ON HANDLING DENSE COLUMNS OF CONSTRAINT MATRIX
IN INTERIOR POINT METHODS
OF LARGE SCALE LINEAR PROGRAMMING

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Une méthode de traitement des colonnes pleines
pour des programmes linéaires de grande taille
dans les méthodes intérieures

RESUME


Mots-clés: Projections de Karmarkar, Programmation Linéaire, Colonnes Pleines

On handling dense columns of constraint matrix
in interior point methods
of large scale linear programming

ABSTRACT

A method is proposed for handling dense columns of the constraint matrix of large scale linear programming problems. Such columns are known to create dense windows in matrix $A A^T$ that has to be inverted at every iteration of the interior point method. Consequently, Cholesky factor of $A A^T$ becomes dense, which degrades the efficiency of
1. Introduction

We are concerned with solving linear programming problem

\begin{align}
\text{minimize} & \quad c^T y , \\
\text{subject to} & \quad Ay = b , \\
& \quad y \geq 0 ,
\end{align}

where $A \in \mathbb{R}^{m \times n}$, $c, y \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We assume that an interior point method derived from a logarithmic barrier approach is applied to it. Algorithmic techniques however are not discussed in this paper (the reader interested in them is referred to papers of Adler et al. (1989b), Gill et al. (1986), Choi et al. (1990) and other references therein).

Instead, a method is proposed for solving equation

$$AA^T x = d,$$

where $AA^T \in \mathbb{R}^{m \times m}$ and $x, d \in \mathbb{R}^m$, which is crucial for the efficiency of the whole LP code. Let us mention that the equation that have to be solved at every iteration of the method has in general slightly different form from (2). For ease of presentation however, we have omitted matrix $D$ in it and the fact that the matrix $A$ that appear in (2) may differ from the constraint matrix (1b) with a bordered rows.

Although the problem constraint matrix $A$ is usually very sparse, matrix $AA^T$ in equation (2) may be quite dense. Any column of matrix $A$ that has $k$ nonzero entries creates creating dense window of size $k \times k$ in matrix $AA^T$ (subject to its symmetric permutation). Standard approaches to solve (2) such as applying QR factorization to matrix $A^T$ or computing Cholesky decomposition of matrix $AA^T$ (see e.g., chapters 6 and 5 of the book of Golub and Van Loan (1983), respectively) may then be completely inefficient, especially if $k$ is comparable with $m$. Matrix $R$ or Cholesky factor $L$ contain in such case triangular dense window of dimension at least $k$ (its size is often larger due to the fill-in i.e. new nonzero elements added.
during the factorization process).

For the above reasons several methods for 'special treatment' of dense columns in the constraint matrix have been proposed. The first one is to avoid formulating $AA^T$ explicitly and to apply iterative procedure of conjugate gradient (see e.g., Golub and Van Loan 1983, chapter 10) to solve (2). A sparse preconditioner $L_sL_s^T = A_sA_s^T$, where $A_s$ is a sparse part of $A$, i.e. it is matrix $A$ with dense columns removed, is used to accelerate the convergence and improve numerical properties of the method (see e.g., Munksgaard 1980). This approach is due to Gill et al. (1986) and was implemented for example by Adler et al. (1989a). Another approach, due to Gill et al. (1988), is to handle dense columns in the form of the Schur complement and to apply Sherman-Morrison-Woodbury formula (see e.g., Hager 1989) to solve equation (2). It is not iterative in character and when carefully implemented (see e.g., Lustig et al. 1990) it proved to be very efficient.

However, dense columns are not so often present in real-life linear programs (in extended Netlib collection of Gay (1985) only 4 of all 86 problems contain columns dense enough to motivate applying special techniques to handle them). A need then arise for rather easily implementable method that does not add significant overhead to the LP code but well prevents degrading influence of dense columns on the speed of computing projections.

In this paper such a method is proposed. A procedure for preprocessing the linear program is suggested which ends up with creating equivalent problem that has better structured constraint matrix $A$. Consequently, the number of nonzero elements of matrix $AA^T$ is remarkably smaller than that of $AA^T$. A significant reduction of nonzero counts of Cholesky factor of $AA^T$ can thus be obtained.

To achieve this, any variable associated with dense column of length $k$ is replicated and the dense column is cut into pieces. Variable $y_i$ is replaced by $p$ variables $y_{i1}, y_{i2}, \ldots, y_{ip}$ each of these being associated
with appropriate part of the long column that has only \( k/p \) nonzero elements. Summing up, instead of \( k^2 \) nonzero elements in \( AA^T \) we have \( p(k/p)^2 = k^2/p \) nonzero elements in \( p \) small dense windows of \( AA^T \) (we neglect terms linear on \( k \)). Additionally, \( p-1 \) new constraints of type

\[
y_{ij} - y_{ij+1} = 0, \quad j = 1,2,\ldots,p-1,
\]

are bordered to a linear program.

Let us observe that the method proposed is particularly easy to implement and that it can be included into the preprocessing of a linear program. Consequently, the overhead added by it to the LP code is negligible.

The method is derived from techniques used widely in multistage stochastic linear programming, where the variables of earlier stages are sometimes replicated to allow formulating all the scenarios independently (see e.g., Rockafellar and Wets 1987 and Lustig et al. 1989) and obtain easily decomposable structures. Mulvey and Ruszczyński (1990) indicate advantages of applying such approach to two specially structured linear programming problems: multistage stochastic and multicommodity network ones.
Finally, Sections 5 and 6 bring numerical results and conclusions, respectively.

2. Dense columns in linear programs

We start this section from showing the degrading influence of the existence of a single dense column in the LP constraint matrix on the efficiency of an interior point method of linear programming. The following example illustrates the problem. Let us suppose that $m = 8$, $n = 9$ and the sparsity pattern of a constraint matrix has the form

$$
A = \begin{bmatrix}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{bmatrix}, \quad AA^T = \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix},
$$

(4)

so $AA^T$ is an $8 \times 8$ full matrix with 28 elements below the diagonal that have to be stored. The Cholesky factor $L$ has also 28 entries below the diagonal. In other words, full matrix technology has to be applied to solve this sparse problem (let us observe that every simplex basis matrix of the problem is triangular and sparse).

Let us now introduce variables $y_{11}$, $y_{12}$ (both equal to $y_1$), split the first column of $A$ into two equal parts and add a constraint of type (3) that ties up replicated variables. The new constraint matrix has now the form

$$
A = \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix}, \quad AA^T = \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{bmatrix}
$$

(5)

so $AA^T$ is a $9 \times 9$ matrix with only 20 elements below the diagonal and so is its Cholesky factor.

Consequently, instead of solving linear program with constraint matrix (4) we may solve better structured problem with matrix (5) and (at every
iteration of the interior point method) take advantage of the lower density of Cholesky factor $L$ of $AA^T$.

3. Some theory of splitting

We show in this section how the splitting can be done in a given linear program to obtain equivalent problem that has better sparsity pattern, i.e. leading to sparser Cholesky factor of matrix $AA^T$.

Let us consider a linear programming problem of the form (1). To simplify the presentation we assume that the first column of its constraint matrix is split into two parts. We thus introduce partition

$$ A = \begin{bmatrix} a_1 & A_0 \end{bmatrix} . $$

and define an $(m+1) \times (n+1)$ matrix

$$ A = \begin{bmatrix} a_{11} & a_{12} & A_0 \\ 1 & -1 & 0 \end{bmatrix} $$

which together with vectors

$$ b = (b^T, 0)^T \in \mathbb{R}^{m+1} , $$

$$ c = (c_1, 0, c_2, \ldots, c_n)^T \in \mathbb{R}^{n+1} , $$

$$ y = (y_{11}, y_{12}, y_2, \ldots, y_n)^T \in \mathbb{R}^{n+1} $$

determine new LP problem

$$ \begin{aligned}
\text{minimize} & \quad c^T y , \\
\text{subject to} & \quad Ay = b , \\
& \quad y \geq 0 , 
\end{aligned} $$

Let us suppose that

$$ a_{11} + a_{12} = a_1 . $$

Observe that $y$ is a feasible point of (1) if and only if $y$ defined by

6
\[ y_1 = y_{11} = y_{12} \]
\[ y_{i} = 0, \quad i = 2, 3, \ldots \]

is a feasible point of (9).

**Proposition 3.1.** Optimal solution of (1) exists if and only if optimal solution of (9) exists and if they both exist, then they satisfy (11).

We omit straightforward proof of this result and pass to discuss its practical significance.

Any dense column of the constraint matrix containing, say, \( k \) nonzero entries can be split into two parts (equation (10)) containing \( q \) and \( k-q \) elements, respectively. Thus, instead of a dense window of size \( k \times k \) in \( AA^T \), we obtain two dense windows of sizes \( (q+1) \times (q+1) \) and \( (k-q+1) \times (k-q+1) \), respectively. The best possible reduction of nonzero elements in \( AA^T \) is obviously obtained when \( q = k/2 \).

Let us observe that part \( a_{12} \) of (7) can be further split into two shorter parts (a new linking constraint will have to be bordered to a constraint matrix then). It is equivalent to splitting the original column \( a_1 \) of (6) into three parts. More generally (as follows from proposition 3.1 by induction), any dense column of length \( k \) can be split into \( p \) equal parts so the dense \( k \times k \) window in \( AA^T \) may be replaced with \( p \) dense windows each of dimension \( k/p+1 \) or \( k/p+2 \) in \( AA^T \) matrix. If we neglect linear terms and omit completely the influence of the rest of the constraint matrix (submatrix \( A_0 \) in equality (6)) on the sparsity structure of \( AA^T \), then we may conclude that instead of \( k^2 \) elements in \( AA^T \) we now deal with \( k^2/p \) nonzero entries in \( AA^T \). Such substantial reduction of the number of entries in the matrix \( AA^T \) is supposed to lead to a remarkable savings of space required by its Cholesky factor and to a significant acceleration of both symbolic and numerical phases of the decomposition of \( AA^T \).
Another advantage of the approach presented is that splitting itself does not cause nonsingularity of $AA^T$.

Proposition 3.2. Let $A$ and $A$ be defined by (6) and (7), respectively. If (10) holds and $\text{rank}(A) = m$, then $\text{rank}(A) = m + 1$.

Proof. The proof follows easily the observation that replacing the second column of (7) by the sum of its two first columns we obtain

$\begin{bmatrix}
    a_{11} & a_1 \\
    1 & 0
\end{bmatrix}
= \begin{bmatrix}
    a_{11} & A \\
    1 & 0
\end{bmatrix}$.

Let us mention that removing dense columns that takes place when a preconditioner for conjugate gradient method is defined or Schur complement approach is applied, may cause serious stability problems. If $\text{rank}(A_0)$ is less than $m$, which is often the case since the removed dense column is usually an integral part of the problem formulation, then $A_0A_0^T$ becomes singular. The Schur complement approach would fail in such a situation while the conjugate gradient method might still work under the condition that the rank deficiency of $A_0A_0^T$ is removed to allow computing the preconditioner (see e.g., Munksgaard 1980). To achieve this, a positive constant is usually added to too small diagonal elements of $A_0A_0^T$ when its Cholesky factor is computed or columns that have too small diagonal elements are simply replaced with appropriate unit vectors (see e.g., Adler et al. 1989b). It leads however to less accurate preconditioner, which may slow down the convergence of the method. It may seem that introducing artificial variables should ensure nonsingularity of $A_0A_0^T$ allowing safe application of Schur complement method. Computational practice of Lustig et al. (1980) did not however confirm this since artificial variables tend to zero when optimum is approached and the instability still manifests itself (an automatical switch
to preconditioned conjugate gradient method is suggested in such case).

In the approach presented in this paper the analogous difficulties are less probable (proposition 3.2. ensures that $AA^T$ is always nonsingular). If they anyway occur, then there still exists the possibility of switching to preconditioned conjugate gradient method.

4. Implementation

The method of splitting dense columns of the constraint matrix has been experimentally implemented. It is advantageous that the analysis of the sparsity pattern of $A$ have to be done only once and that it can be completed before the solution of the problem starts. It thus can be included to preprocessing of the LP coefficient matrix.

Let us mention however that it is not the only way of implementing it. Vanderbei (1990) suggests that it may be preferable to solve the problem in the same form as stated originally and use splitting technique implicitly in a module that solves equation (2). Advantage can then be taken of the smaller size of the problem since the linking constraints of type (3) are not bordered to it. On the other hand, the analysis that is necessary to determine optimal splitting must be repeated in such case at every iteration of the interior point method, which leads to an unnecessary increase of the computation time. Vanderbei (1990) selects the simplest possible splitting
Cholesky factorization. Although the time of splitting in our implementation never exceeds 15% of the time required by the reordering and the symbolic factorization, we want to avoid adding it at every iteration.

Let us now pass to the description of a heuristic that has been applied to determine the partition of a long column. First, let us analyse in more detail the consequences of splitting a column of $A$ for the sparsity pattern of $AA^T$. As before, for ease of presentation we assume that column $a_1$ in (6) is completely dense and that it is cut into two equal parts

$$A = \begin{bmatrix}
a_{11} & 0 & A_1 \\
0 & a_{12} & A_2 \\
1 & -1 & 0 \\
\end{bmatrix}$$

(12)

If column $a_1$ of (6) is not completely dense, then the analysis may always be restricted to a submatrix of $A$ built of those rows only which contain nonzero entries of $a_1$. From (12) we obtain

$$AA^T = \begin{bmatrix}
dense & A_1^T A_2 & a_{11} \\
A_2 A_1^T & dense & a_{12} \\
a_{11} & a_{12} & 2 \\
\end{bmatrix}$$

(13)

and the problem of finding optimal splitting becomes equivalent to choosing such a partition of $a_1$ that ensures minimum number of nonzero entries in parts $A_1^T A_2$ and $A_2 A_1^T$ of (13). We may add here that when $a_1$ of (6) is to be cut into $p > 2$ equal parts, then it is done in $p-1$ successive steps each determining the partition like (12) into $a_{11}$ containing $k/p$ entries and $a_{12}$ containing the rest of nonzeros such that the number of elements in $A_1^T A_2$ and $A_2 A_1^T$ is minimized.

Let us observe that if $A_0$ of (6) is a staircase matrix, as for example in case of multistage stochastic or multicommodity network problems analyzed by Lustig et al. (1989) and by Mulvey and Ruszczynski (1990), then its structure automatically determines optimal column partition for which $A_1^T A_2$...
= 0 and \( A_1A_1^T = 0 \).

Let us now pass to the presentation of the heuristic that determines optimal splitting. Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) define disjoint subsets of row indices of these nonzero elements of column \( a_1 \) that have been included to \( a_{11} \) and \( a_{12} \), respectively. The partition (12) is obtained in the iterative process which starts with a trivial splitting: \( \mathcal{A}_1 \) is empty and \( \mathcal{A}_2 \) is the list of row numbers of all nonzero entries of \( a_1 \). At every step of it one element is removed from list \( \mathcal{A}_2 \) and added to \( \mathcal{A}_1 \). The process is continued until \( \mathcal{A}_1 \) (and consequently \( a_{11} \)) has the required number of entries \((k/2 \text{ if the column is to be split into two equal parts or } k/p \text{ if it is to be cut into } p \text{ equal parts})\). After \( 1-1 \) steps (where \( 1-1 < k/2 \)) we may determine:

\[
\begin{align*}
\text{r}_j & \quad \text{the number of entries of } j\text{-th column of } A_1 \text{ and} \\
\text{s}_j & \quad \text{the number of entries of } j\text{-th column of } A_2, \quad j = 1, 2, \ldots, n-1
\end{align*}
\]

and the number of nonzeros in part \( A_1A_2^T \) of (13)

\[
P = \sum_{j=1}^{n-1} r_j s_j .
\]

(14)

In the next step an element of \( \mathcal{A}_2 \) is looked for such that moving it from \( \mathcal{A}_2 \) to \( \mathcal{A}_1 \) gives the largest possible reduction of the penalty term (14). For a nonzero entry \( (a_{12})_i \) that appears in row \( i \) of \( a_{12} \) we thus determine

\[
\delta P(i) = \sum_{j:w_{ij} \neq 0} (s_j - r_j - 1) ,
\]

(15)

where \( w_{ij} \) defines sparsity pattern of row \( i \) of \( A_2 \) i.e.:

\[
w_{ij} = \begin{cases} 
0, & \text{if } a_{ij} = 0, \\
1, & \text{otherwise}.
\end{cases}
\]

(16)

It is easy to observe that \( P + \delta P(i) \) gives the new value of penalty indicator (14) which will be obtained if an element \( i \) leaves \( \mathcal{A}_2 \) and is added to \( \mathcal{A}_1 \).

We then calculate \( \delta P(i) \) for all elements that are still in \( a_{12} \), determine \( i_* \) for which \( \delta P(i) \) is minimum and "move" index \( i_* \) from \( \mathcal{A}_2 \) to \( \mathcal{A}_1 \).
Updating $r_j$ and $s_j$ completes iteration $1$. This gives the following

**Algorithm 1.**

$\mathcal{A}_1 := \emptyset$

$\mathcal{A}_2 := \{i: (a_i)_1 \neq 0\}$

$r_j := 0, j = 1, 2, \ldots, n-1$

$s_j := \text{number of entries of } j\text{-th column of } A_2, j = 1, 2, \ldots, n-1$

for $l = 1, 2, \ldots, k/2$

for $i \in \mathcal{A}_2$

$$d_i := \sum_{j: w_{ij} \neq 0} (s_j - r_j - 1)$$

$d_i := \min_{i \in \mathcal{A}_2} \{d_i\}$

$\mathcal{A}_1 := \mathcal{A}_1 \cup \{i^*_1\}$

$\mathcal{A}_2 := \mathcal{A}_2 \setminus \{i^*_1\}$

for $j = 1, 2, \ldots, n-1$

if $w_{i^*_1j} = 1$ then

$r_j := r_j + 1$

$s_j := s_j - 1$

end if

The algorithm ends up with determining the partition of $a_1$ into $a_{11}$ and $a_{12}$. The row indices defining this partition are in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

Let us observe that a comfortable access to the sparsity pattern of rows of $A$ is necessary to implement the above algorithm efficiently. It is none restriction however since the access to the constraint matrix of the linear program both by columns and by rows is an elementary requirement satisfied by all codes based on interior point methods.
In our preliminary implementation the analysis is performed while reading MPS formatted data of the problem. All columns of A that are longer than a given threshold length are marked as those to be split. Additionally, space is left in internal data structures of the LP code for new short columns (parts of a long one) and for nonzero entries that will appear in the linking rows of type (3) bordered to a constraint matrix. Row linked lists of positions of row entries are created for the submatrix of A that contains only short columns. They allow a comfortable access to sparsity pattern of rows of this part of A that will remain unchanged as e.g. A_0 in (6) and ensure the efficiency of the execution of the two inner loops of Algorithm 1 (calculating $d_1$ and updating $r_j$ and $s_j$). When the reading of the constraint matrix is completed, all the earlier marked long columns are successively split and its short parts are added to internal data structures. The row linked lists are also updated after every short column addition so all the short columns (including the parts of already partitioned long ones) are taken into account when the optimal splitting of the next long column is looked for.

5. Numerical results

The method described has been tested on two real-life problems from Gay's (1985) Netlib collection (ISRAEL and SEBA) which are widely reported as particularly difficult for the reason of containing dense columns. Problem statistics are given in Table 1.

<table>
<thead>
<tr>
<th>Problem</th>
<th>ROWS</th>
<th>COLS</th>
<th>ELTS</th>
<th>average col. len.</th>
<th>maximum col. len.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ISRAEL</td>
<td>174</td>
<td>142</td>
<td>2269</td>
<td>16</td>
<td>136</td>
</tr>
<tr>
<td>SEBA</td>
<td>515</td>
<td>1028</td>
<td>4352</td>
<td>4</td>
<td>230</td>
</tr>
</tbody>
</table>
Table 2 presents the efficiency of the method of splitting dense columns applied to ISRAEL. Its first column contains the threshold length of the column of $A$ such that any column longer than it has been split. The following columns contain: the number of added constraints, the number of nonzero elements of the lower triangular portion of the $A^T A$ matrix after the splitting, the number of the subdiagonal nonzero elements of its Cholesky factor and the fill-in obtained during the factorization. Before the symbolic factorization was performed the matrix $A^T A$ has been reordered by a symmetric permutation resulting from the minimum degree heuristic (see e.g., chapter 5 of the book of George and Liu (1981) or section 10.9 of the book of Duff et al. (1989)) that is supposed to minimize the fill-in
Table 3. Efficiency of splitting for SEBA.

<table>
<thead>
<tr>
<th>THRESHOLD LENGTH</th>
<th>CONSTRAINTS ADDED</th>
<th>NONZEROS</th>
<th>RELATIVE SAVINGS (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\mathbf{A}^\mathbf{T}$</td>
<td>L</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>51400</td>
<td>53728</td>
</tr>
<tr>
<td>220</td>
<td>2</td>
<td>50038</td>
<td>52887</td>
</tr>
<tr>
<td>210</td>
<td>9</td>
<td>41369</td>
<td>44691</td>
</tr>
<tr>
<td>200</td>
<td>9</td>
<td>41369</td>
<td>44691</td>
</tr>
<tr>
<td>190</td>
<td>13</td>
<td>29453</td>
<td>33006</td>
</tr>
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<td>180</td>
<td>14</td>
<td>28862</td>
<td>32192</td>
</tr>
<tr>
<td>110</td>
<td>16</td>
<td>28587</td>
<td>31322</td>
</tr>
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<td>100</td>
<td>23</td>
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<td>21154</td>
<td>24961</td>
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<td>70</td>
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<td>18182</td>
<td>22994</td>
</tr>
<tr>
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<td>17299</td>
<td>21109</td>
</tr>
<tr>
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<td>51</td>
<td>15409</td>
<td>19489</td>
</tr>
<tr>
<td>40</td>
<td>65</td>
<td>13656</td>
<td>18804</td>
</tr>
<tr>
<td>30</td>
<td>93</td>
<td>12026</td>
<td>19293</td>
</tr>
<tr>
<td>20</td>
<td>137</td>
<td>10299</td>
<td>27537</td>
</tr>
</tbody>
</table>

It easily follows from the analysis of Tables 2 and 3 that the method presented gives significant memory savings when compared with direct solution of equation (2) by applying Cholesky factorization to the matrix $\mathbf{A}^\mathbf{T}$ constructed from the original constraint matrix of the problem. For example, when all columns longer than 50 are split in the problem ISRAEL, then the Cholesky factor of $\mathbf{A}^\mathbf{T}$ has 33% less nonzero entries than the one of $\mathbf{A}^\mathbf{T}$. Similarly, splitting all columns longer than 50 in SEBA reduces three times the number of nonzero elements of the Cholesky matrix. Time savings obtained are also significant since the reduction of nonzeros in $\mathbf{A}^\mathbf{T}$ remarkably simplifies the reordering, the symbolic factorization and, finally, significantly accelerates the numerical factorization. The application of the splitting technique reduced the time of the numerical phase of the decomposition by factors 2.5 and 12 for problems ISRAEL and SEBA, respectively.

Let us also mention that splitting alone took always about 5-10% of the time required by the reordering and the symbolic factorization so its contribution to the time of the whole solution process is negligible.
6. Conclusions

The method of splitting dense columns has the following advantages:
- it is easily implementable (preprocessing LP data);
- it never cause singularity of $AA^T$ (assuming that the original problem is well formulated i.e. it does not have empty rows or linearly dependent constraints);
- it does not require any additional memory;
- it is fast.

It thus seems to be a useful option for including into any LP code based on logarithmic barrier approach.

References


