Orientation of Graphs and Connectivity

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Notions of Connectivity in Digraphs

$k$-arc-connectivity

$\exists k$-arc-disjoint directed paths from any vertex to any other vertex

$\text{Menger } \iff \text{ the size of every arc-cut is at least } k$

$d_{in}(D(X)) \geq k$

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k-arc-connectivity

∃ k-arc-disjoint directed paths from any vertex to any other vertex

Menger ⇔ the size of every arc-cut is at least k

\[ d^\text{in}_D(X) \geq k \]
Notions of Connectivity in Digraphs

**$k$-arc-connectivity**

\[ \exists k \text{-arc-disjoint directed paths from any vertex to any other vertex} \]

the size of every arc-cut is at least $k$

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**$k$-vertex-connectivity**

\[ \exists k \text{-vertex-disjoint directed paths from any vertex to any other vertex} \]

the size of every vertex-cut is at least $k$
Notions of Connectivity in Graphs

k-edge-connectivity

∃k-edge-disjoint paths joining any pair of vertices

Menger ⇔ every edge-cut is of size at least k

dG(X) ≥ k

k-vertex-connectivity

∃k-vertex-disjoint paths joining any pairs of vertices

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**Notions of Connectivity in Graphs**

### $k$-edge-connectivity

$\exists \ k$-edge-disjoint paths joining any pair of vertices

Menger $\iff$ every edge-cut is of size at least $k$

$$d_G(X) \geq k$$

### $k$-vertex-connectivity

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\begin{align*}
& \text{Menger} \\
& \iff \\
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\end{align*}
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& \text{with } X
\end{align*}
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Outline

On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
Outline

On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
$k$-Arc-Connected Orientation

$G$ has a $k$-arc-connected orientation $D$.

Theorem [Nash-Williams 1960]

$G$ has a $k$-arc-connected orientation $D$

$G$ is $2k$-edge-connected

Proved for $k = 1$ by Robbins (1939)
**k-Arc-Connected Orientation**

G has a \( k \)-arc-connected orientation \( D \)

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$k$-Arc-Connected Orientation

$G$ has a $k$-arc-connected orientation $D$ if

$$d^{in}_{D}(X) \geq k \text{ for all } \emptyset \neq X \subset V$$
$k$-Arc-Connected Orientation

G has a $k$-arc-connected orientation $D$ if and only if

\[ d^\text{in}_D(X) \geq k \text{ for all } \emptyset \neq X \subset V \]

\[ d_G(X) = d^\text{in}_D(X) + d^\text{out}_D(X) \geq 2k \text{ for all } \emptyset \neq X \subset V \]

$G$ is $2k$-edge-connected

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**Theorem [Nash-Williams 1960]**

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\[ G \text{ is } 2k\text{-edge-connected} \]

Proved for \( k = 1 \) by Robbins (1939)
The Eulerian case

Eulerian graphs and digraphs

G is Eulerian: \( d_G(v) \) is even \( \forall v \)
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[Diagram of an Eulerian graph with arrows indicating direction and a red arrow highlighting an Eulerian path.]
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\[ \text{Theorem} \]
\( G \) has an Eulerian orientation \( \leftrightarrow \) \( G \) is Eulerian
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$G$ is Eulerian: $d_G(v)$ is even $\forall v$

[Diagram of a graph with Eulerian paths highlighted]

Theorem

$G$ has an Eulerian orientation $\leftrightarrow G$ is Eulerian
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- $G$ is Eulerian: $d_G(v)$ is even $\forall v$
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Theorem

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- $G$ is Eulerian: $d_G(v)$ is even $\forall v$
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Theorem

$G$ has an Eulerian orientation $\iff G$ is Eulerian

\[
\sum_{v \in X} d_D^{in}(v) = d_D^{in}(X) + \left| \{ \text{arcs } uv \text{ such that } u, v \in X \} \right|
\]
\[
\sum_{v \in X} d_D^{out}(v) = d_D^{out}(X) + \left| \{ \text{arcs } uv \text{ such that } u, v \in X \} \right|
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**Eulerian graphs and digraphs**

- **G** is Eulerian: $d_G(v)$ is even $\forall v$
- **D** is Eulerian: $d_{D}^{in}(v) = d_{D}^{out}(v)$ $\forall v$

**Theorem**

- **G** has an Eulerian orientation $\iff$
- **G** is Eulerian

\[
\begin{align*}
  d_{D}^{in}(X) &= d_{D}^{out}(X) \\
  d_{G}(X) &= d_{D}^{in}(X) + d_{D}^{out}(X)
\end{align*}
\]

if **D** is Eulerian

**D** is an orientation of **G**
The Eulerian case

Eulerian graphs and digraphs

*G* is Eulerian: \( d_G(v) \) is even \( \forall v \)

*D* is Eulerian: \( d^\text{in}_D(v) = d^\text{out}_D(v) \forall v \)

Theorem

*G* has an Eulerian orientation

\[ d^\text{in}_D(X) = \frac{1}{2} d_G(X) \]

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Theorem

Any Eulerian orientation of an Eulerian 2\( k \)-connected graph is \( k \)-arc-connected.
Lovász’ proof: Splitting-off

Theorem [Lick 1972]
Every minimally $2k$-edge-connected graph has a vertex of degree $2k$.

Splitting-off (su, sv):
replace su and sv by the edge uv

Splitting-off at s:
split-off pair of edges adjacent to s as long as it is possible

Theorem [Lovász 1979]
If $G$ is $2k$-edge-connected and $d_G(s)$ is even then there exists a complete splitting-off at $s$ that results in a $2k$-edge-connected graph on $V \setminus s$. 

Proof of Nash-Williams’ theorem by induction on $|V| + |E|$.
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Proof of Nash-Williams’ theorem by induction on \(|V| + |E|\)
Covering Crossing Supermodular Functions

An orientation $D$ covers a set-function $p$ if

$$d^\text{in}_D(X) \geq p(X), \forall X$$
Covering Crossing Supermodular Functions

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We are interested in covering

$$h(X) = \begin{cases} 
0 & \text{if } X = \emptyset \text{ or } V \\
k & \text{otherwise} 
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Crossing Supermodular Functions

A set function $p : 2^V \mapsto \mathbb{R}$ is called crossing supermodular if

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$$

holds for all crossing $X, Y \subseteq V$ (ie: none of $X \cap Y$, $X \setminus Y$, $Y \setminus X$, $V \setminus (X \cup Y)$ is empty).
Theorem [Frank 1980]

Let $G$ be a graph and $p$ be a non-negative, integer-valued crossing supermodular set function on $V$ such that $p(V) = p(\emptyset) = 0$. Then there exists an orientation covering $p$ iff

$$e_G(P) \geq \max \left\{ \sum_{X \in P} p(X), \sum_{X \in P} p(V \setminus X) \right\}$$

holds for every partition $P$ of $V$. If $p$ is symmetric then the condition reduces to

$$d_G(X) \geq 2p(X), \forall X$$
Theorem [Nash-Williams 1960]

Every graph has an orientation that preserves at least half (rounded down) of the edge connectivity between any two vertices.
Well-Balanced Orientation

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1. Add to $G$ an odd pairing $M$
2. Take an Eulerian orientation of $G + M$
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3. Remove $M$

Diagram: A graph with arrows indicating orientations.
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1. Add to $G$ an odd pairing $M$
2. Take an Eulerian orientation of $G + M$
3. Remove $M$
4. If $M$ is “good” then the orientation of $G$ is well-balanced (regardless of the Eulerian orientation given at step 2)
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Theorem [Nash-Williams 1960]

Every graph has a “good” odd pairing
Outline

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An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
The Plumbing Problem

Corollary [Frank 1978]

A graph with roots has an orientation satisfying (1) iff it satisfies (2)
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Theorem [Edmonds 1973]

There exists a \( k \) arc-disjoint spanning arborescences iff

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There exists a $k$ arc-disjoint spanning arborescences iff

$$d^{in}_{D}(X) \geq k - |\{\text{roots in } X\}| \quad (1)$$

If there exists $k$ edge-disjoint spanning trees then

$$\sum_{X \in \mathcal{P}} d^{in}_{D}(X) \geq k(|\mathcal{P}| - 1) \quad (2)$$

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Corollary [Frank 1978]

A graph with roots has an orientation satisfying (1) iff it satisfies (2)
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The Plumbing Problem with Matroids

- $G = (V, E)$ is a graph
- $\mathcal{M}$ is a matroid on ground set $S$
- $\pi$ is a placement of $S$ on $V$

$\mathcal{M}$ = uniform matroid of rank 2 on $S = \{s_1, s_2, s_3\}$

Diagram of a graph with nodes $s_1$, $s_2$, $s_3$ and edges connecting them.
The Plumbing Problem with Matroids

- $G = (V, E)$ is a graph
- $\mathcal{M}$ is a matroid on ground set $S$
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- $(T, s)$ is a rooted tree
  - $T$ is a tree
  - $\pi(s) \in V(T)$

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The Plumbing Problem with Matroids

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- $\{(T_1, s_1), \ldots, (T_{|S|}, s_{|S|})\}$ is a matroid-based packing of rooted trees (MBPRT)
  - the $T_i$'s are edge-disjoint
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The Plumbing Problem with Matroids

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- \( \pi \) is *independent* if \( \pi^{-1}(v) \) is independent in \( \mathcal{M} \), for all \( v \in V \)

\[ \mathcal{M} = \text{uniform matroid of rank 2 on } S = \{ s_1, s_2, s_3 \} \]
The Plumbing Problem with Matroids

- \( D = (V, A) \) is a digraph
- \( M \) is a matroid on ground set \( S \)
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- \( (T, s) \) is a \textit{rooted} arborescence
  - \( T \) is an arborescence
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The Plumbing Problem with Matroids

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The Plumbing Problem with Matroids

If there exists a MBPRA then

\[ d_D^{in}(X) \geq r_M(S) - r_M(\pi^{-1}(X)) \]

Corollary of [Frank 1980]

A MBR graph has an orientation satisfying (3) iff it satisfies (4)

\[ M = \text{uniform matroid of rank 2 on } S = \{s_1, s_2, s_3\} \]
The Plumbing Problem with Matroids

**Theorem [DdG, Nguyen, Szigeti 2013]**

There exists a MBPRA iff $\pi$ is independent and

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\[ S_1 \quad S_2 \quad S_3 \]
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Theorem [Katoh, Tanigawa 2013]
There exists a MBPRT iff $\pi$ is independent and

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The Plumbing Problem with Matroids

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An extension [C. Király 2013] replaces:

- \( \{ s_i \in S : v \in V(T_i) \} \) is a base of \( M \) for each \( v \in V \) by
- \( \{ s_i \in S : v \in V(T_i) \} \) is independent and “maximal”

A generalization to “covering intersecting bi-sets families” exists [Bérczi, T. Király and Kobayashi 2013].
Outline

On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
A conjecture of Thomassen

**Conjecture [Thomassen 1989]**

For every $k$, there exists a least integer $f(k)$ such that

\[ G \text{ is } f(k)\text{-vertex-connected} \]

\[ \Downarrow \]

\[ G \text{ has a } k\text{-vertex-connected orientation} \]
A conjecture of Thomassen

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- If $f(k)$ exists then $f(k) \geq 2k$
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- $f(2) \leq 18$ [Jordán 2006]
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A conjecture of Frank

G has a $k$-vertex-connected orientation $D$
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Conjecture [Frank 1995] $G$ has a $k$-vertex-connected orientation $D$

$G$ has a $k$-vertex-connected orientation $D$

Proved for $k = 2$ in the Eulerian case [Berg, Jordán 2006]

Disproved for $k \geq 3$ (even in the Eulerian case) [DdG 2013]

Proved for $k = 2$ [Thomassen 2014]
A conjecture of Frank

\[ G \text{ has a } k\text{-vertex-connected orientation } D \]

\[ D - U \text{ is } (k - |U|)\text{-vertex-connected } \forall U \]

\[ D - U \text{ is } (k - |U|)\text{-arc-connected } \forall U \]
A conjecture of Frank

Conjecture [Frank 1995]

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\[ G - U \text{ is } 2(k - |U|)\text{-edge-connected } \forall U \]
\[ \Updownarrow \]
\[ G \text{ is weakly } 2k\text{-connected} \]
A conjecture of Frank

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\( G - U \) is \(2(k - |U|)\)-edge-connected \( \forall U \)

\( G \) is weakly \( 2k \)-connected

\( G \) is weakly 4-connected:

- \( G \) is 4-edge-connected
- \( G - v \) is 2-edge-connected \( \forall v \)

Proved for \( k = 2 \) in the Eulerian case [Berg, Jordán 2006]

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\[ G \text{ is weakly } 4\text{-connected:} \]
\[ \quad \Rightarrow G \text{ is } 4\text{-edge-connected} \]
\[ \quad \Rightarrow G - v \text{ is } 2\text{-edge-connected } \forall v \]

\[ G \text{ is weakly } 6\text{-connected:} \]
\[ \quad \Rightarrow G \text{ is } 6\text{-edge-connected} \]
\[ \quad \Rightarrow G - v \text{ is } 4\text{-edge-connected } \forall v \]
\[ \quad \Rightarrow G - \{u, v\} \text{ is } 2\text{-edge-connected } \forall u, v \]
A conjecture of Frank

**Conjecture [Frank 1995]**

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\[ G - U \text{ is } 2(k - |U|)\text{-edge-connected } \forall U \]
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Proved for \( k = 2 \) in the Eulerian case [Berg, Jordán 2006]
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A counterexample for $k = 3$
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NP-Completeness

**Theorem [DdG 2013]**

For every $k \geq 3$, the problem of deciding whether a graph has a $k$-vertex-connected orientation is NP-complete.
Graph orientation with connectivity constraints
- is of interest for its (theoretical) applications
- remains challenging (Conjecture of Thomassen for $k \geq 3$)
Thank you for your attention