Orientation of Graphs and Connectivity

Olivier Durand de Gevigney
joint works with: Joseph Cheriyan, Nguyen Viet-Hang and Zoltán Szigeti

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Notions of Connectivity in Digraphs

∃ \( k \)-arc-disjoint directed paths from any vertex to any other vertex

Menger ⇔ the size of every arc-cut is at least \( k \)

∃ \( k \)-vertex-disjoint directed paths from any vertex to any other vertex

Menger ⇔ the size of every vertex-cut is at least \( k \)
**Notions of Connectivity in Digraphs**

**k-arc-connectivity**

\[ \exists k \text{-arc-disjoint directed paths from any vertex to any other vertex} \]

\[ \Longleftrightarrow \text{the size of every arc-cut is at least } k \]

\[ d^\text{in}_D(X) \geq k \]
Notions of Connectivity in Digraphs

**$k$-arc-connectivity**

$\exists$ $k$-arc-disjoint directed paths from any vertex to any other vertex

Menger $\iff$ the size of every arc-cut is at least $k$

$\quad d^{in}_D(X) \geq k$

**$k$-vertex-connectivity**

$\exists$ $k$-vertex-disjoint directed paths from any vertex to any other vertex

Menger $\iff$ the size of every vertex-cut is at least $k$
Notions of Connectivity in Graphs
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**$k$-edge-connectivity**

$\exists$ $k$-edge-disjoint paths joining any pair of vertices

Menger $\iff$ every edge-cut is of size at least $k$

$\quad d_G(X) \geq k$

**$k$-vertex-connectivity**

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Outline

On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
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On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
$k$-Arc-Connected Orientation

$G$ has a $k$-arc-connected orientation $D$

Theorem [Nash-Williams 1960]

$G$ has a $k$-arc-connected orientation $D$

$G$ is $2k$-edge-connected

Proved for $k = 1$ by Robbins (1939)
$k$-Arc-Connected Orientation

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$k$-Arc-Connected Orientation

Let $G$ be a graph and $D$ be an orientation of $G$. $G$ has a $k$-arc-connected orientation $D$ if:

$$d^\text{in}_D(X) \geq k \text{ for all } \emptyset \neq X \subset V$$

Theorem [Nash-Williams 1960]

$G$ has a $k$-arc-connected orientation $D$ if and only if $G$ is $2k$-edge-connected. Proved for $k = 1$ by Robbins (1939).
A graph $G$ has a $k$-arc-connected orientation $D$ if $d^\text{in}_D(X) \geq k$ for all $\emptyset \neq X \subset V$.

Theorem [Nash-Williams 1960]

Proved for $k = 1$ by Robbins (1939)

The graph $G$ is $2k$-edge-connected if $d_G(X) = d^\text{in}_D(X) + d^\text{out}_D(X) \geq 2k$ for all $\emptyset \neq X \subset V$. 

$G$ is $2k$-edge-connected.

Diagram: 

- $X$ is a subset of $V$.
- Arrows denote directed edges with direction indicated by arrowhead.
- Red and blue arrows represent different sets of arcs.
- $d_G(X)$ calculates the sum of in-degree and out-degree for $X$. 

Note: The page number 7/24.
**$k$-Arc-Connected Orientation**

A graph $G$ has a $k$-arc-connected orientation $D$ if the following conditions hold:

- $d_D^\text{in}(X) \geq k$ for all $\emptyset \neq X \subset V$
- $d_G(X) = d_D^\text{in}(X) + d_D^\text{out}(X) \geq 2k$ for all $\emptyset \neq X \subset V$

This implies that $G$ is $2k$-edge-connected.

**Theorem [Nash-Williams 1960]**

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The Eulerian case

Eulerian graphs and digraphs

\[ G \text{ is Eulerian: } d_G(v) \text{ is even } \forall v \]
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Eulerian graphs and digraphs

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\[ \text{Theorem} \quad G \text{ has an Eulerian orientation} \]

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- Theorem: **G** has an Eulerian orientation $\leftrightarrow$ **G** is Eulerian
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- **$G$ is Eulerian**: $d_G(v)$ is even $\forall v$
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- **G** is Eulerian: \( d_G(v) \) is even \( \forall v \)
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Theorem

- **G** has an Eulerian orientation \( \iff \)
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Theorem

- **G has an Eulerian orientation**
  \[ \iff \]
  **G is Eulerian**

\[
\sum_{v \in X} d_D^{in}(v) = d_D^{in}(X) + |\{\text{arcs } uv \text{ such that } u, v \in X\}|
\]
\[
\sum_{v \in X} d_D^{out}(v) = d_D^{out}(X) + |\{\text{arcs } uv \text{ such that } u, v \in X\}|
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Theorem

- $G$ has an Eulerian orientation $\iff$ $G$ is Eulerian

$d_D^{in}(X) = d_D^{out}(X)$ if $D$ is Eulerian

$d_G(X) = d_D^{in}(X) + d_D^{out}(X)$ $D$ is an orientation of $G$
The Eulerian case

**Eulerian graphs and digraphs**

- **G** is Eulerian: $d_G(v)$ is even $\forall v$
- **D** is Eulerian: $d_D^{in}(v) = d_D^{out}(v)$ $\forall v$

**Theorem**

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<table>
<thead>
<tr>
<th>$d_D^{in}(X)$</th>
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$D$ is Eulerian:
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d_D^{in}(X) = \frac{1}{2}d_G(X)
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Theorem

Any Eulerian orientation of an Eulerian $2k$-connected graph is $k$-arc-connected.
Lovász’ proof: Splitting-off

Theorem [Lick 1972]
Every minimally $2k$-edge-connected graph has a vertex of degree $2k$.

Splitting-off

**(Splitting-off)**

**replace** $su$ and $sv$ by the edge $uv$

Theorem [Lovász 1979]
If $G$ is $2k$-edge-connected and $d_G(s)$ is even then there exists a complete splitting-off at $s$ that results in a $2k$-edge-connected graph on $V \setminus s$.

Proof of Nash-Williams’ theorem by induction on $|V| + |E|$.
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Covering Crossing Supermodular Functions

An orientation $D$ covers a set-function $p$ if

$$d_D^\text{in}(X) \geq p(X), \ \forall X$$
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We are interested in covering

$$h(X) = \begin{cases} 
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 k & \text{otherwise}
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Crossing Supermodular Functions

A set function $p : 2^V \mapsto \mathbb{R}$ is called crossing supermodular if

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$$

holds for all crossing $X, Y \subseteq V$ (ie: none of $X \cap Y, X \setminus Y, Y \setminus X, V \setminus (X \cup Y)$ is empty).
Frank’s proof

Theorem [Frank 1980]

Let $G$ be a graph and $p$ be a non-negative, integer-valued crossing supermodular set function on $V$ such that $p(V) = p(\emptyset) = 0$. Then there exists an orientation covering $p$ iff

$$e_G(\mathcal{P}) \geq \max \left\{ \sum_{X \in \mathcal{P}} p(X), \sum_{X \in \mathcal{P}} p(V \setminus X) \right\}$$

holds for every partition $\mathcal{P}$ of $V$. If $p$ is symmetric then the condition reduces to

$$d_G(X) \geq 2p(X), \forall X$$
Well-Balanced Orientation

**Theorem [Nash-Williams 1960]**

Every graph has an orientation that preserves at least half (rounded down) of the edge connectivity between any two vertices.

---

[Diagram of a graph with nodes and edges]
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Every graph has a "good" odd pairing
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![Graph with odd pairing](image)
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On Vertex-Connected Orientations
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Corollary [Frank 1978]
A graph with roots has an orientation satisfying (1) iff it satisfies (2)
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The Plumbing Problem with Matroids

- $G = (V, E)$ is a graph
- $\mathcal{M}$ is a matroid on ground set $S$
- $\pi$ is a placement of $S$ on $V$

$\mathcal{M} = \text{uniform matroid of rank 2 on } S = \{s_1, s_2, s_3\}$
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- \( M \) is a matroid on ground set \( S \)
- \( \pi \) is a placement of \( S \) on \( V \)
- \((T, s)\) is a rooted arborescence
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If there exists a MBPRA then

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Theorem [DdG, Nguyen, Szigeti 2013]

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A MBR graph has an orientation satisfying (3) iff it satisfies (4)

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An extension [C. Király 2013] replaces:
- \( \{ s_i \in S : v \in V(T_i) \} \) is a base of \( \mathcal{M} \) for each \( v \in V \) by
- \( \{ s_i \in S : v \in V(T_i) \} \) is independent and “maximal”

A generalization to “covering intersecting bi-sets families” exists [Bérczi, T. Király and Kobayashi 2013].
Outline

On Arc-Connected Orientations

An Application of Orientation: Packing Trees

On Vertex-Connected Orientations
A conjecture of Thomassen

Conjecture [Thomassen 1989]
For every $k$, there exists a least integer $f(k)$ such that

\[
G \text{ is } f(k)\text{-vertex-connected} \Downarrow \quad G \text{ has a } k\text{-vertex-connected orientation}
\]

If $f(k)$ exists then $f(k) \geq 2$

- $f(1) = 2$ [Robbins 1939]
- $f(2) \leq 18$ [Jordán 2006]
- $f(2) \leq 14$ [Cheriyan, DdG, Szigeti 2012]
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A conjecture of Frank

G has a \( k \)-vertex-connected orientation \( D \)

\[ \forall U \subseteq G \exists D - U \text{ is } (k-|U|) \text{-vertex-connected} \]

\[ \forall U \subseteq G \exists D - U \text{ is } 2(k-|U|) \text{-edge-connected} \]

\[ \text{Conjecture [Frank 1995]} \]

\[ \text{Proved for } k = 2 \text{ in the Eulerian case [Berg, Jordán 2006]} \]

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A conjecture of Frank

$G$ has a $k$-vertex-connected orientation $D$

$D - U$ is $(k - |U|)$-vertex-connected $\forall U$

$D - U$ is $(k - |U|)$-arc-connected $\forall U$

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\[ \forall U \]

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$G$ is weakly 4-connected:
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- $G$ is 4-edge-connected
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$G$ is weakly 6-connected:
- $G$ is 6-edge-connected
- $G - v$ is 4-edge-connected $\forall v$
- $G - \{u, v\}$ is 2-edge-connected $\forall u, v$
A conjecture of Frank

**Conjecture [Frank 1995]**

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\downarrow
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Theorem [DdG 2013]

For every $k \geq 3$, the problem of deciding whether a graph has a $k$-vertex-connected orientation is NP-complete.
Conclusion

Graph orientation with connectivity constraints

- is of interest for its (theoretical) applications
- remains challenging (Conjecture of Thomassen for $k \geq 3$)
Thank you for your attention