounding $U(a)$ to the interval $[0, 1]$.

The ordinal regression consists in the inference of a compatible value function restoring the reference preorder. The transition from a reference preorder to a value function is done according to the following equivalence :

$$\begin{array}{l} U(a_k) > U(a_{k+1}) \;\Leftrightarrow\; a_k \succ a_{k+1} \\ U(a_k) = U(a_{k+1}) \;\Leftrightarrow\; a_k \sim a_{k+1} \end{array} \tag{3}$$

for $k = 1, ..., m - 1$.

In the UTA method, the marginal value functions u_i are assumed to be piecewise linear, so that the intervals $[\alpha_i, \beta_i]$ are divided into $\gamma_i \geq 1$ equal sub-intervals: $[x_i^0, x_i^1]$, $[x_i^1, x_i^2]$, ..., $[x_i^{\gamma_i - 1}, x_i^{\gamma_i}]$, where $x_i^j = \alpha_i + \frac{j(\beta_i - \alpha_i)}{\gamma_i}$, $j = 0, \ldots, \gamma_i$, $i = 1, \ldots, n$. The marginal value (see Figure 1) of an alternative $a \in A$ is approximated by linear interpolation

$$u_i(a) = u_i(x_i^j) + \frac{g_i(a) - x_i^j}{x_i^{j+1} - x_i^j} \left(u_i(x_i^{j+1}) - u_i(x_i^j) \right), \text{ for } g_i(a) \in [x_i^j, x_i^{j+1}] \tag{4}$$

According to (4), the piecewise linear additive model is completely defined by the marginal values at the characteristic points, i.e. $u_i(x_i^0) = u_i(\alpha_i),\ u_i(x_i^1),\ u_i(x_i^2), \ldots, u_i(x_i^{\gamma_i}) = u_i(\beta_i)$.

It is also usual to suppose a kind of normalization, such as, $u_i(\alpha_i) = 0,\ \forall i \in G$, and $\sum_{i=1}^{n} u_i(\beta_i) = 1$. This will bound the value function $U(a)$ in the interval [0,1].

Therefore, a value function $U(a) = \sum_{i=1}^{n} u_i(a)$ is compatible if it satisfies the following set of constraints

$$\begin{array}{l} \left.\begin{array}{l} U(a_k) > U(a_{k+1}) \Leftrightarrow\; a_k \succ a_{k+1} \\ U(a_k) = U(a_{k+1}) \Leftrightarrow\; a_k \sim a_{k+1} \end{array}\right\} \\ k = 1, \ldots, m-1 \\ u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0,\ i = 1, ..., n,\ j = 1, ..., \gamma_i - 1 \\ u_i(\alpha_i) = 0,\ i = 1, ..., n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1 \end{array} \tag{5}$$

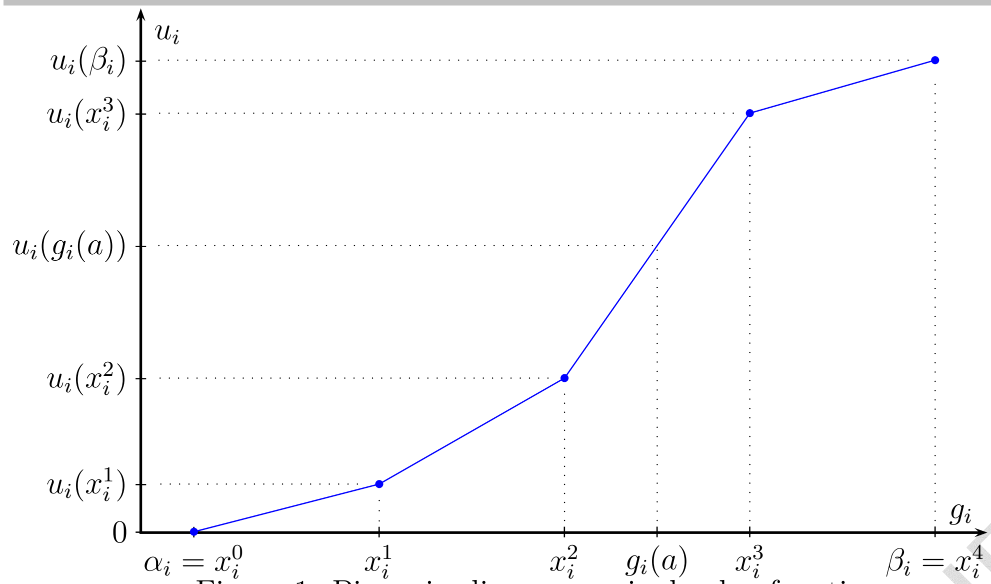

Figure 1: Piecewise linear marginal value function

To verify if a compatible value function $U(a) = \sum_{i=1}^{n} u_i(a)$ restoring the reference preorder $\succsim$ on A^R exists, one can solve the following linear programming problem, where $u_i(x_i^j), i = 1, ..., n,\ j = 1, ..., \gamma_i$, are unknown, and $\sigma^+(a), \sigma^-(a), a \in A^R$ are auxiliary variables:

$$
\begin{array}{ll}
Min \rightarrow & F = \sum_{k=1}^{m} (\sigma^+(a_k) + \sigma^-(a_k)) \\
\text{s.t.} & \\
& \left.\begin{array}{l}
U(a_k) + \sigma^+(a_k) - \sigma^-(a_k) \geq \\
\quad U(a_{k+1}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) + \varepsilon \ \Leftrightarrow \ a_k \succ a_{k+1} \\
U(a_k) + \sigma^+(a_k) - \sigma^-(a_k) = \\
\quad U(a_{k+1}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) \ \Leftrightarrow \ a_k \sim a_{k+1}
\end{array}\right\} k = 1, \ldots, m-1 \\
& u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0,\ i = 1, ..., n,\ j = 1, ..., \gamma_i - 1 \\
& u_i(\alpha_i) = 0,\ i = 1, ..., n \\
& \sum_{i=1}^{n} u_i(\beta_i) = 1 \\
& \sigma^+(a_k),\ \sigma^-(a_k) \geq 0,\ k = 1, ..., m
\end{array}
\qquad (6)
$$

where ε is an arbitrarily small positive value so that $U(a_k)+\sigma^+(a_k)-\sigma^-(a_k) > U(a_{k+1})+\sigma^+(a_{k+1})-\sigma^-(a_{k+1})$ in case of $a_k \succ a_{k+1}$.

If the optimal value of the objective function of the program (6) is equal to zero ($F^* = 0$), then there exists at least one value function $U(a) = \sum_{i=1}^{n} u_i(a)$ satisfying (5), i.e. compatible with the reference preorder on A^R. In other words, this means that the corresponding polyhedron (5) of feasible solutions for $u_i(x_i^j),\ i = 1, ..., n,\ j = 1, ..., \gamma_i$, is not empty.

When the optimal value of the objective function of the program (6) is greater than zero ($F^* > 0$), then there is no value function $U(a) = \sum_{i=1}^{n} u_i(a)$ compatible with the reference preorder on A^R. In such a case, three possible moves can be considered:

- increasing the number of linear pieces γ_i for one or several marginal value function u_i could make it possible to find an additive value function compatible with the reference preorder on A^R,
- revising the reference preorder on A^R could lead to find an additive value function compatible with the new preorder,

- searching over the relaxed domain $F \leq F^* + \eta$ could lead to an additive value function giving a preorder on A^R sufficiently close to the reference preorder (in the sense of Kendall's τ).

3 Existing approaches and motivations for a new method

Our work aims at generalizing the UTA method in order to consider the set of all value functions compatible with the indirect preference information rather than choosing a single value function within the set of compatible ones. The literature concerning MCDA methods involving a set of additive value functions can be viewed from three points of view:

- The methods are designed for different problem statements (problematics, see [24]):
  - choice of the best alternative (*e.g.* [2], [14], [17], [5], [27]),
  - sorting alternatives into predefined categories (*e.g.* [15], [4]),
  - ranking of alternatives from the best to the worst (*e.g.* [10], [12])
- The methods also differ with respect to the kind of the set of value functions and the characteristics of these functions: linear (*e.g.* [14], [12]) or piecewise linear (*e.g.* [3], [10], [4]) or monotone (*e.g.* [1]) value functions.
- The sets of value functions can be:
  - explicitly listed (*e.g.* [28]),
  - defined from stated constraints on the functions (*e.g.* [2], [18]),
  - induced from holistic preference statements concerning alternatives (*e.g.* [15], [32], [1]).

A review of the literature and, particularly, of the methods based on the ordinal regression approach, shows that these methods fail to consider some important issues :

- If the polyhedron of value functions compatible with the stated preference information is not empty, then the choice of a single or few representative value functions is either arbitrary or left to the DM. In the latter case, the DM is supposed to know how to interpret the form of the marginal value functions in order to choose among them, which is not easy for most DMs. Therefore, it seems reasonable to accept existence of all value functions compatible with the preference information provided by the DM and to assess a preference relation in the set of alternatives A with respect to all these functions.
- In most methods, the class of value functions is limited to linear or piecewise linear marginal value functions. To specify the number of characteristic points (breakpoints) is arbitrary and restrictive. It is desirable to consider just monotone marginal value functions which do not involve any parametrization.
- Most methods require that the DM provides constraints on the range of weights of linear marginal value functions, or on the range of variation of piecewise linear marginal value functions. The DM may have, however, difficulties to analyze the link between a specific value function and the resulting ranking. This is why we believe that the DM should be allowed to express preference information in terms of pairwise comparisons of alternatives rather than fixing the above constraints. Providing preference information in this way is consistent with intuitive reasoning of DMs.

- The methods based on ordinal regression are usually considering the preference information provided by the DM as a whole. As a consequence, it is difficult for the DM to associate a piece of his/her preference information with the result and, therefore, to control the impact of each piece of information (s)he provides on the result. As such a control is desirable for a truly interactive process, ordinal regression methods should allow the DM to provide incrementally the preference information by possibly small pieces.

In this paper, we intend to present a new ordinal regression method that accounts for all shortcomings listed above.

4 UTA$^{\text{GMS}}$ – a new UTA-like method

4.1 Presentation of the method

The new UTA$^{\text{GMS}}$ method is an ordinal regression method using a set of additive value functions $U(a) = \sum_{i=1}^{n} u_i(a)$ as a preference model. One of its characteristic features is that it takes into account the set of all value functions compatible with the preference information provided by the DM. Moreover, it considers general non-decreasing marginal value functions instead of piecewise linear only.

We suppose that the DM provides preference information in form of pairwise comparisons of reference alternatives from $A^R \subseteq A$. This preference information is a partial preorder on A^R, denoted by $\succsim$. A value function is called *compatible* if it is able to restore the partial preorder $\succsim$. Moreover, each compatible value function induces a ranking on the whole set A.

In particular, for any two alternatives $a, b \in A$, a compatible value function U ranks a and b in one of the following ways:

- a is preferred to b because $U(a) > U(b)$,
- b is preferred to a because $U(a) < U(b)$,
- a is indifferent to b because $U(a) = U(b)$,

With respect to $a, b \in A$, it is thus reasonable to ask the following two questions:

- are a and b ranked in the same way by <u>all</u> compatible value functions?
- is there <u>at least one</u> compatible value function ranking a at least as good as b (or b at least as good as a)?

Having answers to these questions for all pairs of alternatives $(a, b) \in A \times A$, one gets a *necessary* weak preference relation $\succsim^N$, in case $U(a) \geq U(b)$ for all compatible value functions, and a *possible* weak preference relation $\succsim^P$ in A, in case $U(a) \geq U(b)$ for at least one compatible value function.

Let us remark that preference relations $\succsim^N$ and $\succsim^P$ are meaningful only if there exists at least one compatible value function. Therefore, wherever the contrary is not explicitly stated, we suppose that there exists at least one compatible value function. Observe also that in this case, for any $a, b \in A^R$,

$$a \succsim b \Rightarrow a \succsim^N b$$

and

$$a \succ b \Rightarrow \text{not}(b \succsim^P a).$$

In fact, if $a \succsim b$, then for any compatible value function, $U(a) \geq U(b)$ and, therefore, $a \succsim^N b$. Moreover, if $a \succ b$, then for any compatible value function, $U(a) > U(b)$ and, consequently, there is no compatible value function such that $U(b) \geq U(a)$, which means that not$(b \succsim^P a)$.

Formally, a general additive compatible value function is an additive value function $U(a) = \sum_{i=1}^{n} u_i(a)$ satisfying the following set of constraints:

$$\left.\begin{array}{l} \left.\begin{array}{l} U(c) > U(d) \;\Leftrightarrow\; c \succ d \\ U(c) = U(d) \;\Leftrightarrow\; c \sim d \end{array}\right\} \text{ for all } c, d \in A^R \\ u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\ i = 1, ..., n,\ j = 2, ..., m \\ u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\ i = 1, ..., n, \\ u_i(\alpha_i) = 0,\ i = 1, ..., n \\ \sum_{i=1}^{n} u_i(\beta_i) = 1, \end{array}\right\} (E^{A^R})$$

where τ_i is the permutation on the set of indices of alternatives from A^R that reorders them according to the increasing evaluation on criterion g_i, i.e.

$$g_i(a_{\tau_i(1)}) \leq g_i(a_{\tau_i(2)}) \leq \ldots \leq g_i(a_{\tau_i(m-1)}) \leq g_i(a_{\tau_i(m)})$$

Remark that, due to this formulation of the ordinal regression problem, no linear interpolation is required to express the marginal value of any reference alternative. Thus, one cannot expect that increasing the number of characteristic points will bring some "new" compatible additive value functions. In consequence, UTA$^{\text{GMS}}$ considers all compatible additive value functions while classical UTA ordinal regression (6) deals with a subset of the whole set of compatible additive value functions, more precisely the subset of piecewise linear additive value functions relative to the considered characteristic points.

4.2 Properties of the relations $\succsim^N$ and $\succsim^P$

Binary relations $\succsim^N$ and $\succsim^P$ satisfy the following interesting properties.

Proposition 4.1. $\succsim^P \supseteq \succsim^N$

Proof. If $U(a) \geq U(b)$ for all compatible value functions U, i.e. $a \succsim^N b$, then there is at least one compatible value function U' such that $U'(a) \geq U'(b)$, i.e. $a \succsim^P b$. □

Proposition 4.2. *For all $a, b \in A$, $a \succsim^N b$ or $b \succsim^P a$.*

Proof. Let us denote by $\mathcal{U}$ the set of value functions compatible with $\succsim$. For all $a, b \in A$,

$$U(a) \geq U(b) \text{ for all } U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } U(b) > U(a) \Rightarrow a \succsim^N b \text{ or } b \succsim^P a.$$

□

Let us observe that from Proposition 4.2, we can get the following interesting corollary.

Corollary 4.1. *For all $a, b \in A$,*

1) $not(a \succsim^N b) \Rightarrow b \succsim^P a$,

2) $not(a \succsim^P b) \Rightarrow b \succsim^N a$.

Proof. 1) Since $a \succsim^N b$ or $b \succsim^P a$, if $\text{not}(a \succsim^N b)$, then $b \succsim^P a$.

2) Since $a \succsim^P b$ or $b \succsim^N a$, if $\text{not}(a \succsim^P b)$, then $b \succsim^N a$.

□

Proposition 4.3. $\succsim^N$ *is a partial preorder (i.e. reflexive and transitive).*

Proof. For all $a \in A$, $U(a) = U(a)$. This is true also for all U being compatible value functions, such that $a \succsim^N a$. Let us suppose that for $a, b, c \in A$, we have $a \succsim^N b$ and $b \succsim^N c$. This means that for all compatible value functions U we have $U(a) \geq U(b)$ and $U(b) \geq U(c)$, which implies that for all compatible value functions U we have $U(a) \geq U(c)$, i.e. $a \succsim^N c$. □

Proposition 4.4. $\succsim^P$ *is strongly complete, i.e. for all $a, b \in A$, $a \succsim^P b$ or $b \succsim^P a$, and negatively transitive, i.e. for all $a, b, c \in A$, $not(a \succsim^P b)$ and $not(b \succsim^P c) \Rightarrow not(a \succsim^P c)$.*

Proof. Consider any compatible value function U. For each pair $a, b \in A$, it holds $U(a) \geq U(b)$ or $U(b) \geq U(a)$, i.e. $a \succsim^P b$ or $b \succsim^P a$; therefore $\succsim^P$ is strongly complete.
$not(a \succsim^P b)$ means that there does not exist any compatible value function U such that $U(a) \geq U(b)$.
$not(b \succsim^P c)$ means that there does not exist any compatible value function U such that $U(b) \geq U(c)$.
Therefore, there does not exist any compatible value function U such that $U(a) \geq U(c)$, which means that $not(a \succsim^P c)$. □

Observe that while $\succsim^P$ is negatively transitive, it is not necessarily transitive, i.e. it is possible that for $a, b, c \in A$, $a \succsim^P b$, $b \succsim^P c$ but not $a \succsim^P c$. This can happen because there could exist one compatible value function U such that $U(a) \geq U(b)$ and one compatible value function U' such that $U'(b) \geq U'(c)$, however, there could be no compatible value function U'' such that $U''(a) \geq U''(c)$.

Notice that it is impossible to infer $\succsim^N$ from $\succsim^P$ or vice versa, since $\succsim^N$ and $\succsim^P$ are not dual, i.e. $a \succsim^N b \Leftrightarrow \text{not}(b \succsim^P a)$ does not hold as one could expect. In fact, in case for $a, b \in A$, $U(a) = U(b)$ for all compatible value functions U, we have $a \succsim^N b$ and $b \succsim^P a$.

From the two weak preference relations $\succsim^N$ and $\succsim^P$, one can get preference, indifference and incomparability, in a usual way, i.e.

1) from the necessary weak preference relation $\succsim^N$ one obtains:
   - preference: $a \succ^N b \Leftrightarrow a \succsim^N b$ and $\text{not}(b \succsim^N a)$
   - indifference: $a \sim^N b \Leftrightarrow a \succsim^N b$ and $b \succsim^N a$
   - incomparability: $a ?^N b \Leftrightarrow \text{not}(a \succsim^N b)$ and $\text{not}(b \succsim^N a)$

2) from the possible weak preference relation $\succsim^P$ one obtains:
   - preference: $a \succ^P b \Leftrightarrow a \succsim^P b$ and $\text{not}(b \succsim^P a)$
   - indifference: $a \sim^P b \Leftrightarrow a \succsim^P b$ and $b \succsim^P a$

Observe that in case of $\succsim^P$, incomparability is not considered because, for proposition 4.3, $\succsim^P$ is strongly complete.

The preference relations obtained from $\succsim^N$ constitute the necessary ranking, and the preference relations obtained from $\succsim^P$ constitute the possible ranking; they are presented to the DM as end results of the UTA$^{\text{GMS}}$ method at the current stage of interaction.

4.3 Computation of the relations $\succsim^N$ and $\succsim^P$

In order to compute binary relations $\succsim^P$ and $\succsim^N$ we can proceed as follows. For all alternatives $a, b \in A$, let π_i be a permutation of the indices of alternatives from set $A^R \cup \{a, b\}$ that reorders them according to increasing evaluation on criterion g_i, i.e.

$$g_i(a_{\pi_i(1)}) \leq g_i(a_{\pi_i(2)}) \leq ... \leq g_i(a_{\pi_i(\omega-1)}) \leq g_i(a_{\pi_i(\omega)})$$

where

- if $A^R \cap \{a, b\} = \emptyset$, then $\omega = m + 2$
- if $A^R \cap \{a, b\} = \{a\}$ or $A^R \cap \{a, b\} = \{b\}$, then $\omega = m + 1$
- if $A^R \cap \{a, b\} = \{a, b\}$, then $\omega = m$.

Then, we can fix the characteristic points of u_i, $i = 1, ..., n$, in

$$g_i^0 = \alpha_i, \quad g_i^j = g_i(a_{\pi_i(j)}) \text{ for } j = 1, ..., \omega, \quad g_i^{\omega+1} = \beta_i$$

Let us consider the following set $E(a, b)$ of ordinal regression constraints, with $i = 1, ..., n$, $j = 1, ..., \omega + 1$, as variables:

$$\left.\begin{array}{l}
\left.\begin{array}{l} U(c) \geq U(d) + \varepsilon \;\Leftrightarrow\; c \succ d \\ U(c) = U(d) \;\Leftrightarrow\; c \sim d \end{array}\right\} \text{ for all } c, d \in A^R \\
u_i(g_i^j) - u_i(g_i^{j-1}) \geq 0, \; i = 1, ..., n, \; j = 1, ..., \omega + 1 \\
u_i(g_i^0) = 0, \; i = 1, ..., n \\
\sum_{i=1}^n u_i(g_i^{\omega+1}) = 1,
\end{array}\right\} (E(a, b))$$

where ε is an arbitrarily small positive value, as in (6).

The above set of constraints depends on the pair of alternatives $a, b \in A$ because their evaluations $g_i(a)$ and $g_i(b)$ give coordinates for two of $(\omega + 1)$ characteristic points of marginal value function u_i, for each $i = 1, \ldots, n$. Note that for all $a, b \in A$, $E(a, b) = E(b, a)$.

Let us suppose that the polyhedron defined by the set of constraints $E(a, b)$ is not empty. In this case we have that:

$$a \succsim^N b \;\Leftrightarrow\; d(a, b) \geq 0$$

where:

$$\begin{array}{rcl} d(a, b) & = & Min\{U(a) - U(b)\} \\ & & \text{s.t. set } E(a, b) \text{ of constraints} \end{array} \tag{7}$$

and

$$a \succsim^P b \;\Leftrightarrow\; D(a, b) \geq 0$$

where :
$$\begin{aligned} D(a,b) \;=\; & Max\{U(a)-U(b)\} \\ & \text{s.t. set } E(a,b) \text{ of constraints} \end{aligned} \tag{8}$$

What are the relations between $\succsim^P$ and $\succsim^N$ with respect to computation? In other words, is it necessary to calculate $d(a,b)$ and $D(a,b)$ for all $a, b \in A$? Proposition 4.5 gives a technical result useful for answering this question.

Proposition 4.5. *For all $a, b \in A$, the following equivalences hold:*

$$\begin{aligned} d(a,b) \geq 0 &\Leftrightarrow D(b,a) \leq 0 \\ D(a,b) \geq 0 &\Leftrightarrow d(b,a) \leq 0 \\ d(a,b) = 0 &\Leftrightarrow D(b,a) = 0 \end{aligned}$$

Proof. The proof results from the following equalities:
$d(a,b) = Min_{s.t.E(a,b)}\{U(a)-U(b)\} = -Max_{s.t.E(b,a)}\{U(b)-U(a)\} = -D(b,a)$. □

According to Proposition 4.5, the relation $\succsim^N$ can be computed using either $d(a,b)$ or $D(a,b)$, as shown in Tables 1 and 2. A similar remark concerns the relation $\succsim^P$ which can be computed using either $d(a,b)$ or $D(a,b)$, as shown in Tables 3 and 4.

		$a \succsim^N b$		not$(b \succsim^N a)$
		$d(b,a) > 0$	$d(b,a) = 0$	$d(b,a) < 0$
$a \succsim^N b$	$d(a,b) > 0$			$a \succ^N b$
	$d(a,b) = 0$		$a \sim^N b$	$a \succ^N b$
not$(a \succsim^N b)$	$d(a,b) < 0$	$b \succ^N a$	$b \succ^N a$	$a?b$

Table 1: Necessary ranking computed in terms of $d(a,b)$

		not$(a \succsim^N b)$	$a \succsim^N b$	
		$D(b,a) > 0$	$D(b,a) = 0$	$D(b,a) < 0$
not$(b \succsim^N a)$	$D(a,b) > 0$	$a?b$	$a \succ^N b$	$a \succ^N b$
$b \succsim^N a$	$D(a,b) = 0$	$b \succ^N a$	$a \sim^N b$	
	$D(a,b) < 0$	$b \succ^N a$		

Table 2: Necessary ranking computed in terms of $D(a,b)$

		$b \succsim^P a$		not$(b \succsim^P a)$
		$d(b,a) > 0$	$d(b,a) = 0$	$d(b,a) < 0$
$a \succsim^P b$	$d(a,b) > 0$			$a \succ^P b$
	$d(a,b) = 0$		$a \sim^P b$	$a \sim^P b$
not$(a \succsim^P b)$	$d(a,b) < 0$	$b \succ^P a$	$a \sim^P b$	$a \sim^P b$

Table 3: Possible ranking computed in terms of $d(a,b)$

		not($a \succsim^P b$)	$a \succsim^P b$	
		$D(b,a) > 0$	$D(b,a) = 0$	$D(b,a) < 0$
not($b \succsim^P a$)	$D(a,b) > 0$	$a \sim^P b$	$a \sim^P b$	$a \succ^P b$
$b \succsim^P a$	$D(a,b) = 0$	$a \sim^P b$	$a \sim^P b$	
	$D(a,b) < 0$	$b \succ^P a$		

Table 4: Possible ranking computed in terms of $D(a, b)$

Remark 4.1. *In the absence of any pairwise comparison of reference alternatives, the necessary weak preference relation $\succsim^N$ boils down to the weak dominance relation Δ in A ($a\Delta b$ iff $g_i(a) \geq g_i(b)$, $i = 1, .., n$). Each pairwise comparison provided by the DM, for which the dominance relation does not hold, contributes to enrich $\succsim^N$, i.e. it makes the relation $\succsim^N$ true for at least one more pair of alternatives.*

Remark 4.2. *In the absence of any pairwise comparison of reference alternatives, the possible weak preference relation $\succsim^P$ is a complete relation such that for any pair $(a, b) \in A \times A$:*

- $a \sim^P b$ *(i.e.* $a \succsim^P b$ *and* $b \succsim^P a$*)* $\Leftrightarrow$ *(not($a\Delta b$) and not($b\Delta a$)) or ($a\Delta b$ and $b\Delta a$)*
- $a \succ^P b$ *(i.e.* $a \succsim^P b$ *and not(*$b \succsim^P a$*))* $\Leftrightarrow$ $a\Delta b$ *and not(*$b\Delta a$*)*

Each pairwise comparison provided by the DM, for which the dominance relation does not hold, contributes to impoverish $\succsim^P$, i.e., it makes the relation $\succsim^P$ false for at least one more pair of alternatives.

4.4 Analysis of incompatibility

Let us consider now the case where there is no value function compatible with the preference information. We say, this is the case of incompatibility. In such a case, the polyhedron generated by constraints E^{A^R} is empty. Therefore, the polyhedrons generated by constraints $E(a, b)$, for all $a, b \in A$, are also empty in this case. Such a case may occur in one of the following situations:

- the preferences of the DM do not match the additive model,
- the DM may have made an error in his/her statements; for example stating that $a \succ b$ while $b\Delta a$,
- the statements provided by the DM are contradictory because his/her preferences are unstable, some hidden criteria are taken into account, ...

In such a case, the DM may want either to pursue the analysis with such an incompatibility or to identify its reasons in order to remove it and, therefore, to define a new set of pairwise comparisons B'^R whose corresponding constraints E'^{A^R} generate a non empty polyhedron. Let us consider below the two possible solutions.

4.4.1 Situation where incompatibility is accepted

If the DM wants to pursue the analysis with the incompatibility, he/she has to accept that some of his/her pairwise comparisons of reference alternatives will not be reproduced by any value function. Note that, from a formal viewpoint, if the polyhedron generated by E^{A^R} is empty, then $\succsim^N$ and $\succsim^P$ are meaningless. Thus, the acceptance of the inconsistency means that the DM does not change the preference information represented by $\succsim$ and computes $d(a,b)$ and $D(a,b)$ on a new set of constraints E'^{A^R} differing from the original set E^{A^R} by an additional constraint on the acceptable total error:

$$\left.\begin{array}{l}
\left.\begin{array}{l} U(c)+\sigma^+(c)-\sigma^-(c) > U(d)+\sigma^+(d)-\sigma^-(d) \;\Leftrightarrow\; c \succ d \\ U(c)+\sigma^+(c)-\sigma^-(c) = U(d)+\sigma^+(d)-\sigma^-(d) \;\Leftrightarrow\; c \sim d \end{array}\right\} \text{ for all } c,d \in A^R \\
u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\ i=1,...,n,\ j=2,...,m \\
u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\ i=1,...,n, \\
u_i(\alpha_i)=0,\ i=1,...,n \\
\sum_{i=1}^n u_i(\beta_i)=1, \\
\sigma^+(c)\geq 0,\ \sigma^-(c)\geq 0, \text{ for all } c \in A^R \\
\sum_{c\in A^R}(\sigma^+(c)+\sigma^-(c)) \leq \delta
\end{array}\right\} (E'^{A^R})$$

where $\delta > F^*$, with $F^* = min\sum_{c\in A^R}(\sigma^+(c)+\sigma^-(c))$ subject to E^{A^R}, such that the resulting new set of constraints E'^{A^R} is not empty.

On the basis of E'^{A^R}, for any pair $(a,b)\in A$, the set of constraints $E'(a,b)$ can be built as the union of the constraints E'^{A^R} and the constraints relative to the breakpoints introduced by those alternatives a,b that possibly do not belong to A^R. Then preference relations $\succsim'^N$ and $\succsim'^P$ can be computed by minimizing and maximizing $U(a)-U(b)$ subject to $E'(a,b)$, rather than to $E(a,b)$, respectively. In other words, in this case, $d(a,b)$ and $D(a,b)$ are computed considering $E'(a,b)$ rather than $E(a,b)$.

Obviously, the necessary and possible rankings resulting from these computations will not fully restore the provided pairwise comparisons, i.e. there is at least one couple $a,b\in A^R$ such that

- $a \succsim b$, but it is false that for all the compatible value functions $U(a)\geq U(b)$ (in other words, there exists a compatible value function such that $U(a)<U(b)$ and thus not$(a \succsim'^N b)$), or
- $a \succ b$, but it is false that for all the compatible value functions $U(a)>U(b)$ (in other words, there exists also a value function such that $U(b)\geq U(a)$ and thus $b \succsim'^P a$).

Next result will state that $\succsim'^N$ and $\succsim'^P$ maintain all the main properties of preference relations $\succsim^N$ and $\succsim^P$.

Proposition 4.6.

- $\succsim'^N \subseteq \succsim'^P$,
- $\succsim'^N$ *is a complete preorder (i.e. transitive and strongly complete),*
- $\succsim'^P$ *is strongly complete.*

Proof: $\succsim'^N$ and $\succsim'^P$ are built using the value functions satisfying constraints E'^{A^R}, in the same way as $\succsim^N$ and $\succsim^P$ are built using the value functions satisfying constraints E^{A^R}. Thus the proof is analogous to the proof of Propositions 4.1, 4.3 and 4.4.

4.4.2 Situation where incompatibility is not accepted

If the DM does not want to pursue the analysis with the incompatibility, it is necessary to identify the troublesome pairwise comparisons responsible for this incompatibility, so as to remove some of them. Remark that there may exist several sets of pairwise comparisons which, once removed, make set E^{A^R} of constraints non-empty. Hereafter, we outline the main steps of a procedure which identifies these sets.

Recall that the pairwise comparisons of reference alternatives are represented in the ordinal regression constraints E^{A^R} by linear constraints. Hence, identifying the troublesome pairwise comparisons of reference alternatives amounts at finding a minimal subset of constraints that, once removed from E^{A^R}, leads to a set of constraints generating a non-empty polyhedron of compatible value functions. The identification procedure is to be performed iteratively since there may exist several minimal subsets of this kind.

Let associate with each pairwise comparison of reference alternatives a and b a new binary variable $v_{a,b}$. Using these binary variables, we rewrite the first two constraints of set E^{A^R} as follows:

$$\begin{array}{l} a \succ b \Leftrightarrow U(a) - U(b) + Mv_{a,b} > 0 \\ a \sim b \Leftrightarrow \left\{ \begin{array}{l} U(a) - U(b) + Mv_{a,b} \geq 0 \\ U(b) - U(a) + Mv_{a,b} \geq 0 \end{array} \right. \end{array} \tag{9}$$

where $M > 1$. Remark that if $v_{a,b} = 1$, then the corresponding constraint is satisfied whatever the value function is, which is equivalent to elimination of this constraint. Therefore, identifying a minimal subset of troublesome pairwise comparisons can be performed by solving the following mixed 0-1 linear program:

$$\begin{array}{ll} Min \rightarrow & f = \sum_{a,b \in A^R:\, a \succsim b} v_{a,b} \\ \text{s. t.} & \\ & \left. \begin{array}{l} a \succ b \Leftrightarrow U(a) - U(b) + Mv_{a,b} \geq \varepsilon \\ a \sim b \Leftrightarrow \left\{ \begin{array}{l} U(a) - U(b) + Mv_{a,b} \geq 0 \\ U(b) - U(a) + Mv_{a,b} \geq 0 \end{array} \right. \end{array} \right\} \text{ for all } a, b \in A^R \\ & u_i(g_i(a_{\tau_i(j)})) - u_i(g_i(a_{\tau_i(j-1)})) \geq 0,\ i = 1, ..., n,\ j = 2, ..., m \\ & u_i(g_i(a_{\tau_i(1)})) \geq 0, u_i(g_i(a_{\tau_i(m)})) \leq u_i(\beta_i),\ i = 1, ..., n, \\ & u_i(\alpha_i) = 0,\ i = 1, ..., n \\ & \sum_{i=1}^{n} u_i(\beta_i) = 1, \\ & v_{a,b} \in \{0, 1\} \end{array} \tag{10}$$

The optimal solution of (10) indicates one of the subsets of smallest cardinality being the cause of incompatibility. Alternative subsets of this kind can be found by solving (10) with additional constraint that forbids finding again the same solution. Let f^* be the optimal value of the objective function of (10) and $v^*_{a,b}$ the values of the binary variables at the optimum. Let also $S_1 = \{(a,b) \in A^R \times A^R : a \succsim b \text{ and } v^*_{a,b} = 1\}$. The additional constraint has then the form

$$\sum_{(a,b) \in S_1} v_{a,b} \leq f^* - 1 \tag{11}$$

Continuing in this way, we can identify other subsets, possibly all of them. These subsets of pairwise comparisons are to be presented to the DM as alternative solutions for removing incompatibility. Such a procedure has been described in [21].

5 Extensions

5.1 Specification of pairwise comparisons with gradual confidence levels

The UTA$^{\text{GMS}}$ method presented in the previous section is intended to support the DM in an interactive process. Indeed, defining a large set of pairwise comparisons of reference alternatives at once can be difficult for the DM. Therefore, one way to reduce the difficulty of this task would be to permit the DM an incremental specification of pairwise comparisons. This way of proceeding allows the DM to control the evolution of the necessary and possible weak preference relations.

Another way of reducing the difficulty of the task is to extend the UTA$^{\text{GMS}}$ method so as to account for different confidence levels assigned to pairwise comparisons. Let $\succsim_1 \subseteq \succsim_2 \subseteq ... \subseteq \succsim_s$ be embedded sets of DM's partial preorders of reference alternatives. To each set of partial preorders $\succsim_t, t = 1, \ldots s$, corresponds a set of constraints $E_t^{A^R}$ generating a polyhedron of compatible value functions $P_t^{A^R}$. Polyhedrons $P_t^{A^R}$, $t = 1, \ldots s$, are embedded in the inverse order of the related partial preorders $\succsim_t$, i.e. $P_1^{A^R} \supseteq P_2^{A^R} \supseteq ... \supseteq P_s^{A^R}$. We suppose that $P_s^{A^R} \neq \emptyset$ and, therefore, due to the fact that partial preorders $\succsim_t$ are embedded, $P_t^{A^R} \neq \emptyset$, for all $t = 1, \ldots, s$. If $P_s^{A^R} = \emptyset$ we consider only embedded partial preorders until $\succsim_p$ with $p = max\left\{t : P_t^{A^R} \neq \emptyset\right\}$ and relabel p by s. For all $a, b \in A$, we say that there is a necessary weak preference relation of level t, denoted by $a \succsim_t^N b$ $(t = 1, ..., s)$, if for all value functions U compatible with the partial preorder $\succsim_t$, we have $U(a) \geq U(b)$. Analogously, for all $a, b \in A$, we say that there is a possible weak preference relation of level t, denoted by $a \succsim_t^P b$ $(t = 1, ..., s)$, if for at least one value function U compatible with the partial preorder $\succsim_t$, we have $U(a) \geq U(b)$.

In order to compute possible and necessary weak preference relations $\succsim_t^P$ and $\succsim_t^N$, we can proceed as follows. For all $a, b \in A$, set of constraints $E_t(a, b)$ can be obtained from set $E_t^{A^R}$ by adjoining the constraints relative to the breakpoints introduced by those alternatives a, b that possibly do not belong to A^R. For each $t = 1, \ldots, s$, and binary preference relations $\succsim_t^N$ and $\succsim_t^P$, we have

$$a \succsim_t^N b \;\Leftrightarrow\; d_t(a, b) \geq 0$$

where:

$$d_t(a, b) \;=\; Min\{U(a) - U(b)\} \quad \text{s.t. set } E_t(a, b) \text{ of constraints} \tag{12}$$

and

$$a \succsim_t^P b \;\Leftrightarrow\; D_t(a, b) \geq 0$$

where :

$$D_t(a, b) \;=\; Max\{U(a) - U(b)\} \quad \text{s.t. set } E_t(a, b) \text{ of constraints} \tag{13}$$

Each time we pass from $\succsim_{t-1}$ to $\succsim_t$, $t = 1, \ldots, s-1$, we add to $E_{t-1}^{A^R}$ and, consequently, to $E_{t-1}(a, b)$, new constraints concerning pairs $(c, d) \in A^R \times A^R$, such that $c \succsim_t d$ but not $c \succsim_{t-1} d$, thus the computations of $d_t(a, b)$ and $D_t(a, b)$, for all $a, b \in A \times A$ proceed iteratively.

The following result states that binary preference relations $\succsim_t^N$ and $\succsim_t^P$, $t = 1, \ldots, s$, inherit properties of $\succsim^N$ and $\succsim^P$.

Proposition 5.1.

- $\succsim_t^N \subseteq \succsim_t^P$,

- $\succsim_t^N$ *is a complete preorder (i.e. transitive and strongly complete),*
- $\succsim_t^P$ *is strongly complete and negatively transitive.*

Proof. Analogous to the proof of Propositions 4.1, 4.3 and 4.4 $\square$

An important property of preference relations $\succsim_t^N$ and $\succsim_t^P$, $t = 1, \ldots, s$, is stated by the following proposition.

Proposition 5.2. *$\succsim_t^N$ and $\succsim_t^P$, $t = 1, \ldots, s$, are nested partial preorders: $\succsim_{t-1}^N \subseteq \succsim_t^N$ and $\succsim_t^P \supseteq \succsim_{t-1}^P$, $t = 2, \ldots, s$.*

Proof. $a \succsim_{t-1}^N b$, $a, b \in A$, means that $U(a) \geq U(b)$ for all value functions U satisfying $E_{t-1}^{A^R}$. Since each value function U satisfying $E_t^{A^R}$ satisfies also $E_{t-1}^{A^R}$, we have that $U(a) \geq U(b)$ for all value functions U satisfying $E_t^{A^R}$, from which we get $a \succsim_t^N b$. Thus $a \succsim_{t-1}^N b \Rightarrow a \succsim_t^N b$, i.e. $\succsim_{t-1}^N \subseteq \succsim_t^N$.
$a \succsim_t^P b$, $a, b \in A$, means that there exists at least one value function U satisfying $E_t^{A^R}$ such that $U(a) \geq U(b)$. Let us denote one of these value functions by U^*. Since each value function U satisfying $E_t^{A^R}$ satisfies also $E_{t-1}^{A^R}$, we have that U^* satisfies $E_{t-1}^{A^R}$. Therefore, from $U^*(a) \geq U^*(b)$, we get $a \succsim_{t-1}^P b$. Thus, $a \succsim_t^P b \Rightarrow a \succsim_{t-1}^P b$, i.e. $\succsim_{t-1}^P \supseteq \succsim_t^P$. $\square$

Let λ_t be the confidence level assigned to pairwise comparisons concerning pairs $(c, d) \in A^R \times A^R$, such that $c \succsim_t d$ but not$(c \succsim_{t-1} d)$, $\succsim_0 = \emptyset$, $t = 1, ..., s$, $1 = \lambda_1 > \lambda_2 > \ldots > \lambda_s > 0$. Using partial preorders $\succsim_1, \ldots \succsim_s$ and corresponding $\lambda_1, \lambda_2, \ldots, \lambda_s$, a valued necessary preference relation $R^N : A \times A \mapsto [0, 1]$ or, more precisely, $R^N : A \times A \mapsto \{\lambda_1, \lambda_2, \ldots, \lambda_s, 0\}$, can be built as follows: for all $a, b \in A$

- if there exists one t $(t = 1, \ldots, s)$ such that $a \succsim_t^N b$,
  then $R^N(a, b) = max\left\{\lambda_t, t = 1, \ldots, s, \text{such that } a \succsim_t^N b\right\}$
- if there exists no t $(t = 1, \ldots, s)$ for which $a \succsim_t^N b$, then $R^N(a, b) = 0$.

Analogously, a valued possible preference relation $R^P : A \times A \mapsto [0, 1]$ or, more precisely, $R^P : A \times A \mapsto \{1 - \lambda_1, 1 - \lambda_2, \ldots, 1 - \lambda_s, 1\}$, can be built as follows: for all $a, b \in A$

- if there exists one t $(t = 1, \ldots, s)$ such that $a \succsim_t^P b$,
  then $R^P(a, b) = min\left\{1 - \lambda_t,\ t = 1, \ldots, s,\ \text{such that not}(a \succsim_t^P b)\right\}$
- if $a \succsim_t^P b$ for all t $(t = 1, \ldots, s)$, then $R^P(a, b) = 1$.

Proposition 5.3. *For all $a, b \in A$*

$$R^N(a, b) = \lambda_{t^*} \Leftrightarrow a \succsim_r^N b \quad \textit{for all } r \geq t^* \textit{ and not}(a \succsim_r^N b) \quad \textit{for all } r < t^*;$$

$$R^P(a, b) = 1 - \lambda_{t^*} \Leftrightarrow a \succsim_r^P b \quad \textit{for all } r < t^* \textit{ and not}(a \succsim_r^P b) \quad \textit{for all } r \geq t^*.$$

Proof. Since $R^N(a, b) = max\left\{\lambda_t, t = 1, \ldots, s, \text{such that } a \succsim_t^N b\right\}$, then $R^N(a, b) = \lambda_{t^*}$ implies $a \succsim_{t^*}^N b$. Taking into account that, for Proposition 5.2, $a \succsim_{t-1}^N b \Rightarrow a \succsim_t^N b$, we have $a \succsim_r^N b$ for all $r \geq t^*$. Moreover, $R^N(a, b) = \lambda_{t^*}$ implies not$(a \succsim_r^N b)$ for all r such that $\lambda_r > \lambda_{t^*}$. Taking into account that $\lambda_t > \lambda_{t+1}$ $(t = 1, ..., s - 1)$, we get that $R^N(a, b) = \lambda_{t^*}$ implies not$(a \succsim_r^N b)$ for all $r < t^*$. Thus, we proved that

$$R^N(a, b) = \lambda_{t^*} \Rightarrow a \succsim_r^N b \quad \text{for all } r \geq t^* \text{ and not}(a \succsim_r^N b) \quad \text{for all } r < t^*.$$

For all $a, b \in A$,

$$a \succsim_r^N b \quad \text{for all } r \geq t^* \text{ and not}(a \succsim_r^N b) \quad \text{for all } r < t^* \Rightarrow t^* = min\{t,\ t = 1, \ldots, s, \text{ such that } a \succsim_t^N b\}. \quad (i)$$

Remembering that $\lambda_t > \lambda_{t+1}$ $(t = 1, ..., s-1)$, from (i) we get

$$\lambda_{t^*} = max\left\{\lambda_t, t = 1, \ldots, s, \text{ such that } a \succsim_t^N b\right\},$$

and for the definition of $R^N(a,b)$, $R^N(a,b) = \lambda_{t^*}$. Thus, we proved that

$$a \succsim_r^N b \quad \text{for all } r \geq t^* \text{ and not}(a \succsim_r^N b) \quad \text{for all } r < t^* \Rightarrow R^N(a,b) = \lambda_{t^*},$$

which concludes the proof of

$$R^N(a,b) = \lambda_{t^*} \Leftrightarrow a \succsim_r^N b \quad \text{for all } r \geq t^* \text{ and not}(a \succsim_r^N b) \quad \text{for all } r < t^*.$$

Analogous proof holds for

$$R^P(a,b) = 1 - \lambda_{t^*} \Leftrightarrow a \succsim_r^P b \quad \text{for all } r < t^* \text{ and not}(a \succsim_r^P b) \quad \text{for all } r \geq t^*.$$

□

Proposition 5.3 states that $R^N(a,b) = \lambda_t$ means that $a \succsim_r^N b$ holds only for $r \geq t$, while, for the definition, $R^N(a,b) = 0$ means that $a \succsim_t^N b$ does not hold for any t $(t = 1, \ldots, s)$. Proposition 5.3 also expresses that $R^P(a,b) = 1 - \lambda_t$ means that $a \succsim_r^P b$ holds only for $r < t$, while, for the definition, $R^P(a,b) = 1$ means that $a \succsim_t^P b$ for all t $(t = 1, \ldots, s)$.

It is interesting to investigate the properties of valued binary relations R^N and R^P (for an introduction to valued binary relations and their properties see [6]). Let us remind that a valued binary relation R defined on a set Y, i.e. $R : Y \times Y \mapsto [0,1]$, is

- reflexive, if for all $\alpha \in Y$, $R(\alpha, \alpha) = 1$,
- min-transitive, if for all $\alpha, \beta, \gamma \in Y$, $min(R(\alpha,\beta), R(\beta,\gamma)) \leq R(\alpha,\gamma)$,
- strongly complete, if for all $\alpha, \beta \in Y$, $max(R(\alpha,\beta), R(\beta,\alpha)) = 1$,
- negatively transitive, if for all $\alpha, \beta, \gamma \in Y$, $min((1 - R(\alpha,\beta)), (1 - R(\beta,\gamma))) \leq (1 - (\alpha,\gamma))$.

A valued binary relation which is reflexive and min-transitive is called *fuzzy partial preorder.*

Proposition 5.4. *Valued binary relation R^N is reflexive and min-transitive and, therefore, it is a fuzzy partial preorder. Valued binary relation R^P is strongly complete and negatively transitive.*

Proof. For all $a \in A$, for all value functions U compatible with the partial preorder $\succsim_s$, we have $U(a) = U(a)$, which implies $a \succsim_s^N a$ and $R^N(a,a) = 1$, i.e. R^N is reflexive.

For all $a, b, c \in A$, two cases are possible:
a) $min(R^N(a,b), R^N(b,c)) = 0$,
b) $min(R^N(a,b), R^N(b,c)) > 0$.

Considering that always $R^N(a,c) \geq 0$, in case a) we have

$$R^N(a,c) \geq min(R^N(a,b), R^N(b,c)). \quad (i)$$

In case b), for the definition of R^N, we have

$$min(R^N(a,b), R^N(b,c)) =$$

$$= min\left\{max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that}\ a \succsim_t^N b\right\}, max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that}\ b \succsim_t^N c\right\}\right\} =$$

$$= max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that}\ a \succsim_t^N b\ \ \text{and}\ \ b \succsim_t^N c\right\}.$$

Thus, if $min(R^N(a,b), R^N(b,c)) = \lambda_r$, then $U(a) \geq U(b)$ and $U(b) \geq U(c)$ for all value functions U compatible with $\succsim_r$. Thus, for all value functions U compatible with $\succsim_r$ we have $U(a) \geq U(c)$ and, consequently, $a \succsim_r^N c$. This implies that

$$R^N(a,c) = max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that}\ a \succsim_t^N c\right\} \geq \lambda_r = min(R^N(a,b), R^N(b,c)). \quad (ii)$$

For (i) and (ii), valued binary relation R^N is min-transitive.

Let us suppose that $a \succsim_s^P b$, $a,b \in A$. In this case, for Proposition 5.2, $a \succsim_t^P b$ for all t $(t=1,\dots,s)$ and, therefore, $R^P(a,b)=1$. If, instead, not $(a \succsim_s^P b)$, then for Proposition 4.2, $b \succsim_s^N a$ and, therefore, for Property 1, $b \succsim_s^P a$. In consequence, for Proposition 5.2, $b \succsim_t^P a$ for all t $(t=1,\dots,s)$ and, thus, $R^P(b,a)=1$. This proves completeness of valued binary relation R^P.

For all $a,b,c \in A$, two cases are possible:
a) $max(R^P(a,b), R^P(b,c)) = 1$,
b) $max(R^P(a,b), R^P(b,c)) < 1$.

In case a), we have $R^P(a,b)=1$ or $R^P(b,c)=1$, and thus $1-R^P(a,b)=0$ or $1-R^P(b,c)=0$, such that $min((1-R^P(a,b)),(1-R^P(b,c)))=0$ and considering that always $1-R^P(a,c)\geq 0$, we get

$$min((1-R^P(a,b)),(1-R^P(b,c))) \leq 1-R^P(a,c).$$

In case b), $R^P(a,b)<1$ and $R^P(b,c)<1$, for definition of R^P, we have

$$1-R^P(a,b) = 1-min\left\{1-\lambda_t,\ t=1,\dots,s,\ \text{such that not}(a \succsim_t^P b)\right\}) =$$
$$= max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that not}(a \succsim_t^P b)\right\})$$

as well as

$$1-R^P(b,c) = 1-min\left\{1-\lambda_t,\ t=1,\dots,s,\ \text{such that not}(b \succsim_t^P c)\right\}) =$$

$$= max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that not}(b \succsim_t^P c)\right\}).$$

Thus,

$$min((1-R^P(a,b)),(1-R^P(b,c)))$$
$$=$$
$$min\left\{max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that not}(a \succsim_t^P b)\right\}, max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that not}(b \succsim_t^P c)\right\}\right\}$$
$$=$$
$$max\left\{\lambda_t,\ t=1,\dots,s,\ \text{such that not}(a \succsim_t^P b)\ \text{and not}(b \succsim_t^P c)\right\},$$
$$=$$
$$max\{\lambda_t,\ t=1,\dots,s,\ \text{such that,}\ U(b)>U(a)\ \text{and}\ U(c)>U(b)$$
$$\text{for all value functions}\ U\ \text{compatible with}\ \ \succsim_t\}.$$

If $min((1-R^P(a,b)),(1-R^P(b,c))) = \lambda_r$, then $U(a) < U(c)$ for all value functions compatible with $\succsim_r$, which means that there does not exist any value function U compatible with $\succsim_r$ such that $U(a) \geq U(c)$. Thus, $min\{t,\ t=1,\ldots,s,$ such that not$(a \succsim_t^P c)\} \leq r$ and, therefore,

$$max\{\lambda_t,\ t=1,\ldots,s,\ \text{such that not}(a \succsim_t^P c)\} \geq \lambda_r.$$

Since

$$max\{\lambda_t,\ t=1,\ldots,s,\ \text{such that not}(a \succsim_t^P c)\} =$$
$$= 1 - min\{1-\lambda_t,\ t=1,\ldots,s,\ \text{such that not}(a \succsim_t^P c)\} = 1 - R^P(a,c),$$

we conclude that $(1-R^P(a,c)) \geq \lambda_r = min((1-R^P(a,b)),(1-R^P(b,c)))$.

□

5.2 Accounting for intensity of preference

Another preference information that can be provided by the DM concerns the intensity of preference among two pairs of reference alternatives. Given two pairs of reference alternatives $(c,d) \in \succsim$ and $(c',d') \in \succsim$, such that $c \succ d$ and $c' \succ d'$, the DM can state : "*c is preferred to d at least as much as c' is preferred to d'*". Such statement means that for all compatible value functions U:

$$U(c) - U(d) > U(c') - U(d'). \tag{14}$$

To account for the above preference information, it is sufficient to include condition (14) in set E^{A^R} of constraints. Of course, consequently, condition (14) will be included in constraints $E(a,b)$ for all $a,b \in A$.

Conversely, for all $a,b,a',b' \in A$, it is possible to check whether or not condition

$$U(a) - U(b) > U(a') - U(b') \tag{15}$$

holds for all compatible value functions U.

In order to do so, it is sufficient to check the feasibility of constraints $E(a,b)$ and (15). Such information may enrich the DM's knowledge of his/her preferences.

6 Illustrative example

In this section, we illustrate how a decision aiding process can be supported by the UTA$^{\text{GMS}}$ method. We consider the following hypothetical decision problem. AGRITEC is a medium size firm (350 persons approx.) producing some tools for agriculture. The C.E.O., M$^{\text{r}}$ Becault, intends to double the production and multiply exports by 4 within 5 years. Therefore, he wants to hire a new international sales manager. A recruitment agency has interviewed 17 potential candidates which have been evaluated on 3 criteria (sales management experience, international experience, human qualities) evaluated on a [0,100] scale. The evaluations of candidates are provided in Table 5. Without any further information, the computed partial preorder $\succsim_0^N$ corresponds to the weak dominance relation Δ on the set of alternatives (See Figure 2).

The C.E.O. has attended 4 interviews and can express a confident judgement about theses candidates: Ferret and Frechet are equally good, Fourny is less acceptable than Ferret and Frechet, and Fleichman is even less acceptable than Fourny. This means that the initial reference ranking is the following: Ferret $\sim$ Frechet $\succ$ Fourny $\succ$ Fleichman. For this initial preference information, the partial preorder $\succsim_1^N$ has been computed using UTA$^{\text{GMS}}$ (See Figure 3).

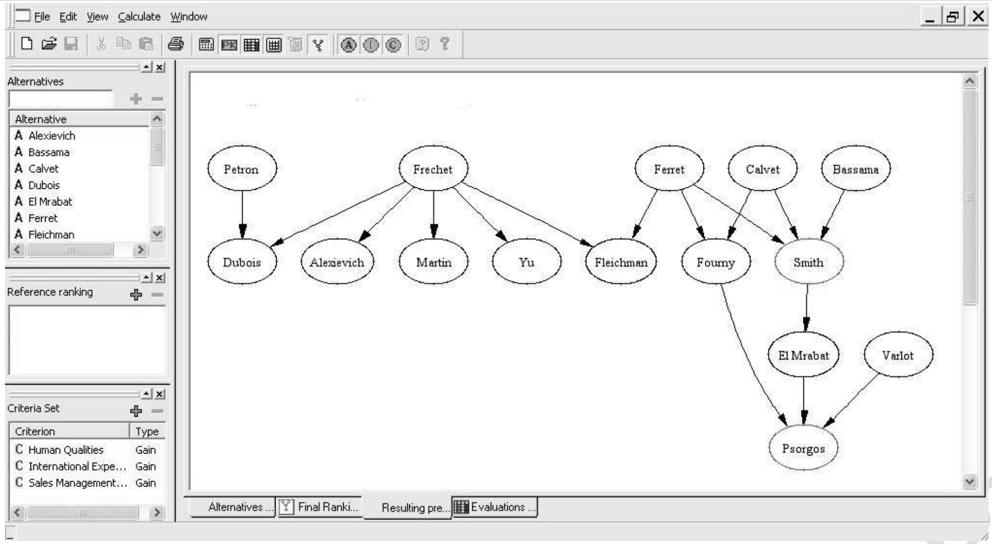

Figure 2: Partial preorder $\succsim_0^N$ corresponding to the weak dominance relation Δ

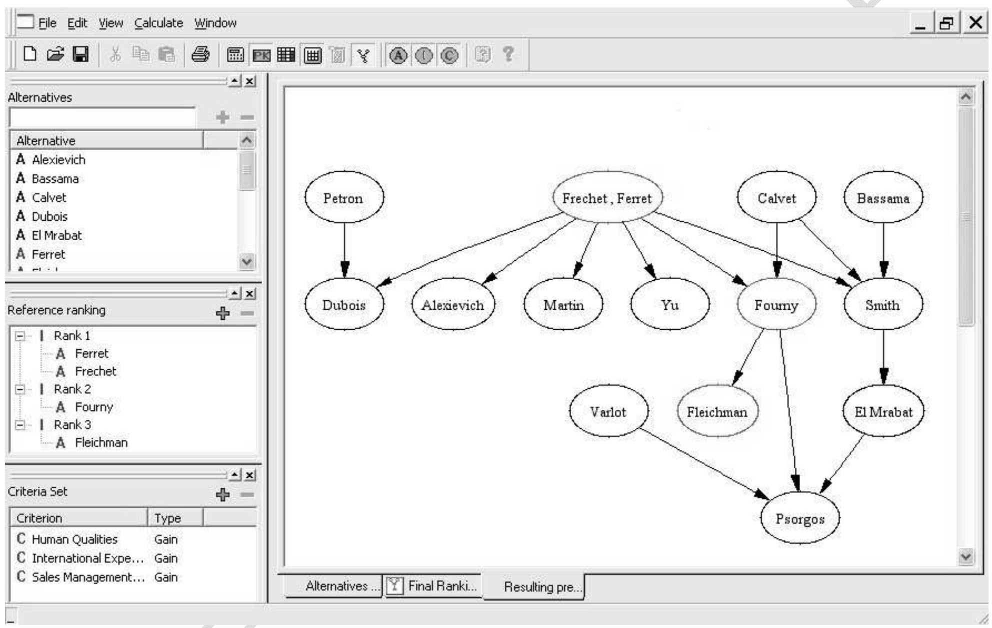

Figure 3: Partial preorder $\succsim_1^N$

Considering this first result, M^r Becault is willing to add further preference information. This results in the following new reference ranking: Ferret $\sim$ Frechet $\succ$ Martin $\succ$ Fourny $\sim$ El Mrabat $\succ$ Fleichman. However, as he did not attend the interview of El Mrabat and Martin, his opinion about the relative ranking of these candidates is not as certain as the initial preference information.

It appears that for the provided information, no additive value function fits the last reference ranking. The analysis of this incompatibility reveals that the statement Ferret $\sim$ Frechet cannot be represented together with the statement Fourny $\sim$ El Mrabat by an additive value function. In other words, it is necessary for M^r Becault to revise one of these statements. As he did not interview El Mrabat, he decides to remove him from the reference ranking which becomes Ferret $\sim$ Frechet $\succ$ Martin $\succ$ Fourny $\succ$ Fleichman. This reference ranking is compatible with a representation by an additive value function. Figure 4 represents two nested partial preorders:

	Crit 1	Crit 2	Crit 3
Alexievich	4	16	63
Bassama	28	18	28
Calvet	26	40	44
Dubois	2	2	68
El Mrabat	18	17	14
Feeret	35	62	25
Fleichman	7	55	12
Fourny	25	30	12
Frechet	9	62	88
Martin	0	24	73
Petron	6	15	100
Psorgos	16	9	0
Smith	26	17	17
Varlot	62	43	0
Yu	1	32	64

Table 5: Evaluation Table

- bold arrows represent partial preorder $\succsim_2^N$ obtained for the most certain preference information only, i.e., Ferret $\sim$ Frechet $\succ$ Fourny $\succ$ Fleichman,
- dashed arrows represent partial preorder $\succsim_3^N$ obtained for the consistent preference information composed of the most certain preference information and the less confident preference information about Martin, i.e., Ferret $\sim$ Frechet $\succ$ Martin $\succ$ Fourny $\succ$ Fleichman.

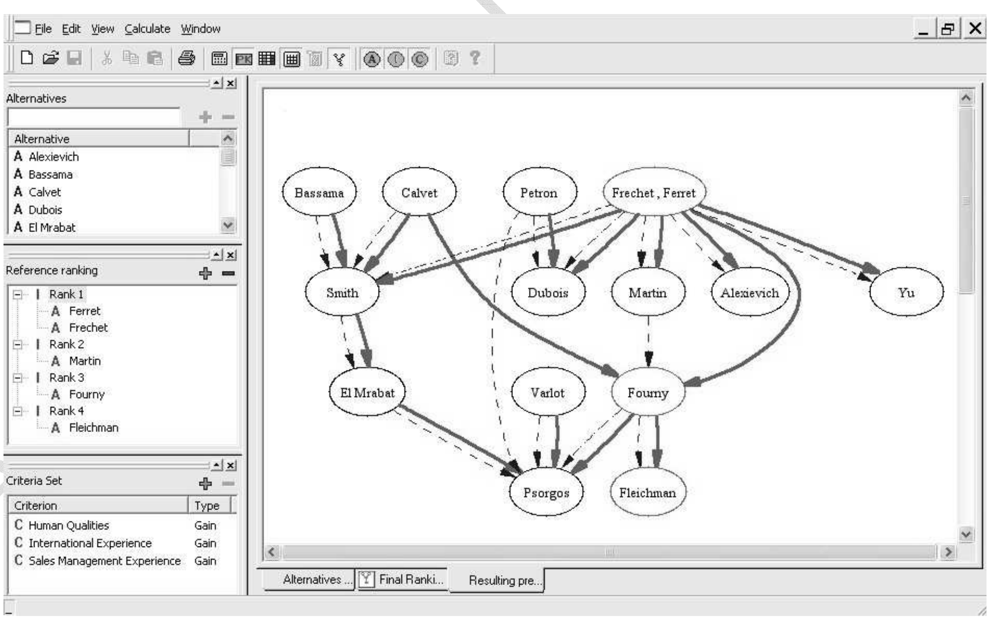

Figure 4: Nested partial preorders $\succsim_2^N$ (bold) and $\succsim_3^N$ (dashed)

The interactive process can be pursued, $\mathrm{M^r}$ Becault adding in iteration t some new pairwise comparisons of reference alternatives, thus enriching the resulting partial preorder $\succsim_t^N$, until it is decisive enough for the C.E.O. to make his choice.

7 Conclusion

The new UTA$^{\mathrm{GMS}}$ method presented in this paper is an ordinal regression method supporting multiple criteria ranking of alternatives; it is distinguished from previous methods of this kind by the following new features:

- the method considers general additive value functions rather than piecewise linear ones,
- the final rankings are defined using all value functions compatible with the provided preference information,
- the method provides two final rankings: the necessary ranking identifies "sure" preference statements while the possible ranking identifies "possible" preference statements,
- distinguishing necessary and possible consequences of using all value functions compatible with preference information, UTA$^{\mathrm{GMS}}$ includes a kind of robustness analysis instead of using a single "best-fit" value function,
- the necessary and possible preference relations considered in UTA$^{\mathrm{GMS}}$ have several properties of general interest for MCDA,
- when the DM provides preference information that cannot be represented by an additive model, the method identifies which pieces of the information underly this impossibility,
- the method does not require the DM to interpret (and even look at) the marginal value functions,
- the DM can assign confidence levels to pieces of preference information, which yields a valued necessary preference relation (proved to be a fuzzy partial preorder) and a valued possible preference relation (proved to be a strongly complete and negatively transitive valued binary relation).

We envisage the following future developments of the presented methodology:

- application to multicriteria sorting problems,
- application to group decision problems,
- application to interactive multiobjective optimization.

References

[1] C.A. Bana e Costa and J.C.. Vansnick. MACBETH - an interactive path towards the construction of cardinal value fonctions. *International transactions in operational Research*, 1:489–500, 1994.

[2] Barron, F.H. Selecting a best multiattribute alternative with partial information about attribute weights *Acta Psychologica*, 80:91–103, 1992.

[3] V. Belton and T. Stewart. *Multiple Criteria Decision Analysis: An integrated approach.* Kluwer Academic Publishers, Dordrecht, 2002.

[4] J.M. Devaud, G. Groussaud, and E. Jacquet-Lagreze. UTADIS: Une méthode de construction de fonctions d'utilité additives rendant compte de jugements globaux. In *Proceedings of the European working group on MCDA, Bochum , Germany*, 1980.

[5] L. C. Dias and J. N. Clímaco. Additive Aggregation with Variable Interdependent Parameters: the VIP Analysis Software. *Journal of the Operational Research Society*, 51:1070-1082, 2000.

[6] J. Fodor and M. Roubens. *Fuzzy preference modelling and multicriteria decision support.* Kluwer Academic Publishers, Dordrecht, 1994.

[7] S. Greco, B. Matarazzo, and R. Slowinski. The use of rough sets and fuzzy sets in MCDM. In T. Gal, T. Stewart, and T. Hanne, editors, *Advances in MCDM models, Algorithms, Theory, and Applications*, pages 14.1–14.59. Kluwer Academic Publishers, Dordrecht, 1999.

[8] S. Greco, B. Matarazzo, and R. Slowinski. Rough sets theory for multicriteria decision analysis. *European Journal of Operational Research*, 129:1–47, 2001.

[9] S. Greco., B. Matarazzo, and R. Slowinski. Decision rule approach. In J. Figueira, S. Greco, M. Ehrgott, editors, *Multiple Criteria Decision Analysis, State of the Art Surveys*, chapter 13, pages 507–561. Springer, New York, 2005.

[10] E. Jacquet-Lagrèze and Y. Siskos. Assessing a set of additive utility functions for multicriteria decision making: the UTA method. *European Journal of Operational Research*, 10:151–164, 1982.

[11] R.L. Keeney and H. Raiffa *Decisions with multiple objectives: Preferences and value tradeoffs.* J. Wiley, New York, 1976.

[12] C.W. Kirkwood and R.K. Sarin. Ranking with partial information: A method and an application. *Operations Research*, 33(1):38–48, January-February 1985.

[13] L. Kiss, J.M. Martel, and R. Nadeau. ELECCALC - an interactive software for modelling the decision maker's preferences. *Decision Support Systems*, 12(4-5):757–777, November 1994.

[14] M. Koksalan and V. Taner. An approach for finding the most preferred alternative in the presence of multiple criteria. *European Journal of Operational Research*, 60(1):53–60, 1992.

[15] M. Koksalan and C. Ulu. An interactive approach for placing alternatives in preference classes. *European Journal of Operational Research*, 144:429–439, 2003.

[16] D. Krantz, D. Luce, P Suppes and A Tversky. *Foundations of measurement: Vol. 1. Additive and polynomial representations* , New York, Academic Press, 1971.

[17] R. Lahdelma, J. Hokkanen and P. Salminen, SMAA - Stochastic multiobjective acceptability analysis, *European Journal of Operational Research* , 106:137-143, 1998.

[18] K.S. Lee, K.S. Park, and S.H. Kim. Dominance, potential optimality, imprecise information, and hierarchical structure in multi-criteria analysis. *Computers and Operations Research*, 29:1267–1281, 2002.

[19] J.G. March. Bounded rationality, ambiguity and the engineering of choice. *Bell Journal of Economics*, 9:587–608, 1978.

[20] R.S. Michalski, I. Bratko, and M. Kubat (*eds.*). *Machine learning and datamining - Methods and applications*. J. Wiley, New York, 1998.

[21] V. Mousseau, J. Figueira, L.C. Dias, C. Gomes da Silva, and J.N. Clímaco. Resolving inconsistencies among constraints on the parameters of an MCDA model. *European Journal of Operational Research*, 147(1):72–93, May 2003.

[22] V. Mousseau and R. Slowinski. Inferring an ELECTRE TRI model from assignment examples. *Journal of Global Optimization*, 12(2):157–174, 1998.

[23] D. Pekelman and S.K. Sen. Mathematical programming models for the determination of attribute weights. *Management Science*, 20(8):1217–1229, April 1974.

[24] B. Roy. *Multicriteria Methodology for Decision Aiding*. Kluwer Academic Publishers, Dordrecht, 1996.

[25] B. Roy and D. Bouyssou. *Aide Multicritère à la Décision: Méthodes et Cas*. Economica, Paris, 1993.

[26] T.L. Saaty. The Analytic Hierarchy and Analytic Network Processes for the Measurement of Intangible Criteria and for Decision-Making. in J. Figueira and S. Greco and M. Ehrgott (ed.) *IEEE Multiple Criteria Decision Analysis: State of the Art Surveys*, 345-408, Springer, Berlin, 2005.

[27] A.A. Salo and R.P. Hamalainen. Preference ratio in multiattribute evaluation (PRIME) - elicitation and decision procedures under incomplete information. *IEEE Transactions on Systems, Man and Cybertnetics: Part A*, 31(6):533–545, November 2001.

[28] Y. Siskos. A way to deal with fuzzy preferences in multicriteria decision problems. *European Journal of Operational Research*, 10(3):314–324, 1982.

[29] Y. Siskos, E. Grigoroudis, and N.F. Matsatsinis. UTA Methods. In J. Figueira, S. Greco, M. Ehrgott, editors, *State of the Art in Multiple Criteria Decision Analysis*. Springer, Berlin, 2005.

[30] V. Srinivasan and A.D. Shocker. Estimating the weights for multiple attributes in a composite criterion using pairwise judgments. *Psychometrika*, 38(4):473–493, 1973.

[31] P. Wakker. *Additive Representations of Preferences, A New Foundation of Decision Analysis*. Kluwer Academic Publishers, Dordrecht, 1989.

[32] M. Weber. A method of multiattribute decision making with incomplete information. *Management Science*, 31(11):1365–1371, 1985.