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Abstract

We study completeness in differential approximability classes. In differential approximation, the quality of an approximation algorithm is the measure of how far is the solution computed from a worst one and how close is it to an optimal one. The main classes considered are DAM, the differential counterpart of APX, including the NP optimization problems approximable in polynomial time within constant differential approximation ratio and the DGLO, the differential counterpart of GLO, including problems for which their local optima guarantee constant differential approximation ratio. We define natural approximation preserving reductions and prove completeness results for the class of the NP optimization problems (class NPO), as well as for DAM and for a natural subclass of DGLO. We also define class 0-APX of the NPO problems that are not differentially approximable within any ratio strictly greater than 0 unless P = NP. This class is very natural for differential approximation, although has no sense for the standard one. Finally, we prove the existence of hard problems for a subclass of DPTAS, the differential counterpart of PTAS, the class of NPO problems solvable by polynomial time differential approximation schema.

1 Introduction

An optimization problem II is in NPO if the decision version of II is in NP. Given an instance \(x\) of II and a feasible solution \(y\) for \(x\), we denote by \(\text{opt}(x)\) the value of an optimal solution of \(x\) and by \(\omega(x)\) the value of a worst solution of \(x\). In what follows, the worst-value solution \(\omega(x)\) of an instance \(x\) of an NPO problem II, will be defined as the optimum solution for \(x\) when seen as instance of the NPO problem II', where II and II' have the same set of instances and the same solution-set for any instance, and the goal of II is \(\min\) (resp., \(\max\)), if the goal of II is \(\max\) (resp., \(\min\)). Worst solutions are not always easy to compute. For instance, for the minimization version of travelling salesman problem, the worst solution is a Hamiltonian cycle of maximum total distance, i.e., the optimum solution of maximum travelling salesman problem. The computation of such a solution is not trivial since the latter problem is NP-hard. On the contrary, examples of problems for which a worst solution is easily computed are maximum independent set where the worst solution is the empty set, minimum vertex cover, or minimum graph-coloring, where the worst solution is the entire vertex-set of the input-graph.

Polynomial approximation deals with polynomial computation of “good”, with respect to a predefined criterion, feasible solutions for hard NPO problems. Two main such criteria have
been used until now: the standard approximation ratio measuring the “nearness” of a solution to an optimal one, and the differential approximation ratio measuring how a solution is ranged in the interval between a worst solution and an optimal one. More formally, for an approximation algorithm \( A \) computing a feasible solution \( y \) for \( x \) with value \( m_A(x, y) \), its standard approximation ratio is defined as \( \gamma_A(x, y) = m_A(x, y)/\text{opt}(x) \) and its differential one as \( \delta_A(x, y) = |\omega(x) - m_A(x, y)|/|\omega(x) - \text{opt}(x)| \). In what follows, whenever it is understood, reference to problem II will be dropped. Finally note that, for any problem II and for any algorithm \( \Lambda \), \( 0 \leq \delta_\Lambda \leq 1 \).

An approximation measure \( \mu \) is called cost-respecting ([22]) if given two solutions \( y_1 \) and \( y_2 \) for an instance \( x \) of an optimization problem II, the fact that \( y_1 \) is worse than \( y_2 \) implies that \( \mu(y_1) \) is worse than \( \mu(y_2) \). Obviously, both standard and differential approximation ratios are cost-respecting measures.

Regarding the type of approximation results, NPO problems can be classified with respect to the approximation ratio known for them. So, for example, APX (DAPX) is the class of NPO problems polynomially approximable within constant (differential) ratio; PTAS (DPTAS) is the class of problems polynomially approximable by standard (differential) polynomial time approximation schemata, i.e., within standard (differential) ratios arbitrarily close to 1; in other words, the ratios achieved by these schemata are of the form \( 1 - \epsilon \) (for maximization problems in standard approximation and for any problem in differential one) or \( 1 + \epsilon \) (for minimization problems in standard approximation) for any \( \epsilon > 0 \); finally, FPTAS (DFPTAS) is the class of problems approximable by standard (differential) fully polynomial time approximation schemata, i.e., within ratios arbitrarily close to 1 in time polynomial in both the size of their instances and in \( 1/\epsilon \). Other standard approximability classes have also been defined in the literature as, for example, LOG-APX (problems polynomially approximable within logarithmic approximation ratio), or POLY-APX (here the ratio achieved is a polynomial of the size of the instance), etc., (see, for instance, [2, 18] for more details about such standard approximation classes).

In this paper we study completeness for NPO, under the differential approximation ratio, for DAPX, for a subclass of DPTAS, the one of maximization problems the worst solution of which is computable in polynomial time, as well as for a subclass of DAPX, the class of problems whose local optima ensure a guaranteed differential approximation ratio with respect to their global optima.

2 Preliminaries

Formally, an NP optimization problem II is defined as a four-tuple \((I, \text{sol}, m, \text{opt})\) such that: \( I \) is the set of instances of II and it can be recognized in polynomial time; given \( x \in I, \text{sol}(x) \) denotes the set of feasible solutions of \( x \); for every \( y \in \text{sol}(x) \), \(|y|\) is polynomial in \(|x|\); given any \( x \) and any \( y \) polynomial in \(|x|\), one can decide in polynomial time if \( y \in \text{sol}(x) \); given \( x \in I \) and \( y \in \text{sol}(x) \), \( m(x, y) \) denotes the value of \( y \) for \( x \); \( m \) is polynomially computable and is commonly called feasible value, or objective value; finally, \( \text{opt} \in \{\max, \min\} \). The set of NP optimization problems forms the class NPO. An NPO problem II is said to be polynomially bounded, if, for any instance \( x \) of II, the value of the optimum solution of \( x \) is bounded by a polynomial in \(|x|\). The set of polynomially bounded problems of NPO forms the class NPO-PB. Following the previous notations, the worst-value solution of an instance \( x \) of an NPO problem \( II = (I, \text{sol}, m, \text{opt}) \), \( \omega(x) \) is the optimum solution for \( x \) with respect to the NPO problem \( II' = (I, \text{sol}, m, \text{opt}') \) where \( \text{opt}' = \max \), if \( \text{opt} = \min \) and \( \text{opt}' = \min \), if \( \text{opt} = \max \).

Since the beginning of the 80’s, researchers have been highly interested in providing a structure in standard approximation by defining suitable approximation preserving reductions in order to study completeness in approximability classes. Note that completeness for an approximability class can be seen as a way to provide lower bounds in the approximability of its members, in
the sense that such result means that no problem of this class can have better approximability behavior unless \( P = \text{NP} \). Pioneering works in this direction, used in this paper, are, among others, the ones in [12, 22, 23]. In [22] several natural minimization problems have been shown to be \text{NPO-complete} under an approximation preserving reduction called strict-reduction, dealing with any cost-respecting approximation measure \( r \).

**Definition 1.** Consider two \text{NPO} problems \( \Pi = (I, \text{sol}, m, \text{opt}) \) and \( \Pi' = (I', \text{sol}', m', \text{opt}) \). A strict-reduction is a pair \((f, g)\) of polynomially computable functions, \( f : I \rightarrow I' \) and \( g : I \times \text{sol}' \rightarrow \text{sol} \) such that:

- \( \forall x \in I, x \mapsto f(x) \in I' \);

- \( \forall y \in \text{sol}'(f(x)), y \mapsto g(x, y) \in \text{sol}(x) \);

- if \( r \) is an approximation measure, then \( r_{\Pi}(x, g(x, y)) \) is as good as \( r_{\Pi'}(f(x), y) \).

Throughout the paper, for any reduction \( R \), we will denote by \( \Pi \leq_R \Pi' \) the fact that \( \Pi \) \( R \)-reduces to \( \Pi' \).

In [23], the subclass \text{MAX-SNP} of \text{APX} has been introduced and complete problems have been provided for it, under \text{L-reduction}. In [12], a polynomial time approximation schema preserving reduction, called \text{P-reduction} there, has been introduced and the existence of \text{APX-complete} problems has been shown. In what follows, we borrow the term \text{PTAS} from [3, 13] and we will use it instead of \( \text{P} \).

**Definition 2.** Consider two \text{NPO} maximization problems \( \Pi \) and \( \Pi' \). Then, \( \Pi \leq_{\text{PTAS}} \Pi' \) if there exist three functions \( f, g \) and \( c \), computable in polynomial time, such that:

- \( \forall x \in I_\Pi, \forall \epsilon \in ]0, 1[ \cap \mathbb{Q}, f(x, \epsilon) \in I_{\Pi'} \);

- \( \forall x \in I_\Pi, \forall \epsilon \in ]0, 1[ \cap \mathbb{Q}, \forall y \in \text{sol}_{\Pi'}(f(x, \epsilon)), g(x, y, \epsilon) \in \text{sol}_{\Pi}(x) \);

- \( c : ]0, 1[ \cap \mathbb{Q} \rightarrow ]0, 1[ \cap \mathbb{Q} \);

- \( \forall x \in I_\Pi, \forall \epsilon \in ]0, 1[ \cap \mathbb{Q}, \forall y \in \text{sol}_{\Pi'}(f(x, \epsilon)), \gamma_{\Pi'}(f(x, \epsilon), y) \geq 1 - c(\epsilon) \Rightarrow \gamma_{\Pi}(x, g(x, y, \epsilon)) \geq 1 - \epsilon \).

Furthermore, another reduction called \text{F} has been defined in [12] by means of which \text{PTAS-complete} problems have been provided.

Surprisingly enough, differential approximation, although introduced in [4] since 1977, has not been systematically used until the 90’s ([5, 1, 6, 25] are, to our knowledge, the most notable uses of it) when a formal framework for it and a more systematic use started to be drawn ([15, 16]). In any case, no structural approach to the study of differential approximability has been developed until now. This is the main objective of this paper.

First of all, let us observe that the strict reduction in [22] is also approximation preserving with respect to differential approximation. This, as we will see in Section 3, shows the existence of \text{NPO-complete} problems also in the framework of the differential approximation. In [9], it is shown that there exist problems for which no polynomial time algorithm can guarantee that any solution computed will be even slightly far from a worst one, unless \( P = \text{NP} \). In other words, there exist problems for which the differential ratio of any polynomial time algorithm is equal to 0. This is the case for example for minimum independent dominating set. Such a behavior represents a worst case for the differential approximability of an \text{NPO} problem and draws a subclass of \text{NPO} called \text{0-DAPX} in Section 3. Denote by \text{0-DAPX-complete}, the problems of \text{0-DAPX} sharing the following property: if one of them can be polynomially
solved within (even non-constant) differential approximation ratio strictly greater than 0, then all of 0-DAPX can. We prove in Section 3 that under the strict-reduction NPO-complete = 0-DAPX-complete ⊆ 0-DAPX ⊆ NPO.

In Section 4, we tackle the question of the existence of complete problems for DAPX. We define a suitable reduction, called DPTAS-reduction and show that under it many natural NPO problems are DAPX-complete.

Besides PTAS, the two most notable classes of APX in the literature are MAX-SNP and GLO. The first one, introduced, as we have already mentioned in [23], is defined in logical terms and, furthermore, independently on any approximability property of its members; henceforth, MAX-SNP is notorious for differential approximation also without need of defining any differential counterpart for it. The latter one, GLO, is, roughly speaking, the class of the NPO-PB problems whose all locally optimal solutions (with respect to a suitable neighborhood) guarantee constant standard approximation ratio. It is introduced in [8] where a local optima preserving (LOP) reduction, which is a special case of L-reduction provided with some suitable local optimality properties, is also defined.

**Definition 3.** Let \( \Pi' \) and \( \Pi \) be two NPO problems. An L-reduction \( h = (f, g) \) is a local optima preserving reduction (LOP-reduction) if the following properties hold:

- for any \( x \in \mathcal{I}_{\Pi'} \), there exists a subset \( \text{sol}'_{\Pi}(f(x)) \subseteq \text{sol}_{\Pi}(f(x)) \) such that:
  \[
g(x, \text{sol}'_{\Pi}(f(x))) \supseteq \text{sol}_{\Pi}(x) \quad \text{(surjectivity)};
\]
  \[
  \forall y, z \in \text{sol}_{\Pi}(f(x)), m_{\Pi}(f(x), y) \leq m_{\Pi}(f(x), z) \Rightarrow m_{\Pi}(x, g(x, y)) \leq m_{\Pi}(x, g(x, z)) \quad \text{(partial monotonicity)};
\]
- for every constant \( k \), there exists a constant \( h \) such that:
  
  given a \( k \)-bounded neighborhood \( N \) for \( \Pi \), there exists an \( h \)-bounded neighborhood \( N' \) for \( \Pi' \) such that for any instance \( x \in \mathcal{I}_{\Pi'} \) and \( u, z \in \text{sol}'_{\Pi}(f(x)) \), \( u \in N(f(x), z) \Rightarrow g(x, u) \in N'(x, g(x, z)) \) (locality);
  
  \[
  \forall z \in \text{sol}'_{\Pi}(f(x)), \text{if } z \text{ is the optimum over } \text{sol}'_{\Pi}(f(x)) \cap N(f(x), z), \text{then it is the optimum over } N(f(x), z) \quad \text{(dominance).}
\]

In Section 6, we first define the differential counterpart of GLO, denoted by DGLO, as well as, two subclasses, namely DGLO\(_{0}\), the class of maximization problems of MAX-SNP for which value 0 is a feasible one (obviously, the worst one), and DGLO\(_{AF}\), the class of the minimization problems \( \Pi \) of MAX-SNP for which there exists \( \Pi' \in \text{DGLO}_{0} \) such that \( \Pi \) is affinely transformable to \( \Pi' \). We also devise a local optima preserving reduction strongly inspired from the LOP-reduction of [8] and, under this new reduction we prove the existence of natural complete problems for DGLO\(_{0} \cup \text{DGLO}_{AF}\). Finally, by devising an appropriate reduction, we show the existence of hard problems for a natural subclass of DPTAS.

In what follows, a number of NPO problems is mentioned and/or discussed. Their definitions together with specifications of their worst solutions are given in Appendix A.

### 3 Differential NPO-completeness

We study in this section NPO-completeness with respect to differential approximation. Based upon Definition 1 of [22], given in Section 2, we define a particular strict-reduction, called D-reduction, which we use in the sequel for proving NPO-completeness.
Definition 4. A D-reduction is a strict reduction up to the replacement of the third item in Definition 1 by the condition $\delta_{II}(x, g(x, y)) \geq \delta_{II}(f(x, y), \delta > 0$. Two optimization problems II and II' are D-equivalent if II D-reduces to II' and II' D-reduces to II. 

Theorem 1. MAX WSAT and MIN WSAT are D-equivalent.

Proof. We construct a differential reduction from MAX WSAT to MIN WSAT. Let $\varphi$ be an instance of MAX WSAT on $n$ variables and $m$ clauses. The instance $\varphi'$ of MIN WSAT contains $m$ clauses and the same set of $n$ variables. With each clause $\ell_1 \lor \ldots \lor \ell_k$ of $\varphi$ we associate in $\varphi'$ the conjunction $\ell_1 \lor \ldots \lor \ell_k$, where $\ell_i = \bar{x}_j$ if $\ell_i = x_j$ and $\ell_i = x_j$ if $\ell_i = \bar{x}_j$. It is easy to see that if an assignment $y$ satisfies the instance $\varphi$ then the complement of $y$ satisfies $\varphi'$. It is easy to see that $\text{opt}(\varphi') = \sum_{i=1}^{n} w(x_i) - \text{opt}(\varphi)$ and $w(\varphi') = \sum_{i=1}^{n} w(x_i) - w(\varphi)$. Also, if $m(\varphi', y')$ is the value of the solution $y'$ in $\varphi'$, then the complement of this solution $y$ has in $\varphi$ the value $m(\varphi, y) = \sum_{i=1}^{n} w(x_i) - m(\varphi', y)$. Thus, $\delta(\varphi, y) = \delta(\varphi', y)$. The reduction from MIN WSAT to MAX WSAT is completely analogous.

As usually, ([11, 22]), we denote by MAX NPO and MIN NPO, the classes of maximization and minimization NPO problems, respectively.

Theorem 2. MAX WSAT is MAX NPO-complete and MIN WSAT is MIN NPO-complete under $\leq_D$. MAX NPO-hard and MIN NPO-hard (under $\leq_D$) coincide and form the class of NPO-hard problems.

Proof. For the completeness of MIN WSAT, remark that Theorem 3.1 in [22], based upon an extension of Cook's proof ([10]) of SAT NP-completeness to optimization problems, holds for any cost-respecting approximation measure and that the differential ratio is such a measure. Furthermore, solution $\text{triv}$, as defined in [22] is indeed a worst-value one for MIN WSAT. The MAX NPO-completeness of MAX WSAT can be proved in exactly the same way. Finally, the last assertion follows as an immediate assertion of Theorem 1.

In a completely analogous way, as in Theorem 1 one can prove the D-equivalence of MIN $\{0,1\}$ INTEGER PROGRAMMING and MAX $\{0,1\}$ INTEGER PROGRAMMING.

Corollary 1. MIN $\{0,1\}$ INTEGER PROGRAMMING and MAX $\{0,1\}$ INTEGER PROGRAMMING are NPO-complete, under $\leq_D$.

We note here that, the result of [22] about the MIN NPO-completeness of MIN TSP (Theorem 3.3) can be erroneously seen as in "glaring contradiction" to a result of [19, 17] where it is proved that MIN TSP on graphs with polygonally bounded edge-distances is in DAPX. In fact, there is no contradiction at all. Solution $\text{triv}$ for MIN TSP adopted in [22], is considered as a tour containing exclusively edges of maximum distance. But such a solution is not always feasible for any instance of MIN TSP (the worst-value solution for this problem is an optimal solution of MAX TSP); hence the strict reduction of Theorem 3.3 in [22] is not a D-one. Finally, solution $\text{triv}$ adopted in [22] for MIN $\{0,1\}$ INTEGER PROGRAMMING coincides with the worst-value one as it has been adopted here in Section 1; so, the strict reduction of Theorem 3.4 in [22] is a D-one.

We now introduce an approximation class, called $0$-DAPX in what follows, that seems very natural for differential approximation while has no sense in the standard case.

Definition 5. $0$-DAPX is the class of NPO problems II for which approximation within any differential approximation ratio $\delta > 0$ would entail $\mathbf{P} = \mathbf{NP}$. A problem II is said to be $0$-DAPX-hard, if approximation of II within any strictly positive differential approximation ratio would imply approximation of any other $0$-DAPX problem within strictly positive approximation ratios.
Let $\Pi'$ be an NP-complete decision problem and $\Pi$ an NPO problem. The underlying idea for proving inclusion of $\Pi$ in 0-DAPX is, starting from an instance of $\Pi'$, to construct instances for $\Pi$ that have only two distinct feasible values and to prove that any differential $\delta$-approximation for $\Pi'$, $\delta > 0$, could distinguish between positive instances and negative instances for $\Pi'$. Intuitively, for any 0-DAPX problem, there exists at least one instance for which, unless $P = NP$, any polynomial time approximation algorithm cannot compute a solution other than a worst one. Remark that inclusion in 0-DAPX is rather a negative than a positive approximation result. This seems quite natural since 0-approximability represents the worst intractability level for an NPO problem in the differential approach.

Example 1. (MIN INDEPENDENT DOMINATING SET) In [9] it is proved that if $P \neq NP$, then, for any decreasing $\delta : \mathbb{N} \to (0,1)$, MIN INDEPENDENT DOMINATING SET is not differential $\delta$-approximable in polynomial time. Given an instance $\varphi$ of SAT with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$ we construct a graph $G$, instance of MIN INDEPENDENT DOMINATING SET associating with any positive literal $x_i$ a vertex $u_i$ and with any negative literal $\bar{x}_i$ a vertex $v_i$. For $i = 1, \ldots, n$ we draw edges $u_i v_i$. For any clause $C_j$ we add in $G$ a vertex $w_j$ and an edge between $w_j$ and each vertex corresponding to a literal contained in $C_j$. Finally, we add edges in $G$ in order to obtain a complete graph on $w_1, \ldots, w_m$. An independent set of $G$ contains at most $n+1$ vertices since it contains at most one vertex among $w_1, \ldots, w_m$ and at most one vertex among $u_i$ and $v_i$ for $i = 1, \ldots, n$. An independent dominating set containing the vertices corresponding to true literals of a non-satisfiable assignment and one vertex corresponding to a clause not satisfied by this assignment is a worst solution of $G$ of size $n+1$. If $\varphi$ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying $\varphi$ is an independent dominating set (each vertex $w_j$ is dominated by a vertex corresponding to a true literal of $C_j$) of minimum size. On the other hand, if $\varphi$ is not satisfiable then $\text{opt}(G) = n+1$. So, any independent dominating set of $G$ has cardinality either $n$, or $n+1$. Hence, if an approximation algorithm achieves approximation ratio strictly greater than 0, i.e., if it computes a solution strictly better than the worst one, then it will compute an optimal solution for $G$ (of size $n$), then one can correctly deduce that $\varphi$ is satisfiable, while if it returns a solution of size $n+1$, then one can correctly conclude that $\varphi$ is not satisfiable.

By analogous reductions, it is proved in [24] that for any $k > 3$, polynomially bounded MAX Wk-SAT-B as well as the general minimization and maximization versions of integer-linear programming are in 0-DAPX.

Theorem 3. Under $\leq_D$, NPO-complete $\subseteq$ 0-DAPX-complete $\subseteq$ 0-DAPX.

Proof. We first prove NPO-complete $\subseteq$ 0-DAPX. Fix a problem $\Pi \in$ NPO-complete. Then, for any $\Pi' \in$ NPO, $\Pi' \leq_D \Pi$. Assume that $\text{max Wk-SAT-B}$ stands for $\Pi'$. Then, from [24] and by the fact $\text{max Wk-SAT-B} \leq_D \Pi$, we conclude that $\Pi \in$ 0-DAPX.

We now prove NPO-complete = 0-DAPX-complete. Fix a problem $\Pi$ NPO-complete. For any problem $\Pi'' \in$ NPO, $\Pi'' \leq_D \Pi$. Also, by the fact NPO-complete $\subseteq$ 0-DAPX, $\Pi \in$ 0-DAPX. Since 0-DAPX $\subseteq$ NPO, then for any problem $\Pi' \in$ 0-DAPX, $\Pi' \leq_D \Pi$ and thus $\Pi$ is 0-DAPX-hard. In the other direction, fix a problem $\Pi$ 0-DAPX-complete. For any problem $\Pi'' \in$ 0-DAPX, $\Pi'' \leq_D \Pi$; in particular $\text{max WSAT} \leq_D \Pi$. Since $\text{max WSAT}$ is NPO-complete, then for any $\Pi' \in$ NPO, $\Pi' \leq_D \text{max WSAT}$. Thus, $\Pi' \leq_D \Pi$ and, consequently, $\Pi$ is NPO-complete.

Finally, the inclusion 0-DAPX-complete $\subseteq$ 0-DAPX is immediate from Definition 5.

A natural question rising from th above is: what is the relation between NPO-complete and 0-DAPX? Taking into consideration the fact that 0-DAPX is the hardest differential approximability class in NPO, one might guess that NPO-complete $\equiv$ 0-DAPX, but in order
to prove it we need a stronger reducibility. For instance, remark that Definition 5 could be stated by means of a reduction. For example, consider a kind of Turing-reduction and call it \( T_0 \)-reduction. Call a problem \( \Pi \) \( T_0 \)-reducible to a problem \( \Pi' \) (\( \Pi \leq T_0 \Pi' \)), if there exists a Turing-reduction which, by addressing one or, eventually, a polynomial number of queries to an oracle for \( \Pi' \), transforms any \( \delta \)-differential approximation algorithm for \( \Pi \), \( \delta > 0 \) into an optimal (exact) algorithm for \( \Pi' \). Then, \( 0-DAPX \) is the class of \( NPO \) problems \( \Pi \) for which there exists an \( NP \)-complete problem \( \Pi' \) such that \( \Pi' \leq T_0 \Pi \). Moreover, a problem is \( 0-DAPX \)-hard, if any problem in \( 0-DAPX \) \( T_0 \)-reduces to it. Note finally that \( T_0 \)-reduction can be seen as a particular case of both \( D \)-reduction and the strict reduction of [22]. Using then \( \leq T_0 \), one can prove that \( NPO \)-complete = \( 0-DAPX \)-complete in Theorem 3 always works. On the other hand, in order to prove \( 0-DAPX \)-complete = \( 0-DAPX \), inclusion \( 0-DAPX \)-complete \( \subseteq \) \( 0-DAPX \) is immediate. For \( 0-DAPX \)-complete \( \supseteq \) \( 0-DAPX \), we fix a \( 0-DAPX \) problem \( \Pi \) and prove that any \( 0-DAPX \)-complete problem \( \Pi' \) \( T_0 \)-reduces to \( \Pi \). Let \( \Pi'' \) be the \( NP \)-complete problem that has entailed the inclusion of \( \Pi \) in \( 0-DAPX \).

Remark now that since \( \Pi' \) is \( NP \)-hard the optimization and decision versions of \( \Pi' \) are polynomial time equivalent ([3]); these versions are \( NP \)-hard and \( NP \)-complete and denote them by \( \Pi^*_O \) and \( \Pi^*_D \), respectively. The \( NP \)-completeness of \( \Pi'' \) and \( \Pi^*_D \) implies \( \Pi^*_D \leq K \) \( \Pi'' \). On the other hand, we have \( \Pi'' \leq T_0 \Pi \). In all, (i) \( \Pi'' = \Pi^*_O \equiv \Pi^*_D \leq K \) \( P \leq T_0 \Pi \), where by \( \leq K \) we denote the usual Karp-reduction from \( \Pi^*_D \) to \( P \), and (ii) \( \equiv \leq K \equiv \leq T_0 \) is a \( T_0 \)-reduction. So, \( 0-DAPX \)-complete \( \supseteq \) \( 0-DAPX \).

Figure 1 gives a summary of the class-inclusions discussed above under \( \leq D \). Note that problems as TSP, in both maximization and minimization versions, \( MAX \ INDEPENDENT \ SET \), \( MIN \ COLORING \), etc., are, under \( \leq D \), \( NPO \)-intermediate, i.e., they belong to \( NPO \setminus 0-DAPX \).

![Diagram](image.png)

**Figure 1:** Class inclusions and completeness under \( D \) reduction.

4 Differential APX-completeness

Let us now address the problem of completeness in the class \( DAPX \). Note first that a careful reading of the proof of the standard \( APX \)-completeness of \( MAX \ WSAT-B \) given in [12] establishes also the following proposition which will be used in what follows.

**Proposition 1.** ([12]) Let \( \Pi \in APX \). There exist \( 3 \) functions \( f, g \) and \( c \) such that \( \forall x \in \mathcal{I}_\Pi \),
∀z ∈ sol_Π(x), ∀ρ ∈ ]0, 1[: 

1. \( f(x, z, ρ) = (φ_{x,z,ρ}, W_{x,z,ρ}, w_{x,z,ρ}) \) with \((φ_{x,z,ρ}, W_{x,z,ρ}, w_{x,z,ρ}) ∈ I_{MAX WSAT}; f \) is polynomially computable;

2. ∀y ∈ sol_{MAX WSAT}(f(x, z, ρ)), g(x, z, ρ, y) ∈ sol_Π(x); g is polynomially computable;

3. \( c_ρ : ]0, 1[\cap Q → ]0, 1[\cap Q; \)

4. if \( γ_Π(x, z) ≥ ρ \), then \( f(x, z, ρ) ∈ I_{MAX WSAT-B} \) and, ∀y ∈ sol_{MAX WSAT-B}(f(x, z, ρ)), if \( γ_{MAX WSAT-B}(f(x, z, ρ), y) ≥ 1 - c_ρ(ε) \), then \( γ_Π(x, g(x, z, ρ, y)) ≥ 1 - ε. \)

Note that \( f \) and \( g \) are not functions of \( ε \) and that the reduction (the computation of \( f \)) and the interpretation (computation of \( g \)) are computable in polynomial time. The fact that \( f \) is a \( ρ \)-approximation and, consequently, that \( Π \) is in \( APX \), allows to consider instance \( f(x, z, ρ) \) as an instance of \( MAX WSAT-B \) and to consider that \( g \) preserves polynomial time approximation schemata.

We first define a notion of polynomial time differential approximation schemata preserving reducibility, called \( DPTAS \)-reduction in what follows.

**Definition 6.** Consider two \( NPO \) problems \( Π \) and \( Π' \). Then, \( Π \leq_{DPTAS} Π' \) if there exist three functions \( f, g \) and \( c \), computable in polynomial time, such that:

- ∀\( x ∈ I_Π, ∀ε ∈ ]0, 1[\cap Q, f(x, ε) ∈ I_Π; f \) is possibly multivalued;
- ∀\( x ∈ I_Π, ∀ε ∈ ]0, 1[\cap Q, ∀y ∈ sol_Π(f(x, ε)), g(x, y, ε) ∈ sol_Π(x); \)
- \( c : ]0, 1[\cap Q → ]0, 1[\cap Q; \)
- ∀\( x ∈ I_Π, ∀ε ∈ ]0, 1[\cap Q, ∀y ∈ sol_Π(f(x, ε)), δ_Π(f(x, ε), y) ≥ 1 - c(ε) ⇒ δ_Π(x, g(x, y, ε)) ≥ 1 - ε; \) if \( f \) is multivalued, i.e., \( f = (f_1, \ldots, f_i) \), for some \( i \) polynomial in \( |x| \), then the former implication becomes: ∀\( x ∈ I_Π, ∀ε ∈ ]0, 1[\cap Q, ∀y ∈ sol_Π((f_1, \ldots, f_i)(x, ε)), ∃j ≤ i \) such that \( δ_Π(f_j(x, ε), y) ≥ 1 - c(ε) ⇒ δ_Π(x, g(x, y, ε)) ≥ 1 - ε. \)

The following proposition can be easily shown.

**Proposition 2.** Given two \( NPO \) problems \( Π \) and \( Π' \), if \( Π \leq_{DPTAS} Π' \) and \( Π' ∈ DAPX \), then \( Π ∈ DAPX. \)

Let \( Π ∈ DAPX \) and let \( T \) be a differential \( ρ \)-approximation algorithm for \( Π \), with \( ρ ∈ ]0, 1[. \) There exists a polynomial \( p \) such that ∀\( x ∈ I_Π, |ω(x) - opt(x)| ≤ 2p(|x|) \). An instance \( x ∈ I_Π \) can be written in terms of an integer linear program as:

\[
x : \begin{cases} 
\text{opt} & v(y) \\
y ∈ C_x 
\end{cases}
\]

where \( C_x \) is the constraint-set of \( x \). For any \( i ∈ \{0, \ldots, p(|x|)\} \) and for any \( l ∈ \mathbb{N} \), we define \( x_{i,l} \) by:

- if \( Π \) is a maximization problem, then

\[
x_{i,l} : \begin{cases} 
\text{max} & \left[ v_{i,l}(y) = \left\lfloor \frac{v(y)}{2^l} \right\rfloor - l \right] \\
y ∈ C_x 
\end{cases}
\]
• if $\Pi$ is a minimization problem, then

$$x_{i,l} : \begin{cases} \min_{y \in C_x} \left[v_{i,l}(y) = l - \left\lfloor \frac{v(y)}{2^i} \right\rfloor \right] \
\end{cases}$$

Any $x_{i,l}$ can be considered as an instance of an NPO problem denoted by $\Pi_{i,l}$.

**Proposition 3.** Let $\epsilon < \min \{ \rho / 2, 1/2 \}$ and $x \in I_{\Pi}$. Assume $(i, l) \in \{1, \ldots, p(|x|)\} \times \mathbb{N}$ is such that $2^i \leq \epsilon \text{opt}(x) - \omega(x) \leq 2^{i+1}$ and set $l = \left\lfloor \omega(x)/2^i \right\rfloor$. Then, for any $y \in \text{sol}_{\Pi}(x) = \text{sol}_{\Pi_{i,l}}(x_{i,l})$:

1. $\delta_{\Pi_{i,l}}(x_{i,l}, y) \geq (1 - \epsilon) \implies \delta_{\Pi}(x, y) \geq 1 - 3\epsilon$;

2. $\delta_{\Pi}(x, y) \geq \rho \implies \delta_{\Pi_{i,l}}(x_{i,l}, y) \geq (\rho - \epsilon)/(1 + \epsilon)$.

**Proof.** Remark that for any $(i, l)$ defined as in the statement of the proposition and $\forall y \in \text{sol}_{\Pi}(x)$, $v_{i,l}(y) = \left\lfloor v(y)/2^i \right\rfloor - l$, in the maximization case, and $v_{i,l}(y) = l - \left\lfloor v(y)/2^i \right\rfloor$, in the minimization one. Furthermore, remark that $\delta_{\Pi}(x, y) = \delta_{\Pi_{i,l}}(x_{i,l}, y)$. For now on, we assume $\Pi$ a maximization problem (the proof in the case of minimization is similar).

In order to prove item 1, note that

$$\frac{v(y)}{2^i} - \left\lfloor \frac{\omega(x)}{2^i} \right\rfloor \leq \frac{v(y) - \omega(x)}{2^i} + 1 \quad (1)$$

$$\frac{\text{opt}(x)}{2^i} - \left\lfloor \frac{\omega(x)}{2^i} \right\rfloor \geq \frac{\text{opt}(x) - \omega(x)}{2^i} - 1 \geq \frac{\text{opt}(x) - \omega(x)}{2^i}(1 - \epsilon) \quad (2)$$

where (2) is true because $(\text{opt}(x) - \omega(x))/2^i \geq 1/\epsilon$. Combination of (1) and (2) gives $\delta_{\Pi}(x, y) \geq (1 - \epsilon)^2 - 2^i(\text{opt}(x) - \omega(x)) \geq (1 - \epsilon)^2 - \epsilon \geq 1 - 3\epsilon$, and the proof of item 1 is complete.

For the proof of item 2 remark that

$$\frac{v(y)}{2^i} - \left\lfloor \frac{\omega(x)}{2^i} \right\rfloor \geq \frac{v(y) - \omega(x)}{2^i} - 1 \quad (3)$$

$$\frac{\text{opt}(x)}{2^i} - \left\lfloor \frac{\omega(x)}{2^i} \right\rfloor \leq \frac{\text{opt}(x) - \omega(x)}{2^i} + 1 \leq \frac{\text{opt}(x) - \omega(x)}{2^i}(1 + \epsilon) \quad (4)$$

and combination of (3) and (4) achieves $\delta_{\Pi_{i,l}}(x_{i,l}, y) \geq (\rho - \epsilon)/(1 + \epsilon)$ that completes the proof of item 2 and of the proposition.

The proof of the existence of a DAPX-complete problem in the following Theorem 4, will be performed along the following schema. We first prove that any DAPX problem $\Pi$ is reducible to MAX WSAT-$B$ by a reduction transforming a PTAS for MAX WSAT-$B$ into a DPTAS for $\Pi$; we denote it by $\leq_5^\text{D}$. Next, we consider a particular APX-complete problem $\Pi'$, say MAX INDEPENDENT SET-$B$; MAX WSAT-$B$ that is in APX is PTAS-reducible to MAX INDEPENDENT SET-$B$. MAX INDEPENDENT SET-$B$ is both in APX and in DAPX and, moreover, standard and differential approximation ratios coincide for it; this coincidence draws a trivial reduction called lD-reduction; it trivially transforms a differential polynomial time approximation schema into a standard polynomial time approximation schema. The composition of the three reductions specified (i.e., the one from $\Pi$ to MAX WSAT-$B$, the one from MAX WSAT-$B$ to MAX INDEPENDENT SET-$B$ and the lD-reduction) is a DPTAS reduction transforming a differential polynomial time approximation schema for MAX INDEPENDENT SET-$B$ into a differential polynomial time approximation schema for $\Pi$, i.e., MAX INDEPENDENT SET-$B \in$ DAPX-complete. Figure 2 depicts the sketch just above.

**Theorem 4.** MAX INDEPENDENT SET-$B$ is DAPX-complete.
Proof. We first show that any integer valued problem $\Pi \in \text{DAPX}$ reduces to $\text{MAX WSAT-B}$ by a reduction transforming a standard to a differential approximation schema (extension to the case of rational values is immediate).

Remark first that given a formula $\varphi$, a variable-weight system $\vec{w}$ and a constant $B$, one can decide in polynomial time if $(\varphi, B, \vec{w}) \in I_{\text{MAX WSAT-B}}$. Since $\Pi$ is in $\text{DAPX}$, let $T$ be a polynomial algorithm that guarantees differential ratio $\rho \in [0, 1]$. Let $\epsilon < \min\{\rho, 1/2\}$.

For any $\zeta > 0$, we denote by $0_\zeta$ an oracle that, for any instance $x$ of $\text{MAX WSAT-B}$, computes a feasible solution $0_\zeta(x) \in \text{sol}_{\text{MAX WSAT-B}}$ guaranteeing $\gamma_{\text{MAX WSAT-B}}(x, 0_\zeta) \geq 1 - \zeta$. We construct an algorithm $A$ (reduction) using this oracle such that:

- $A$ guarantees differential approximation ratio $1 - \epsilon$ for $\Pi$ and
- in the case where $0_\zeta$ is polynomial (in other words, $0_\zeta$ can be seen as a polynomial time approximation schema), $A$ is also polynomial.

The $\leq_D^S$-reduction claimed is based upon the construction of a family $\mathcal{F}$ of instances $x_{i,l}$: $\mathcal{F} = \{x_{i,l} : (i, l) \in F\}$, where $F$ is of polynomial size and contains a pair $(i_0, l_0)$ such that:

- either $i_0 \neq 0$, $2^{i_0} \leq \epsilon |\text{opt}(x) - \omega(x)| \leq 2^{i_0+1}$ and $l_0 = |\omega(x)/2^{i_0}|$, or
- or $i_0 = 0$, $\epsilon |\text{opt}(x) - \omega(x)| \leq 2$ and $l_0 = \omega(x)$.

Remark 1. For instance $x_{i_0,l_0}$ the worst value is 0; henceforth standard and differential ratios coincide. In other words, $\delta_{\Pi_{i_0,l_0}}(x_{i_0,l_0}, z) = \gamma_{\Pi_{i_0,l_0}}(x_{i_0,l_0}, z)$, for all feasible $z$. 

Remark 2. For $i_0 = 0$, $\delta_{\Pi}(x, z) = \delta_{\Pi_{0,\omega(x)}}(x_{0,\omega(x)}, z) = \gamma_{\Pi_{0,\omega(x)}}(x_{0,\omega(x)}, z)$.

Suppose first that such a set $F$ can be constructed in polynomial time. For each $(i, l) \in F$, we consider the three functions $g_{i,l}$, $f_{i,l}$ and $c_{i,l}$ (Proposition 1) for the instance $x_{i,l}$. We set $\epsilon' = \min \{(c_{i,l})_{\rho}(\epsilon), (c_{i,l})_{(\rho-\epsilon)/(1+\epsilon)}(\epsilon/3) : (i, l) \in F\}$ and define, for $(i, l) \in F$,

$$\eta = \begin{cases} \rho & i = 0 \\ \frac{\rho - \epsilon}{1+\epsilon} & \text{otherwise} \end{cases}$$

Let $z = T(x)$. For any $(i, l) \in F$, we set

$$z_{i,l} = \begin{cases} g_{i,l}(x_{i,l}, z, \eta, 0_{\epsilon'}(f_{i,l}(x_{i,l}, z, \eta))) & \text{if } f_{i,l}(x_{i,l}, z, \eta) \text{ is an instance of MAX WSAT-B} \\ \text{otherwise} \end{cases}$$

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Remark that \( z_{i,l} \) is a feasible solution for \( x_{i,l} \) and, consequently, for \( x \). In all, \( A \) constructs \( z_{i,l} \) for each \((i,l) \in F \) and selects the best among them as solution for \( x \).

We now prove that \( A \) achieves differential approximation ratio \( 1 - \epsilon \). Remark that for \((i,l) = (i_0,l_0)\), \( z = T(x) \) guarantees ratio (standard or differential) \( \eta \) for \( x_{i_0,l_0} \) (by item 2 of Proposition 3 for the case \( i_0 \neq 0 \), by Remark 2 otherwise). By item 4 of Proposition 1 applied for problem \( \Pi_{i_0,l_0} \) and for \( \rho = \eta \), we have: \( f_{i_0,l_0}(x_{i_0,l_0}, z, \eta) \in T_{\text{MAX WSAT-B}} \). Hence by definition of \( z_{i_0,l_0} \), \( z_{i_0,l_0} = g_{i_0,l_0}(x_{i_0,l_0}, \eta, 0, \omega(f_{i_0,l_0}(x_{i_0,l_0}, z, \eta))) \). On the other hand, recall that \( \gamma_{\text{MAX WSAT-B}}(0, \omega(f_{i_0,l_0}(x_{i_0,l_0}, z, \eta))) \geq 1 - \epsilon \). We distinguish two cases:

- if \( i_0 = 0 \), then by item 4 of Proposition 1 and since \( 1 - \epsilon' \geq 1 - (c_{i_0,\omega(x)}) \rho(\epsilon) \), we have \( \gamma_{\Pi_{i_0,l_0}}(x_{0,\omega(x)}, z_0, \omega(x)) \geq 1 - \epsilon \) and Remark 2 implies \( \delta_{\Pi}(x, z_{0,\omega(x)}) \geq 1 - \epsilon \);

- if \( i_0 \neq 0 \), then by item 4 of Proposition 1 and since \( 1 - \epsilon' \geq 1 - (c_{i_0,l_0})^{(p-\epsilon)/(1+\epsilon)}(\epsilon/3) \), we have \( \gamma_{\Pi_{i_0,l_0}}(x_{i_0,l_0}, z_{i_0,l_0}) \geq 1 - (\epsilon/3) \) and by item 1 of Proposition 3 we have \( \delta_{\Pi}(x, z_{i_0,l_0}) \geq 1 - \epsilon \).

Since \((i_0,l_0) \in F\), \( A \) has already computed the solution \( z_{i_0,l_0} \). By taking into account that the solution finally returned by \( A \) is the best among the computed ones, we immediately conclude that it is at least as good as \( z_{i_0,l_0} \). Therefore, it guarantees the ratio \( 1 - \epsilon \).

We finally prove that \( F \) can be constructed in polynomial time. For this, we consider two cases:

- if \( \epsilon|\omega(x) - \text{opt}(x)| \leq 2 \), then \( |\omega(x) - \text{opt}(x)| \leq \lfloor 2/\epsilon \rfloor \); in this case \( \exists k \in \{-\lfloor 2/\epsilon \rfloor, \ldots, \lfloor 2/\epsilon \rfloor \} \)
  such that \( \omega(x) = k + v(z) \); then \( i_0 = 0 \) et \( l_0 = k + v(z) \);

- if \( \exists i \in \{1, \ldots, p(x)\} \) such that \( 2^i \leq \epsilon|\text{opt}(x) - \omega(x)| \leq 2^{i+1} \), then \( |\text{opt}(x)/2^i - (\omega(x)/2^i)| \leq \lfloor 2/\epsilon + 1 \rfloor \); so \( \exists k \in \{-\lfloor 2/\epsilon + 1 \rfloor, \ldots, \lfloor 2/\epsilon + 1 \rfloor \} \)
  such that \( \omega(x_{i_0}) = \lfloor \omega(x)/2^i \rfloor = v_{i_0}(z) + k \); in this case \( i_0 = i \) and \( l_0 = v_{i_0}(z) + k \).

Consequently, \( F \) can be defined as:

\[
F = \left\{ (0, v(z) + k) : k \in \left\{ -\left\lfloor \frac{2}{\epsilon} \right\rfloor, \ldots, \left\lfloor \frac{2}{\epsilon} \right\rfloor \right\} \right\} \\
\cup \left\{ (1, \ldots, p(x)) \times \left\{ \begin{array}{c} v_{i_0}(z) - \left\lfloor \frac{2}{\epsilon} + 1 \right\rfloor, \ldots, v_{i_0}(z) + \left\lfloor \frac{2}{\epsilon} + 1 \right\rfloor \end{array} \right\} \right\}.
\]

Remark that since \( F \) is of polynomial cardinality, the number of oracle’s calls is also polynomial. So the total complexity is polynomial. For simplicity, we simply use \( \leq \) to denote the reduction just exhibited; in all we have shown that \( \forall \Pi \in \text{DAPX}, \)

\[
\Pi \leq^S_{\text{DAPX}} \text{MAX WSAT-B} \tag{5}
\]

and the transformation of a PTAS for \text{MAX WSAT-B} into a DPTAS for any problem \( \Pi \in \text{DAPX} \) is complete.

Since \text{MAX INDEPENDENT SET-B} is APX-complete, any problem in APX, a fortiori \text{MAX WSAT-B}, reduces to \text{MAX INDEPENDENT SET-B}, i.e.,

\[
\text{MAX WSAT-B} \leq_{\text{PTAS}} \text{MAX INDEPENDENT SET-B} \tag{6}
\]

On the other hand, since, for any instance \( G \) of \text{MAX INDEPENDENT SET}, \( \omega(G) = 0 \), standard and differential approximation ratios coincide; hence,

\[
\text{MAX INDEPENDENT SET-B} \leq_{\text{ID}} \text{MAX INDEPENDENT SET-B} \tag{7}
\]
The composition of reductions in (5), (6) and (7), i.e., $\leq^D_5 \circ \leq^\text{PTAS} \circ \leq^\text{ID}$ clearly fits Definition 6; therefore, this composition is a DPTAS-reduction.

In all, we have shown that, $\forall \Pi \in \text{DAPX}$, $\Pi \leq^\text{DPTAS} \text{MAX INDEPENDENT SET} - B$. Furthermore, MAX INDEPENDENT SET - B is in DAPX, since any algorithm computing a maximal (for the inclusion) independent set guarantees a differential approximation ratio $1/(B + 1)$ (recall that $\omega(G) = 0$ for any $G$). Consequently, MAX INDEPENDENT SET - B is DAPX-complete and the proof of the theorem is complete. ■

**Theorem 5.** MIN VERTEX COVER - B, MAX SET PACKING - B, MIN SET COVER - B, are DAPX-complete under DPTAS-reductions.

**Proof.** Note first that MAX SET PACKING - B, MIN VERTEX COVER - B and MIN SET COVER - B belong to DAPX. As for MAX INDEPENDENT SET - B, any approximation algorithm computing a maximal (for the inclusion) set packing achieves differential approximation ratio 1/B (recall that the worst-value solution for MAX SET PACKING - B is the empty set; so, standard and differential approximation ratios coincide). MIN VERTEX COVER is equivalent to MAX INDEPENDENT SET via affine transformation of their objective functions and the differential approximation ratio is stable for such transformations; henceforth, MIN VERTEX COVER - B is approximable within differential approximation ratio 1/B. Finally, the proof of the inclusion of MIN SET COVER - B in DAPX follows from [14, 16] where in [16] it is proved that when $|C| \leq |S|$, then MIN SET COVER is approximable within differential approximation ratio 1/2, while in [14] it is proved that for $|C| \geq |S|$, MIN SET COVER - B is approximable within differential ratio 1/B.

The DAPX-hardness for MIN VERTEX COVER is immediate. On the other hand, MAX SET PACKING, is approximable equivalent for both standard and differential approximations under reductions preserving constant approximation ratios (as well as ratios depending on the order of the input-graph) to MAX INDEPENDENT SET.

The hardness of MIN SET COVER - B for DAPX can be proved by the remark that MIN VERTEX COVER - B is the restriction of MIN SET COVER - B in set systems where any element of the ground set $C$ belongs to exactly two sets of the family $S$ (considering a graph as a set-system where any vertex is a set containing its adjacent edges). ■

**Theorem 6.** MAX INDEPENDENT SET, MIN VERTEX COVER, MAX SET PACKING, MIN SET COVER, MAX CLIQUE and MAX $\ell$-COLORABLE INDUCED SUBGRAPH, are DAPX-hard under DPTAS-reductions.

**Proof.** The hardness of the first four problems is immediate from Theorem 5. The DAPX-hardness for MAX CLIQUE comes from the fact that MAX INDEPENDENT SET and MAX CLIQUE are approximable equivalent for both standard and differential approximations under reductions preserving constant approximation ratios (as well as ratios depending on the order of the input-graph).

For the hardness of MAX $\ell$-COLORABLE INDUCED SUBGRAPH consider a graph $G(V, E)$, instance of MAX INDEPENDENT SET and consider graph $G_{\ell}(V_{\ell}, E_{\ell})$ consisting of $\ell$ copies of $G$, any two distinct copies being linked completely. Consider $B_{\ell}$ a solution for MAX $\ell$-COLORABLE INDUCED SUBGRAPH on $G_{\ell}$ and denote by $V'_{\ell}$ its vertex-set. Obviously, the greatest among the colors in $V'_{\ell}$, denoted by $V'$, is an independent set of $G$; hence $m(G, V') \geq m(G_{\ell}, V'_{\ell})/\ell$. On the other hand, given a maximum independent set of $G$ the solution consisting of taking one copy of it in each of the $\ell$ copies of $G$ in $G_{\ell}$ is feasible for MAX $\ell$-COLORABLE INDUCED SUBGRAPH; so, $\text{opt}(G_{\ell}) \geq \ell \text{opt}(G)$. Combining expressions above and taking into account that for both problems worst-values are 0, one concludes that the reduction just described is a DPTAS-one. ■

Let us note that if we restrict ourselves in the class of DAPX problems with $\omega(x)$ computable in polynomial time, denoted by DAPX$_p$, then the proof of the existence of DAPX$_p$-complete
problems can be much simplified. In fact, consider an instance \( x \) of such a problem \( \Pi \) (assume without loss of generality that \( \Pi \) is a maximization problem) and an instance \( x' \) of problem \( \Pi' \) which is identical to \( \Pi \), i.e., both problems are defined on the same inputs and have identical sets of constraints; consequently they have the same set of solutions. Consider finally that 
\[
m_{\Pi}(x', y) = m_{\Pi}(x, y) - \omega(x).
\]
Since differential ratio is stable under affine transformations of the objective function, if \( \Pi \in \text{DAPX} \), then \( \Pi' \in \text{DAPX} \). Moreover, it is easy to see that 
\[
\delta_{\Pi}(x, y) = \delta_{\Pi}(x', y) = \gamma_{\Pi}(x', y);\]
in other words \( \Pi' \) belongs to both \( \text{DAPX} \) and \( \text{APX} \). We so have: \( \Pi \equiv_D \Pi' \leq_{\text{PTAS}} \text{MAX INDEPENDENT SET-B} \), where \( \equiv_D \) denotes the differential approximation equivalence between \( \Pi \) and \( \Pi' \); since this later problem is in \( \text{DAPX}_\text{P} \) and \( \equiv_D \circ \leq_{\text{PTAS}} \) is a \( \text{DPTAS} \)-reduction, we get that \( \text{MAX INDEPENDENT SET-B} \) is \( \text{DAPX}_\text{P} \)-complete.

The additional complexity of the proof of Theorem 4 is for taking into account problems for which worst solution is not polynomially computable.

5 Differential PTAS-hardness

In this section, we will take into consideration the class \( \text{DPTAS} \) and we will address the problem of completeness in such class.

Consider the following reduction preserving fully polynomial time differential approximation schemata, denoted by \( \text{DFPTAS} \)-reduction in what follows.

**Definition 7.** Assume two \( \text{NPO} \) problems \( \Pi \) and \( \Pi' \). Then, \( \Pi \leq_{\text{DFPTAS}} \Pi' \), if there exist three functions \( f, g \) and \( c \) such that:

- \( f \) and \( g \) are as in Definition 2;
- \( c : [0, 1[\cap\mathbb{Q}] \times \mathcal{I}_\Pi \rightarrow ]0, 1[\cap\mathbb{Q}] \); its time complexity and its value are polynomial in both \( |x| \) and \( 1/\epsilon \);
- \( \forall x \in \mathcal{I}_\Pi, \forall \epsilon \in ]0, 1[\cap\mathbb{Q}, \forall y \in \text{sol}_{\Pi'}(f(x, \epsilon)), \delta_{\Pi'}(f(x, \epsilon), y) \geq 1 - c(\epsilon, x) \Rightarrow \delta_{\Pi}(x, g(x, y, \epsilon)) \geq 1 - \epsilon. \]

**Proposition 4.** Given two \( \text{NPO} \) problems \( \Pi \) and \( \Pi' \), if \( \Pi \leq_{\text{DFPTAS}} \Pi' \) and \( \Pi' \in \text{DPTAS} \), then \( \Pi \in \text{DPTAS} \).

In the following we will apply the simplification drawn at the end of Section 4, in order to study completeness, not for the whole class \( \text{DPTAS} \) but for a subclass \( \text{DPTAS}_\text{P} \) consisting of the maximization problems of \( \text{PTAS} \) the worst-value of which is computable in polynomial time (this class includes, in particular, maximization problems with worst value 0). Recall that, the first problem proved \( \text{PTAS} \)-complete (under \( \text{FPTAS} \) reduction) is \( \text{MAX LINEAR WSAT-B} \) ([12]).

Consider a problem \( \Pi \in \text{DPTAS}_\text{P} \) and an instance \( x \in \mathcal{I}_\Pi \) expressed, in terms of an integer linear program as:

\[
x : \begin{cases} \text{opt } v(y) \\ y \in C_x \end{cases}
\]

Consider also a new problem \( \Pi' \) and an instance \( x' \in \mathcal{I}_{\Pi'} \) whose integer-linear programming formulation is

\[
x' : \begin{cases} \text{opt } v(y') - \omega(x) \\ y' \in C_{x'} \\ C_x = C_{x'} \end{cases}
\]

Obviously, \( y = y' \), consequently, \( \text{sol}_{\Pi}(x) = \text{sol}_{\Pi'}(x') \), \( \Pi' \in \text{DPTAS}_\text{P} \) (more precisely, for any \( x' \in \mathcal{I}_{\Pi'}, \omega(x') = 0 \) and,

\[
\delta_{\Pi}(x, y) = \delta_{\Pi'}(x', y') = \gamma_{\Pi'}(x', y')
\]

(8)
Furthermore, since \( \Pi \in \text{DPTAS}_p \), so does \( \Pi' \) (the differential ratio is stable under affine transformations of the objective function). On the other hand, by (8), \( \Pi' \in \text{PTAS} \); hence \( \Pi' \leq \text{FPTAS} \) MAX LINEAR WSAT-\( B \). In all, for any \( \Pi \in \text{DPTAS}_p \), \( \Pi \equiv \Pi' \leq \text{FPTAS} \) MAX LINEAR WSAT-\( B \), where \( \Pi' \) is defined as above and \( \equiv \) denotes, as previously, the differential approximation equivalence between \( \Pi \) and \( \Pi' \). Obviously, reduction \( \equiv \leq \text{FPTAS} \) is a \( \text{DFPTAS} \)-reduction.

Consider now the closure of \( \text{DPTAS}_p \) under affine transformations of objective functions of its problems:

\[
\text{DPTAS}_p^{\text{AF}} = \{ \Pi \in \text{DPTAS} : \exists \Pi' \in \text{DPTAS}_p, \Pi' \leq_{\text{AF}} \Pi \}
\]

Let us note that \( \text{DPTAS}_p^{\text{AF}} \) does not coincide with \( \text{DPTAS}_p \). Indeed, MAX INDEPENDENT SET in planar graphs, being in \( \text{PTAS} \) and having worst value 0, is in \( \text{DPTAS}_p \); henceforth, \( \text{MIN VERTEX COVER} \) in planar graphs is in \( \text{DPTAS}_p^{\text{AF}} \).

Let any \( \Pi'' \in \text{DPTAS}_p^{\text{AF}} \) and \( \Pi \) its “affine mate” in \( \text{DPTAS}_p \). Then, \( \Pi'' \leq_{\text{AF}} \Pi \equiv \Pi' \leq_{\text{FPTAS}} \) MAX LINEAR WSAT-\( B \) and since, obviously, the reduction \( \leq_{\text{AF}} \circ \equiv \leq_{\text{FPTAS}} \) is a \( \text{DFPTAS} \)-one, the following proposition holds.

**Proposition 5.** MAX LINEAR WSAT-\( B \) is \( \text{DPTAS}_p^{\text{AF}} \)-hard, under \( \leq_{\text{DFPTAS}} \).

6 MAX-SNP and differential GLO

In the theory of approximability of optimization problems based upon the standard approximation ratio interesting results have been obtained by studying the behavior of local search heuristics and the degree of approximation that such heuristics can achieve. In particular, in [8, 7], the class GLO is defined as the class of polynomially bounded optimization problems whose local optima have a guaranteed quality with respect to the global optima. More precisely, given an NPO problem \( \Pi \), a neighborhood is a function \( N : \mathcal{I}_\Pi \times \text{sol}(\mathcal{I}_\Pi) \to 2^{\text{sol}(\mathcal{I}_\Pi)} \), computable in polynomial time, such that for any \( x \in \mathcal{I}_\Pi \) and \( y \in \text{sol}(x) \), \( N(x, y) \subseteq \text{sol}(x) \). Solution \( y \) is a local optimum of \( x \) w.r.t. \( N \) if for every \( y' \in N(x, y) \), \( m(x, y) \) is better than \( m(x, y') \). Problem \( \Pi \) has guaranteed local optima, if there exists a neighborhood \( N \) and a constant \( r \) such that for any \( x \in \mathcal{I}_\Pi \) any local optimum \( y \) for \( x \) guarantees standard approximation ratio \( r \). Given \( y \) and \( y' \) in \( \text{sol}(x) \), \( y' \) is an \( h \)-bounded neighbor of \( y \) if it is obtained from \( y \) by changing at most \( h \) elements of it. A neighborhood \( N \) is called \( h \)-bounded if there exists an integer constant \( h \), such that any \( y' \in N(x, y) \) is an \( h \)-bounded neighbor of \( y \). Based upon the above concepts the class GLO is defined as follows.

**Definition 8.** Let \( \Pi \) be a polynomially bounded NPO problem. Then, \( \Pi \in \text{GLO} \) if (i) at least one feasible \( y \in \text{sol}(x) \) can be computed in polynomial time, for every \( x \in \mathcal{I}_\Pi \) and (ii) there exist an integer \( h \in \mathbb{N} \) and a suitable \( h \)-bounded neighborhood \( N \) such that \( \Pi \) has guaranteed local optima with respect to \( N \). \( \blacksquare \)

Of course, the differential counterpart of GLO, called DGLO in what follows, can be defined analogously. In [20] it is shown that MAX CUT, MIN DOMINATING SET-\( B \), MAX INDEPENDENT SET-\( B \), MIN VERTEX COVER-\( B \), MAX SET PACKING-\( B \), MIN COLORING, MIN SET COVER-\( B \), MIN SET \( W(K) \) COVER-\( B \), MIN FEEDBACK EDGE SET, MIN FEEDBACK VERTEX SET-\( B \) and MIN MULTIPROCESSOR SCHEDULING, are included in DGLO. Furthermore in [21] it is proved that both MIN and MAX TSP on graphs with polynomially bounded edge-distances are also included in DGLO. In the opposite, as it is shown in [20], for any \( h > 0 \) general MIN FEEDBACK VERTEX SET is not in DGLO with respect to \( h \)-bounded neighborhoods; for any \( k \) and any \( h < k \), MAX \( k \)-SAT is not in DGLO with respect to \( h \)-bounded neighborhoods; B-KNAPSACK\((n^k)\) is not in
DGLO with respect to $h$-bounded neighborhoods, for any $h > 0$ (we have here a polynomial problem whose local optima do not have guaranteed quality); Subset Sum is not in DGLO with respect to $h$-bounded neighborhoods, for any $h > 0$.

Let us now consider the relationship of DGLO with respect to the differential approximability class DAPX. Let

$$\text{GLO}^{\text{PTAS}} = \{ \Pi' : \Pi' \in \text{NPO}, \exists \Pi \in \text{GLO}, \Pi' \leq_{\text{PTAS}} \Pi \}$$

be the closure of GLO under $\leq_{\text{PTAS}}$. Analogously, set

$$\text{DGLO}^{\text{PTAS}} = \{ \Pi' : \Pi' \in \text{NPO}, \exists \Pi \in \text{DGLO}, \Pi' \leq_{\text{PTAS}} \Pi \}$$

the closure of DGLO under $\leq_{\text{PTAS}}$. In [7] it is proved that $\text{GLO}^{\text{PTAS}} = \text{APX}$. It is easy to show that the same holds for differential approximation.

Proposition 6. $\text{DAPX} = \text{DGLO}^{\text{PTAS}}$.

Proof. Since DGLO $\subseteq$ DAPX and PTASS-reduction preserves differential approximation ratios, then DGLO $^{\text{PTAS}}$ $\subseteq$ DAPX. On the other hand, since Max Independent Set-B is DAPX-complete then for any problem $A \in$ DAPX, $A \leq_{\text{PTAS}}$ Max Independent Set-B.

This last problem belongs to DGLO, hence DAPX $\subseteq$ DGLO$^{\text{PTAS}}$ and the proof of the proposition is complete. 

Among other interesting properties of the class GLO, in [8] it is proved that Max 3-SAT is complete in GLO $\cap$ MAX-SNP with respect to reductions that preserve the quality of local optima (called LOP-reduction; see Definition 3 in Section 2). A related result in [18] shows that MAX-SNP $\subseteq$ Non-Oblivious GLO, a variant of the class GLO defined by means of local search algorithms that are allowed to use more general kinds of objective functions, rather than the natural objective function of the given problem, for improving the quality of the solution (note that by making use of the LOP-reduction to Max 3-SAT, presented in [8], all problems in MAX-SNP can be solved by non-oblivious local search up to a solution of guaranteed quality). Such result is indeed based upon the existence of a MAX-SNP problem (Max k-CSP) which is general enough that all MAX-SNP problems can be formulated as subclasses of it and which therefore plays again the role of a complete problem.

In what follows, we show the existence of complete problems for a large, natural subclass of DGLO. As one can see from Definition 3 in Section 2, the local optimality preserving properties do not depend on the approximation measure adopted. Hence, in an analogous way, we define here a reduction called DLOP which is a PTAS-one with the same local optimality preserving properties as the ones in Definition 3.

Definition 9. A DLOP-reduction is a PTAS-reduction with the same surjectivity, partial monotonicity, locality and dominance properties as an LOP-reduction.

Proposition 7. Given two NPO problems $\Pi$ and $\Pi'$, if $\Pi \leq_{\text{DLOP}} \Pi'$ and $\Pi' \in$ DGLO, then $\Pi \in$ DGLO.

Let DGLO$_0$ be the class of MAX-SNP maximization problems that belong to DGLO and for which the worst value 0 is feasible for any instance (Max Independent Set-B, for example, is such a problem). Note that for the problems of DGLO$_0$, the standard and differential approximation ratios coincide. Now let us consider the closure of DGLO$_0$ under affine transformations. This leads to the following definition.

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Definition 10. Let $\Pi$ be a polynomially bounded NPO problem. Then, $\Pi \in \text{DGLO}'$ if (i) it belongs to $\text{DGLO}_0$, or (ii) it can be transformed into a problem in $\text{DGLO}_0$ by means of an affine transformation; in other words, $\text{DGLO}' = \text{DGLO}_0^{\text{AF}}$. [1]

Now we can prove the following result.

Theorem 7. For any problem $\Pi \in \text{DGLO}'$, $\Pi \leq_{\text{DLOP}} \text{MAX INDEPENDENT SET-B}$.

Proof. Assume $\Pi \in \text{DGLO}'$. We then have the following two cases: (i) $\Pi \in \text{DGLO}_0$ or (ii) can be transformed into a problem in $\text{DGLO}_0$ by means of an affine transformation.

We first deal with case (i), i.e., we consider $\Pi \in \text{DGLO}_0$. Then, the following assertions hold:

- $\Pi \in \text{DGLO}_0$ implies $\Pi \in \text{GLO}$ (by the way $\text{DGLO}_0$ has been defined just above);
- $\Pi \in \text{GLO}$ implies $\Pi \in \text{MAX-SNP}$ (by the definition of $\text{GLO}$);
- by the result of [8]:
  
  \[
  \Pi \leq_{\text{LOP}} \text{MAX 3-SAT}
  \]  
  (9)

Observe next that the L-reduction of [23] between $\text{MAX 3-SAT}$ and $\text{MAX 3-SAT-B}$, transforms any feasible solution for $\text{MAX 3-SAT-B}$ of value $m$ to a solution for $\text{MAX 3-SAT}$ of value $m' = m - K$ where $K$ is a function of $c$ (the rate of the expander used) and of the optimal value of the instance of $\text{MAX 3-SAT}$. Hence, the L-reduction of $\text{MAX 3-SAT}$ to $\text{MAX 3-SAT-B}$ is an $\text{LOP}$-one:

\[
\text{MAX 3-SAT} \leq_{\text{LOP}} \text{MAX 3-SAT-B}
\]  
  (10)

Finally, remark that the L-reduction of [23] between $\text{MAX 3-SAT-B}$ and $\text{MAX INDEPENDENT SET-B}$ is also an $\text{LOP}$-one since it preserves equality of feasible values:

\[
\text{MAX 3-SAT-B} \leq_{\text{LOP}} \text{MAX INDEPENDENT SET-B}
\]  
  (11)

From (9), (10) and (11), and since $\text{LOP}$-reductions are transitive, one concludes that for any $\Pi \in \text{DGLO}_0$, $\Pi \leq_{\text{LOP}} \text{MAX INDEPENDENT SET-B}$. Finally, it is easy to see that for $\text{DGLO}_0$, an $\text{LOP}$-reduction is also a $\text{DLOP}$-one.

We now deal with case (ii), i.e., we assume that $\Pi$ is obtained by means of an affine transformation from a problem $\Pi' \in \text{DGLO}_0$. Since an affine transformation is a $\text{DLOP}$-reduction, $\Pi \leq_{\text{LOP}} \Pi'$ and by case (i) of the proof, $\Pi' \leq_{\text{DLOP}} \text{MAX INDEPENDENT SET-B}$. The proof of the theorem is then completed. [1]

Proposition 8. $\text{MAX CUT}$, $\text{MIN VERTEX COVER-B}$, $\text{MAX SET PACKING-B}$, $\text{MIN SET COVER-B}$ are $\text{DGLO}'$-complete, under $\text{DLOP}$-reductions.

Proof. The result for $\text{MAX CUT}$ and $\text{MAX SET PACKING-B}$ follows from the fact that both problems belong to $\text{DGLO}_0$ and from Theorem 7. The result for $\text{MIN VERTEX COVER-B}$ follows from the fact that $\text{MAX INDEPENDENT SET}$ is affinely transformable to it. Finally, the result for $\text{MIN SET COVER-B}$ follows from the fact that it is a generalization of $\text{MIN VERTEX COVER-B}$. [1]

In all, Figure 3 summarizes the image of the several classes discussed in this section and of their relationships. Note that $\text{MIN MULTIPROCESSOR SCHEDULING}$, or even $\text{MIN}$ and $\text{MAX TSP}$ on graphs with polynomially bounded edge-distances belong to $\text{DGLO}$ ([20, 21]) but neither to $\text{GLO}$, nor to $\text{DGLO}'$. On the other hand, $\text{MIN VERTEX COVER-B}$ belongs to $\text{DGLO}'$ but not to $\text{MAX-SNP}$.
7 Concluding remarks

We have proposed a structure for differential approximability classes. One of the interesting
points in this work is that the results obtained confirm our intuition that differential approxi-
mation is rich enough to motivate further both “structural” and “computational” research. We
have defined natural reductions and proved completeness of natural NPO problems for DAPX
and for a subclass of DGLO. Moreover, we have exhibited the hard problems for DPTAS with-
out, unfortunately, providing complete problems for it. The existence of such problems is an
interesting open problem. Another point for further investigation is the existence of interme-
diate problems in differential approximability classes, i.e., problems which, unless P = NP, or
another complexity theoretic hypothesis, they belong to a certain approximability class and they
are neither complete for it, nor belong to an inferior class.

References

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A A list of NPO problems

We present the list of NPO problems mentioned and/or discussed in the paper, together with a characterization of their worst-value solutions. For most of these problems, comments about their approximability in standard approximation can be found in [2]. MAX WSAT-B and MAX LINEAR WSAT-B are also dealt in [3, 12], respectively.

Maximum and minimum weighted satisfiability (MAX WSAT and MIN WSAT).

Given a boolean formula $\varphi$ with non-negative integer weights $w(x)$ on any variable $x$ appearing in $\varphi$, MAX WSAT consists of computing a truth assignment to the variables of $\varphi$ that both satisfies $\varphi$ and maximizes the sum of the weights of the variables set to 1. We consider that the assignment setting all the variables to 0, even if it does not satisfy $\varphi$, is feasible and represents the worst-value solution for MAX WSAT. In MIN WSAT, the objective is to minimize the total weight of an assignment satisfying $\varphi$. Here we assume the assignment setting any variable to 1 to be feasible (even if it does not satisfy $\varphi$) and to represent the worst-value solution for MIN WSAT. By MAX WSAT-B, we denote the version of MAX WSAT where the variable-weights are polynomially bounded and their sum lies in the interval $[B, 2B]$. By MAX wk-sat, we denote the version of MAX WSAT where no clause contains more than $k$ literals. Finally, by MAX LINEAR WSAT-B we will denote the version of MAX WSAT where the variable-weights are polynomially bounded and their sum lies in the interval $[B, (n/(n - 1))B]$. For both MAX WSAT-B and MAX LINEAR WSAT-B it is assumed that the assignment setting all variables to 0 is feasible and that its value is $B$. Obviously, this assignment represents the worst feasible value for these problems.

Maximum and minimum \{0,1\} integer programming (MAX \{0,1\} INTEGER PROGRAMMING and MIN \{0,1\} INTEGER PROGRAMMING).

Given an integer matrix $A$, an integer vector $b$ and a positive integer vector $c$, the objective is to determine a \{0,1\}-vector $\bar{x}$ satisfying $A \cdot \bar{x} \leq \bar{b}$ and maximizing $c \cdot \bar{x}$ (for MAX \{0,1\} INTEGER PROGRAMMING), or satisfying $A \cdot \bar{x} \geq \bar{b}$ and minimizing $c \cdot \bar{x}$ (for MIN \{0,1\} INTEGER PROGRAMMING). Worst-value solutions here are vectors $\{0\}^n$ and $\{1\}^n$, respectively.

Maximum satisfiability (MAX SAT).

Given a boolean CNF $\varphi$, the objective is to compute a truth assignment to the variables maximizing the number of clauses satisfied. By MAX $k$-SAT, we denote the version of MAX SAT where no clause contains more than $k$ literals. Worst-value solution; the optimal solution of MIN SAT (where we wish to compute a truth assignment to the variables minimizing the number of clauses satisfied) on $\varphi$.

Maximum independent set (MAX INDEPENDENT SET).

Given a graph $G(V, E)$, an independent set is a subset $V' \subseteq V$ such that whenever


\{v_i, v_j\} \subseteq V', v_i v_j \notin E$, and \textsc{max independent set} consists in finding an independent set of maximum size. By \textsc{max independent set}-\textit{B}, we denote \textsc{max independent set} in bounded-degree graphs. Worst-value solution: the empty set.

**Maximum clique (\textsc{max clique}).**

Consider a graph $G(V, E)$. A \textit{clique} of $G$ is a subset $V' \subseteq V$ such that every pair of vertices of $V'$ are linked by an edge in $E$, and \textsc{max clique} consists in finding a maximum size set $V'$ inducing a clique in $G$ (a maximum-size clique). Worst-value solution: the empty set.

**Minimum coloring (\textsc{min coloring}).**

Given a graph $G(V, E)$, we wish to color $V$ with as few colors as possible so that no two adjacent vertices receive the same color. Worst-value solution: $V$.

**Maximum $\ell$-colorable induced subgraph (\textsc{max $\ell$-colorable induced subgraph}).**

Given $\ell < \Delta(G)$ (the maximum graph-degree), \textsc{max $\ell$-colorable induced subgraph} consists of finding, in a graph $G(V, E)$, a maximum-order subgraph $G'$ of $G$ that is $\ell$-colorable. Worst-value solution: the empty set.

**Minimum vertex-covering (\textsc{min vertex cover}).**

Given a graph $G(V, E)$, a \textit{vertex cover} is a subset $V' \subseteq V$ such that, $\forall uv \in E$, either $u \in V'$, or $v \in V'$, and \textsc{min vertex cover} consists of determining a minimum-size vertex cover. Worst-value solution: $V$.

**Minimum set-covering (\textsc{min set cover}).**

Given a collection $\mathcal{S}$ of subsets of a finite set $C$, a \textit{set cover} is a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ such that $\bigcup_{S_i \in \mathcal{S}'} S_i = C$, and \textsc{min set cover} consists of finding a cover of minimum size. We denote by $\textsc{min set cover}-\textit{B}$ the restriction of \textsc{min set cover} on set-systems where $|S_i| \leq B$, $S_i \in \mathcal{S}'$. Also, we denote by $\textsc{min set cover}(K)$ where, given a fixed constant $K$, integer weights less than, or equal to, $K$ are assigned to the sets of $\mathcal{S}$ and the objective consists of minimizing the total weight of a set cover. Worst-value solution for $\textsc{min set cover}$: $\min\{|S|, |C|\}$ and for its weighted version the total weight of $\mathcal{S}$.

**Maximum set-packing (\textsc{max set packing}).**

Given a collection $\mathcal{S}$ of subsets of a finite set $C$, a \textit{set packing} is a sub-collection of mutually disjoint sets of $\mathcal{S}'$ and \textsc{max set packing} consists of determining a set packing of maximum size. We denote by $\textsc{max set packing}-\textit{B}$ the restriction of \textsc{max set-packing} on set-systems where $|S_i| \leq B$, $S_i \in \mathcal{S}'$. Worst-value solution: the empty set.

**Minimum dominating set (\textsc{min dominating set}).**

Given a graph $G(V, E)$, the objective is to compute a subset $V' \subseteq V$ such that $\forall u \in V \setminus V'$, there exists $v \in V'$ for which $uv \in E$. By \textsc{min dominating set}-\textit{B}, we denote \textsc{min dominating set} in bounded-degree graphs. Worst-value solution: $V$.

**Minimum independent dominating set (\textsc{min independent dominating set}).**

Given a graph $G(V, E)$, the objective is to compute a maximal (for the inclusion) independent set of minimum size. Worst-value solution: a maximum independent set.

**Maximum cut (\textsc{max cut}).**

Given a graph $G(V, E)$, the objective is to determine a subset $V' \subseteq V$ maximizing the number of edges whose one endpoint is in $V'$ and the other one in $V \setminus V'$. Worst-value solution: the empty set.
Minimum feedback edge set (MIN FEEDBACK EDGE SET).
Given a graph $G(V, E)$, the objective is to determine a minimum size subset $E' \subseteq E$ such that any cycle in $G$ uses at least one edge in $E'$. Worst-value solution: $E$.

Minimum feedback vertex set (MIN FEEDBACK VERTEX SET).
Given a graph $G(V, E)$, the objective is to determine a minimum size subset $V' \subseteq V$ such that any cycle in $G$ uses at least one vertex in $V'$. By MIN FEEDBACK VERTEX SET-$B$, we denote the version of MIN FEEDBACK VERTEX SET in bounded-degree graphs. Worst-value solution: $V$.

Minimum multiprocessor scheduling (MIN MULTIPROCESSOR SCHEDULING).
Given $n$ tasks $T_1, \ldots, T_n$ together with their respective execution lengths $\ell_1, \ldots, \ell_n$, we search for allocating the tasks to $m$ processors in such a way that the overall deadline of the most used processor is minimized (there are no precedence constraints). We assume that execution lengths are bounded by a polynomial of $n$. Worst-value solution is the one where one allocates all the tasks to only one processor; the value of such solution is $\sum_{i=1}^n \ell_i$.

Binary knapsack (B-KNAPSACK).
Given two integer $n$-vectors $\vec{a}$ and $\vec{b}$ and an integer $B$, the objective is to determine a vector $\vec{x} \in \{0, 1\}^n$ which, under the constraint $\vec{b} \cdot \vec{x} \leq B$, maximizes the scalar $\vec{a} \cdot \vec{x}$. We denote by B-KNAPSACK($n^k$), the restriction of B-KNAPSACK in instances where the greatest number is at most $n^k$. Remark that B-KNAPSACK($n^k$) $\in \mathbf{P}$. Worst-value solution: the vector $\{0\}^n$.

Subset sum (SUBSET SUM).
This is as B-KNAPSACK with the additional constraint that vectors $\vec{a}$ and $\vec{b}$ coincide. Worst-value solution: the vector $\{0\}^n$.

Minimum and maximum traveling salesman problem (MIN and MAX TSP).
Given a complete graph on $n$ vertices, denoted by $K_n$, with positive distances on its edges, MIN TSP (resp., MAX TSP) consists of minimizing (resp., maximizing) the cost of a Hamiltonian cycle (an ordering $(v_1, v_2, \ldots, v_n)$ of $V$ such that $v_nv_1 \in E$ and, for $1 \leq i < n$, $v_iv_{i+1} \in E$), the cost of such a cycle being the sum of the distances of its edges. Worst-value solution of the former is an optimal solution for the latter in the same instance, and vice-versa.

Note that MAX INDEPENDENT SET, MAX SET PACKING and MAX CLIQUE are approximate equivalent under both standard and differential approximation ratios; moreover, the two ratios coincide for each of these problems. For MAX $\ell$-COLORABLE INDUCED SUBGRAPH standard and differential approximation ratios coincide. Furthermore, the differential ratio for MIN VERTEX COVER coincides with the standard and differential ratios for MAX INDEPENDENT SET.