A THEORETICAL FRAMEWORK FOR ANALYSING
THE NOTION OF RELATIVE IMPORTANCE
OF CRITERIA

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UN CADRE DE TRAVAIL THÉORIQUE POUR L’ANALYSE
DE LA NOTION D’IMPORTANCE RELATIVE DES CRITÈRES

Résumé :

L’aide multicritère à la décision requiert le plus souvent l’utilisation de poids, coefficients d’importance, ou bien d’une hiérarchie de critères, de seuils de veto, ... Ceux-ci sont des paramètres d’importance qui visent à différencier le rôle dévolu à chaque critère dans la construction des préférences globales. De nombreux auteurs ont proposé des méthodes d’évaluation de tels paramètres mais peu d’entre eux ont cherché à analyser en détail ce qui est sous-jacent à la notion d’importance relative des critères et à en donner une définition formelle précise.

Dans ce travail, notre objectif est de définir, pour cette notion, un cadre d’analyse valides dans des conditions très générales. Dans ce cadre, il apparaît clairement que l’importance des critères est prise en compte de manière très différente dans les diverses procédures d’agrégation. Ce cadre permet d’analyser sous un nouvel angle des questions fondamentales telles que :
- dans quelles conditions est-il possible d’affirmer qu’un critère est au moins aussi important qu’un autre ?
- les paramètres d’importance des diverses procédures d’agrégation sont-ils dépendants ou non du codage des critères ?
- Quels sont les liens existant entre les notions d’importance des critères et de plus ou moins grande compensation des préférences ?

Ce cadre de travail théorique semble suffisamment général pour poursuivre les recherches en vue de définir des méthodes d’évaluation des paramètres d’importance valides d’un point de vue théorique.

Mots clé : Aide Multicritère à la Décision, Importance des critères, Analyse Théorique.
A THEORETICAL FRAMEWORK FOR ANALYSING
THE NOTION OF RELATIVE IMPORTANCE OF CRITERIA
INTRODUCTION

This paper deals with Multiple Criteria Decision Aid (MCDA). In the present work, we will suppose defined:
- a set \( A \) of alternatives,
- a family \( F \) of \( n \) semi-criteria \( g_1, g_2, \ldots, g_n \) (\( n \geq 3 \)) pertinent in a given decision context (\( F = \{1, 2, \ldots, n\} \)).

More precisely, we suppose that the preferences of the Decision Maker (DM) concerning the alternatives of \( A \) are formed and argued, in the context considered, by reference to \( n \) points of view adequately reflected by the criteria contained in \( F \).

We denote by \( g_j(a) \) the value attributed to the alternative \( a \) on the \( j \)-th criterion; we will call this value the \( j \)-th performance of \( a \); \( g_j(a) \) is a real number. The greater it is, the better the alternative \( a \) is for the DM. Specifically, we only suppose that \( g_j(a) \geq g_j(b) \) implies that the alternative \( a \) is at least as good as the alternative \( b \) with respect to the \( j \)-th point of view. When considering the only point of view formalized by \( g_j \) the imprecision, and/or the uncertainty, and/or the inaccurate determination of performances may lead the DM to judge that \( a \) is indifferent to \( b \) when \( g_j(a) = g_j(b) \), \( \forall i \neq j \), even if \( g_j(a) \neq g_j(b) \). So as to account for such a statement, criteria contained in \( F \) are usually considered to be semi-criteria, i.e., such that:

\[
\begin{align*}
  a & \succP_j b \iff g_j(a) > g_j(b) + q_j \\
  a & \simI_j b \iff |g_j(a) - g_j(b)| \leq q_j
\end{align*}
\]

where
- \( q_j \) the indifference threshold, represents the maximum difference of performance compatible with an indifference situation\(^1\);
- \( \succP_j b \) is to be interpreted as a preference for \( a \) over \( b \) on criterion \( g_j \);
- \( \simI_j b \) is to be interpreted as an indifference between \( a \) and \( b \) on criterion \( g_j \).

\( P_j \) is called the preference relation restricted to the \( j \)-th criterion (the same terminology holds for \( I_j \)). \((I_j, P_j)\) defines a semi-order on \( A \).

From a comprehensive point of view, we refer to a position that accounts for the \( n \) points of view simultaneously. From such a comprehensive point of view, when considering two performance vectors \((g_1(a), g_2(a), \ldots, g_n(a))\) and \((g_1(b), g_2(b), \ldots, g_n(b))\), the DM can have
These difficulties stem from zones of imprecision and conflict in the DM's mind. So as not to compel the DM to make arbitrary judgments, it is prudent and usual (see [Roy 85]) to introduce a third relation in order to model the DM's preferences at a comprehensive level. This relation, incomparability, will be denoted by R which reflects the absence of positive reasons for choosing one of the three possibilities given above or the willingness to put off such a choice. Two alternatives a and b are said to be incomparable (aRb) if, of the three assertions aIb, aPb and bPa, none is valid. Thus, for every pair of alternatives (a, b), one and only one of the four following assertions is valid:

\[
\begin{align*}
\text{aPb} \\
\text{aIb} \\
\text{bPa} & \quad [1] \\
\text{bRa}
\end{align*}
\]

Let us consider the outranking relation S=\text{P} \sqcup \text{I}, aSb being interpreted as "a is at least as good as b." As aPb \Rightarrow aSb \Rightarrow \text{not}[bPa], the four assertions of system [1] correspond to the following assertions:

\[
\begin{align*}
\text{aPb} \\
\text{aSb and not[aPb]} \\
\text{bPa} & \quad [2] \\
\text{not[aSb] and not[bPa]}
\end{align*}
\]

In this paper, we consider a comprehensive preference system using three relations (I, P, R).² Let us recall that a standard approach to MCDA consists of grounding the construction of the comprehensive preferences on the restricted preferences (I, and P, \forall j \in F) corresponding to the n criteria. This is done through a so-called Multiple Criteria Aggregation Procedure (MCAP). In the MCAP, all criteria are not supposed to play the same role; the criteria are commonly said not to have the same importance. This is why there are, in the MCAP, parameters that aim at specifying the role of each criterion in the aggregation of performances. We will call such parameters importance parameters. They aim at accounting for how the DM attaches importance to the points of view modelled by the criteria.

The nature of these parameters varies across MCAP. The way of formalizing the relative importance of each criterion differs from one aggregation model to another. All this is done, for instance, by means of:

- scaling constants in Multiattribute Utility Theory (see [Keeney & Raiffa 76,93]),
- a weak-order on F in lexicographic techniques or a complete pre-order on F in the ORESTE method (see [Roubens 82]),
- intrinsic weights in PROMETHEE methods (see [Brans et al. 84]),
- intrinsic weights combined with veto thresholds in ELECTRE methods (see [Roy & Bertier 73] and [Roy 91]).

² The reader will find more details concerning the basic concepts of preference modelling in [Vincke 90].
- tradeoffs or substitution rates in the Zionts and Wallenius procedures (see [Zionts & Wallenius 76, 83]),
- aspiration levels in interactive methods (see [Vanderpooten & Vincke 89]),
- eigen vectors of a pairwise comparison matrix in the AHP method (see [Saaty 80]).

What exactly is covered by this notion of importance? What does a decision-maker mean by assertions such as "criterion g_j is more important than criterion g_i," "criterion g_j has a much greater importance than criterion g_i?", etc. Let us recall, by way of comparison, that the assertion "b is preferred to a" reflects the fact that, if the decision-maker must choose between b and a, he is supposed to decide in favor of b. So, it is possible to test whether this assertion is valid or not. There is no similar possibility for comparing the importance of criteria. These remarks lead more generally to question us on what gives meaning in practice to this notion of importance. We will come back to this question at the end of the paper by showing that the framework presented in the paper helps to progress in the understanding of how to elicit this notion. Moreover, the way this notion is taken into account, within the framework of different models mentioned above, by means of importance parameters reveals deep differences, as will be shown below.

This paper has a double purpose:

i) to propose a formal framework allowing us to give sense to the notion of relative importance of criteria under very general conditions,

ii) to use this formal framework to give some partial answers to questions such as:
   - under what conditions is it possible to state that one criterion is more important than another?
   - are importance parameters of the various MCAPs dependent on or independent of the encoding of criteria?
   - what are the links between the two concepts of importance of criteria and the compensatoriness of preferences?

It seems to us that the proposed theoretical framework increases our capability to interpret the information obtained through interacting with the DM in order to express formally his positions concerning the respective role of the various criteria through numerical values assigned to importance parameters. This question is not directly addressed in this paper (certain aspects of this problem are studied in [Mousseau 93]). In any case, the theoretical framework proposed here highlights some basic traps to be avoided when assigning values to importance parameters.

We will introduce, in section 1, a formal definition of the notion of relative importance of criteria as well as the theoretical framework we propose to analyse it. In section 2, we will present some results on the theoretical framework introduced. This will lead us to specify, in
section 3, how the relative importance of criteria is taken into account in some basic aggregation procedures. In the fourth section, we will show how the proposed theoretical framework sheds new light on certain fundamental problems. In the last section, we will consider the two ways in which it is possible to give meaning to the notion of importance of criteria and show the interest of the proposed framework in both approaches.
Basic empirical hypothesis: The relative importance of criterion $g_i$ is characterized by:
- the set of "situations" in which $aP_b$ and $aP_b$ hold simultaneously,
- the set of "situations" in which $aP_b$ and $aS_b$ hold simultaneously,
- the set of "situations" in which $aP_b$ and not($bPa$) hold simultaneously.

1.2. Notations

Let us denote by:
- $X_i$ the ordered set of possible performances on criterion $g_i$ ($X_i \subseteq \mathbb{R}$).
- $X = \prod_{i=1}^{n} X_i$ the set of performance vectors.
- $x = (x_1, x_2, ..., x_n) \in X$ a performance vector corresponding to an alternative $a$ such that $g_i(a) = x_i, \forall i \in F$.

$X_j = \prod_{i \neq j} X_i$ the set of performance vectors from which the $j^{th}$ component is removed.

$x_{-j} = (x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n) \in X_{-j}$

In order to simplify the notations, we will denote by:
- $x = (x_{-j}, x_j)$
- $(x_{-j}, x_j) \mathcal{P} (x_{-j}, y_j) \iff x \mathcal{P} y$

The dominance relation $\Delta$ relatively to $F$ is defined by:
- $\forall x, y \in X^2, x \Delta y \iff x_i \geq y_i, \forall i \in F$

The data of the preferences $(I, \mathcal{P})$ restricted to the $n$ viewpoints, together with the comprehensive preference system $\Psi$ (I, P, R), define what we call a preference structure $\Psi$ on $X$.

Considering a preference structure $\Psi$ on $X$, the following sets are defined for all $x_j, y_j \in X_j^2$ such that $x_j \mathcal{P}_j y_j$:
- $\Delta^I_j(x_j, y_j) = \{ (x_{-j}, x_j), (x_{-j}, y_j) \in X_j^2 \text{ such that } (x_{-j}, x_j) \mathcal{P} (x_{-j}, y_j) \}$
- $\Delta^S_j(x_j, y_j) = \{ (x_{-j}, x_j), (x_{-j}, y_j) \in X_j^2 \text{ such that } (x_{-j}, x_j) \mathcal{S} (x_{-j}, y_j) \}$
- $\Delta^N_j(x_j, y_j) = \{ (x_{-j}, x_j), (x_{-j}, y_j) \in X_j^2 \text{ such that } \text{not}((x_{-j}, y_j) \mathcal{P} (x_{-j}, x_j)) \}$

The set $\Delta^I_j(x_j, y_j)$ ($\Delta^S_j(x_j, y_j)$ and $\Delta^N_j(x_j, y_j)$, respectively) contains all combinations of performances $(x_{-j}, x_j)$ and $(x_{-j}, y_j)$ that it is possible to combine with $x_j$ and $y_j$ when $x_j \mathcal{P}_j y_j$, so as to obtain $x \mathcal{P} y$ ($x \mathcal{S} y$ and $\text{not} y \mathcal{P} x$, respectively) on the comprehensive preference level.

5 In this characterization the contribution of criterion $g_i$ to the preferences at the comprehensive level is considered only through the situations of strict preference at the restricted level of criterion $g_i$. This does not seem restrictive when all criteria are semi-criteria, but it can become restrictive when $F$ contains pseudo-criteria (see [Roy & Vincze 84]).

6 The question of where does the comprehensive preferences come from will be addressed in section 5.
Since those sets are defined only $\forall x_j, y_j \in X^2_j$ such that $x_j P y_j$, we will not mention this condition of definition explicitly each time we refer to those sets.

1.3. Formal definition of the relative importance of criteria

The following definition consists of a simplification of the one proposed in [Roy 91a] analyzing the more general case of pseudo-criteria. Using the notations given above, this definition formalizes the basic empirical hypothesis of section 1.1.

**Definition 1.1:** the importance of a semi-criterion $g_j$ is characterized by the sets $\Delta^y_j(x_j, y_j), \Delta^S_j(x_j, y_j), \Delta^E_j(x_j, y_j)$ (defined $\forall x_j, y_j \in X^2_j$ such that $x_j P y_j$). The values attributed to the importance parameters must be grounded on the range and the shape of the preceding sets.

This highlights the restricted nature of the importance parameters relative to the complexity of the notion of RIC.

Let us illustrate this definition with some important specific cases:

- $\Delta^y_j(x_j, y_j) = X^2_j$ means that $(x_j, x_j) P (y_j, y_j)$ holds $\forall x_j, y_j \in X^2_j$ such that $x_j P y_j$ and $\forall (x_j, y_j) \in X^2_j$. Criterion $g_j$ imposes its viewpoint and is thus a dictator (see 3.2).

- $\Delta^S_j(x_j, y_j) = \{(x_j, y_j) \in X^2_j$ such that $\forall i \in F[j] x_i \geq y_i - q_i$ (i.e., such that $x_j S y_j)$\} means that the only cases in which $x_j P y_j$ and $(x_j, x_j) P (y_j, y_j)$ hold simultaneously are the cases in which no other criterion is opposed to $g_j$ ($x_j S y_j \iff \not x_j P x_j$). In other words, criterion $g_j$ imposes its decision on the comprehensive level only when no other criterion is opposed to it.

- $\Delta^E_j(x_j, y_j) = X^2_j(S)$ and $\Delta^E_j(x_j, y_j) \subseteq \Delta^y_j(x_j, y_j)$ means that the importance of criterion $g_j$ is reduced (relative to the preceding case), since even when no criterion is opposed to it, $g_j$ does not always impose a strict preference on the comprehensive level.

- $\Delta^E_j(x_j, y_j) = \Delta^y_j(x_j, y_j) \forall x_j, y_j \in X^2_j$ such that $x_j P y_j$ means that the only cases in which $x_j P y_j$ appears compatible with $y_j P x$ are those where $x_j S y_j$ holds. Let us remark that other cases can exist, particularly when a criterion $g_j$ opposes a veto to $y_j S x$ (see 2.2.4).

2. INTERPRETATION AND PROPERTIES

2.1. Interpretation

As $a P b \Rightarrow a S b \Rightarrow \not b P a$, the following sequence of inclusions holds (see 2.2.1):

$\forall x_j, y_j \in X^2_j$, $\Delta^y_j(x_j, y_j) \subseteq \Delta^S_j(x_j, y_j) \subseteq \Delta^E_j(x_j, y_j)$

If $F$ satisfies the cohesion axiom (see appendix A), $\Delta^E_j(x_j, y_j) \supseteq X^2_j(S)$

Moreover, $\Delta^y_j(x_j, y_j) \supseteq X^2_j(S)$ holds if $\forall j \in F$, $[x_j P y_j$ and $x_j = y_j \forall i \neq j] \Rightarrow x_j P y_j$. 

6
From this fact, it is possible to give a concrete interpretation of the differences (see figure 1):

\[ C_j(x_j,y_j) = \{ (x_{j'},y_{j'}) \in \Delta_j(x_j,y_j) \text{ such that } (x_{j'},y_{j'}) \notin X_j^2(S) \} \]

Reflects the ability of criterion \( g_j \) to convince, i.e., the cases in which the comprehensive preference between two alternatives is in favor of the partial preference on criterion \( g_j \) in spite of the opposition of other criteria.

\[ E_j(x_j,y_j) = \Delta_j^S(x_j,y_j) \Delta_j^p(x_j,y_j) \]

Reflects the ability of criterion \( g_j \) to equilibrate, i.e., the cases in which the partial preference on criterion \( g_j \) allows us to obtain an indifference between the two alternatives in spite of the opposition of criteria.

\[ O_j(x_j,y_j) = \Delta_j^S(x_j,y_j) \Delta_j^p(x_j,y_j) \]

Reflects the ability of criterion \( g_j \) to oppose, i.e., the cases in which the partial preference on criterion \( g_j \) is opposed to other criteria in such a way that the two alternatives are incomparable on the comprehensive level (see result 2.2.4).

**Figure 1**: Interpretation of the sets \( C_j(x_j,y_j), E_j(x_j,y_j) \) and \( O_j(x_j,y_j) \) in the case where \( \Delta_j^p(x_j,y_j) \supseteq X_j^2(S) \).

The illustrations presented in section 3 will allow the reader to verify that it is interesting to compare the relative size of the sets \( C_j(x_j,y_j), E_j(x_j,y_j) \) and \( O_j(x_j,y_j) \) in order to analyse how these three different aspects of the notion of importance are taken into account in several MCAPs.
2.2. Results

In what follows, the family of criteria F will be supposed consistent, i.e., will verify three axioms given in appendix A. The proofs of the results presented in this section can be found in appendix B.

2.2.1 Embedded nature of the sets \( \Delta^H_j(x_p,y_j), H \in \{P,S,V\} \).

Result 2.1: \( \forall j \in F, \forall x_p,y_j \in X^2_j \), it holds

\[
\Delta^P_j(x_p,y_j) \subseteq \Delta^S_j(x_p,y_j) \subseteq \Delta^V_j(x_p,y_j) \\
\Delta^S_j(x_p,y_j) \supseteq X^2_j(S)
\]

Moreover, if criterion \( g_j \) is inductive, i.e., such that \([g(a)=g(b) \forall i \neq j \text{ and } aP,b] \Rightarrow aPb\) then:

\( \Delta^S_j(x_p,y_j) \supseteq X^2_j(S) \)

2.2.2 Expanding nature of the sets \( \Delta^H_j(x_j,y_j), H \in \{P,S,V\} \).

Result 2.2: \( \forall x_j,y_j, u_j,v_j \in X^2_j \) such that \( u_j \geq x_j, y_j \geq v_j, x_jP_jy_j \),

\( \Delta^H_j(x_j,y_j) \subseteq \Delta^H_j(u_j,v_j), \forall H \in \{P,S,V\} \)

2.2.3 Borderline of the sets \( \Delta^H_j(x_j,y_j), H \in \{P,S,V\} \).

As \( \Delta \) denotes the dominance relation relatively to \( F \) (see 1.2), we denote by \( \Delta_j \) the dominance relation defined on \( X^2_j \).

Result 2.3: If \((x_j,y_j) \in \Delta^H_j(x_p,y_j)\),

then, \( \forall u_j,v_j \in X^2_j \) verifying

\[
\begin{bmatrix}
  u_j & x_j & X_j \\
  v_j & \Delta_j & v_j
\end{bmatrix}
\]

it holds \((u_j,v_j) \in \Delta^H_j(x_p,y_j) \ (H \in \{P,S,V\})\)

As a consequence, there exists a minimal subset from which it is possible to derive all the other elements of \( \Delta^H_j(x_j,y_j) \). This subset will be called the borderline of \( \Delta^H_j(x_j,y_j) \).

Definition 2.1\(^7\): \((x_j,y_j)\) will be a borderline element of \( \Delta^H_j(x_j,y_j) \) iff

\( \forall (u_j,v_j) \in \Delta^H_j(x_j,y_j), \frac{x_j \Delta_j u_j}{v_j \Delta_j v_j} \Rightarrow (u_j,y_j)=(x_j,y_j) \)

The knowledge of the set of all the borderline elements (i.e., the borderline of \( \Delta^H_j(x_j,y_j) \)) is sufficient to reconstruct the whole set \( \Delta^H_j(x_j,y_j), H \in \{P,S,V\} \). An illustrative example of borderline is given in the appendix C.

---

\(^7\) In the case of a family of criteria containing some criteria for which \( X_j \) is continuous, if \( \Delta^H_j(x_j,y_j) \) is an open set, it is necessary to replace this set by its topological closure.
Let us denote by $C^j(x_j, y_j)$, $E^j(x_j, y_j)$ and $O^j(x_j, y_j)$ the borderlines of $\Delta^j(x_j, y_j)$, $\Delta^j(x_j, y_j)$ and $\Delta^j(x_j, y_j)$, respectively. If $\forall j \in F$, $X_j$ is discrete, we have the following inclusions:

$$\forall j \in F, \forall x_j, y_j \in X_j \left[ C^j(x_j, y_j) \subseteq C^j(x_j, y_j), \quad E^j(x_j, y_j) \subseteq E^j(x_j, y_j), \quad O^j(x_j, y_j) \subseteq O^j(x_j, y_j) \right]$$

Thus, we will call these three borderlines, the conviction, the equilibration and the opposition borderlines, respectively. Since the knowledge of these three borderlines allows us to characterize the form and the extent of the sets $\Delta^j(x_j, y_j)$ (H=P,S,V), we can claim, according to definition 1.1, that all the information concerning the importance of the criterion $g_j$ is contained in $C^j(x_j, y_j)$, $E^j(x_j, y_j)$ and $O^j(x_j, y_j)$, $\forall x_j, y_j \in X_j$.

Consequently, when decision aid is grounded on a given MCAP, the numerical values attributed to the importance parameters of this MCAP are sufficient to specify the three borderlines. Hence, the numerical values for importance parameters must be chosen in the light of the three resulting borderlines.

### 2.2.4 Opposition-incomparability result

Considering the comprehensive relational system (L,P,R) which underlies the sets $\Delta^j(x_j, y_j)$ (H=P,S,V), we have the following result:

Result 2.4: $\forall j \in F, \forall x_j, y_j \in X_j$ such that $x_jP y_j$,

$$O^j(x_j, y_j) = \emptyset \iff \exists x_jx_jx_jx_j \in X_j \text{ such that } (x_j, y_j)R(y_j, y_j)$$

As a consequence, we have (see 3.1, 3.2 and 3.3):

$$\forall j \in F, \forall x_j, y_j \in X_j \text{ such that } x_jP y_j, \Delta^j(x_j, y_j) = \Delta^j(x_j, y_j) \iff R = \emptyset$$

### 3. CHARACTERIZATION OF THE SETS $\Delta^j(x_j, y_j)$ IN SOME BASIC MCAPs

For a given MCAP, the sets $\Delta^j(x_j, y_j)$ (H=P,S,V) are completely defined when numerical values are attributed to importance parameters. Conversely, the knowledge of all the sets $\Delta^j(x_j, y_j)$ defines an MCAP in a unique way. In fact, the sets $\Delta^j(x_j, y_j)$ contain the comprehensive preference induced by any couple of performance vectors. There is a one-to-one correspondence between an MCAP (with given numerical values for its importance parameters) and the sets $\Delta^j(x_j, y_j)$.

So as to highlight the link between importance parameters and the shape of the sets $\Delta^j(x_j, y_j)$ (or their respective borderlines), we will characterize these sets below in the case of basic simple MCAPs.

---

8 if not, the following inclusions still hold by substituting to $C_j$, $E_j$ and $O_j$ their topological closure.
3.1 Weighted sum

The weighted sum aggregation uses importance coefficients $k_i (k_i \geq 0)$ such that:

\[
\begin{align*}
\text{xPy} & \Leftrightarrow \sum_{i=1}^{n} k_i x_i > \sum_{i=1}^{n} k_i y_i \\
\text{xLy} & \Leftrightarrow \sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} k_i y_i
\end{align*}
\]

This MCAP implies that all criteria are true criteria. It follows that the sets $\Delta^I_j(x_i, y_j)$ are characterized by:

\[
\Delta^E_j(x_i, y_j) = \{(x_j, y_j) \in X^2_j \text{ such that } \sum_{i=1}^{n} k_i (x_i - y_i) > k_j (y_j - x_j)\}
\]

\[
\Delta^S_j(x_i, y_j) = \{(x_j, y_j) \in X^2_j \text{ such that } \sum_{i=1}^{n} k_i (x_i - y_i) \geq k_j (y_j - x_j)\}
\]

\[
\Delta^U_j(x_i, y_j) = \Delta^S_j(x_i, y_j)
\]

In this MCAP, the importance is taken into account essentially through the ability of criteria to convince. The ability to equilibrate is very weak: the only couples $(x_j, y_j)$ contained in $E_j(x_j, y_j)$ and in $E(x_j, y_j)$ are those verifying $\sum_{i=1}^{n} k_i (x_i - y_i) = k_j (y_j - x_j)$. No ability to oppose is attached to the criteria $(O_j(x_j, y_j)=O_j(x_j, y_j)=\emptyset)$

3.2 Lexicographic aggregation

In this MCAP, the importance attached to each criterion is characterized by its position in a hierarchy such that $g_1$ is the most important criterion and $g_n$ the least important one. More precisely, in this MCAP, we have:

\[
\begin{align*}
x_i > y_1 & \Rightarrow \text{xPy} \text{ whatever } x_j, y_j \\
x_i = y_1, \ldots, x_{i-1} = y_{i-1}, x_i > y_i & \Rightarrow \text{xPy} \text{ whatever } x_{i+1}, \ldots, x_n, y_{i+1}, \ldots, y_n \\
x_i = y_1, \ldots, x_n = y_n & \Rightarrow \text{xLy}
\end{align*}
\]

This implies\(^9\) that all criteria are true criteria ($q_i=0$). In this MCAP, the importance parameter takes the form of a permutation on the set of criteria. The sets $\Delta^U_j(x_i, y_j)$ are characterized by:

---

\(^9\) A direct transposition of these formulae to the case of quasi-criteria leads to a comprehensive preference relation which is not transitive
\[
\begin{align*}
\Delta^p_i(x_i, y_i) &= \Delta^p_i(x_i, y_i) = \Delta^p_i(x_i, y_i) = \{(x_i, y_i) \in X_i \} \\
\Delta^p_i(x_i, y_i) &= \Delta^p_i(x_i, y_i) = \Delta^p_i(x_i, y_i) = \{(x_i, y_i) \in X_i \text{ such that } x_i \geq y_i \} \\
\Delta^p_i(x_i, y_i) &= \Delta^p_i(x_i, y_i) = \Delta^p_i(x_i, y_i) = \{(x_i, y_i) \in X_i \text{ such that } (x_i = y_i \text{ and } x_i \geq y_i) \text{ or } (x_i > y_i) \} \\
\end{align*}
\]

In this MCAP, the RIC is only taken into account through the ability of criteria to convince \((E_i(x,y) \Rightarrow O_i(x,y) = \emptyset)\). We can observe that \(g_i\) is a dictator.

### 3.3 Majority rule

This MCAP is defined by:

\[
\begin{align*}
\forall xS\forall y \Leftrightarrow \sum_{i \in C(xS \forall y)} k_i \geq s \\
\forall xP \forall y \Leftrightarrow \forall xS \forall y \text{ and not} [\forall y S \forall x]
\end{align*}
\]

where:  
\(s = \text{majority level, } s \in [0,1]\)  
\(C[\forall xS \forall y] = \text{coalition of criteria such that } x_iS_iy\)  
\(k_i = \text{weight of criterion } g_i, \sum_{i=1}^{n} k_i = 1\)

In this MCAP, it holds:

\[
\begin{align*}
\Delta^p_i(x_i, y_i) &= \{(x_i, y_i) \in X_i \text{ such that } \sum_{i \in j \text{ and } x_i \geq y_i = q_i} k_i \geq s - k_i \} \\
\Delta^p_i(x_i, y_i) &= \Delta^p_i(x_i, y_i) \\
\Delta^p_i(x_i, y_i) &= \{(x_i, y_i) \in \Delta^p_i(x_i, y_i) \text{ such that } \sum_{y_i \geq x_i = q_i} k_i \leq s - k_i \}
\end{align*}
\]

In this aggregation procedure, the importance of criteria is taken into account both through the ability of criteria to convince and to equilibrate (no ability to oppose is attached to criteria: \(O_i(x_i, y_i) = \emptyset\)).

### 3.4 Electre I MCAP extended to semi-criteria\(^{10}\)

In this MCAP, the importance parameters are of two types: for each criterion \(g_i\), there is a weight \(k_i\) and a veto threshold \(v_j \geq q_j\). \(\forall xS\forall y\) holds iff:

i) the majority rule holds for a given majority level \(s\),

ii) \(\exists i \in F\) such that \(y_i > x_i + v_i\) (non-veto condition).

---

\(^{10}\) see [Roy & Bouyssou 93] or [Vinck 92].
In this MCAP, it holds:

\[
\Delta_j^S(x, y) = \{(x_i, y_j) \in X_j^2 \text{ such that } \sum_{i \neq j} k_i \geq s_k - k_j \text{ and } y_i \leq x_i + v_i \forall i \neq j\}
\]

\[
\Delta_j^Y(x, y) = \begin{cases} 
X_j^2 & \text{if } x_j > y_j + v_j \\
\Delta_j^S(x, y) \cup V(X_j^2) & \text{otherwise}
\end{cases}
\]
This empirical hypothesis leads us to propose hereafter a formal condition for \( g_i \) to be at least as important as \( g_i \) grounded on the sets \( \Delta_j^H(x_j, y_j) \) and \( \Delta_i^H(x_i, y_i) \). However, we are confronted with difficulties linked to the fact that these two sets are neither defined on the same space (\( X_j^2 \) and \( X_i^2 \), respectively), nor under the same conditions (\( \forall x_j, y_j \in X_j^2 \) such that \( x_j, y_j \) and \( \forall x_i, y_i \in X_i^2 \) such that \( x_i, y_i \), respectively). So as to circumvent these difficulties, we will introduce the following additional notations and definitions:

\[
X_{-ij} = \prod_{k \neq i,j} X_k
\]

\[
\Delta^H_j(x_j, y_j/ y_i P x_i) = \text{Restriction of } \Delta^H_j(x_j, y_j) \text{ to couples } (x_j, y_j) \text{ verifying } y_i P x_i
\]

Let \( D \subseteq X_j^2 \), we define \( \text{Proj}_j(D) \) as the subset of \( X_{-ij}^2 \) obtained by deleting in each element of \( D \) the evaluations \( x_i \) and \( y_i \).

On such basis, we propose the following condition.

**Condition 4.1.**

A necessary condition for \( g_j \) to be at least as important as \( g_i \) is:

\[
\forall H \in \{P, S, V\}, \ \forall z_i, t_i \in X_i^2 \text{ such that } z_i P t_i, \ \exists z_j, t_j \text{ verifying :}
\]

\[
\begin{bmatrix}
    z_j P t_j \\
    \text{Proj}_j[\Delta^H_j(z_i, t_i/y_j P x_j)] \subseteq \text{Proj}_i[\Delta^H_i(z_j, t_j/y_i P x_i)]
\end{bmatrix}
\]

for \( H=\mathbb{P} \) this condition corresponds to the following statement:

\[
\forall y \in Y^2 \text{ such that } z P y \text{ with } z P t_i \text{ and } x_j P y
\]
The above condition cannot be considered as a sufficient one because it can be verified while the empirical idea of importance leads to reject this assertion. For example, it is the cases in Electre (see 3.4) with \( k_i = k_j \) and \( v_i = q_i \) and \( v_j \) much greater than \( q_j \). In this case, condition 4.1 holds not only for validating "\( g_i \) is as least as important as \( g_j \)" and but also for validating "\( g_j \) is as least as important as \( g_i \)" (which is contradicts the empirical idea of importance). Let us remark that if the ability to oppose a veto exists for \( g_j \) and not for \( g_i \) (still
1- If these equalities hold without changing the values of importance parameters, the characterization of the sets $\Delta^H_j(x_j, y_j)$ is, in the MCAP under consideration, invariant when the encoding of $g_j$ is changed (i.e., when we substitute $\phi(g_j)$ for $g_j$). Considering the definition proposed for importance parameters, it appears that these parameters are independent of the encoding of $g_j$; such parameters are said to be intrinsic to the significance axis of $g_j$.

2- In the opposite case, it is necessary to adapt the value of some importance parameters in order to keep the sets $\Delta^H_j(x_j, y_j)$ unchanged when the encoding of $g_j$ is transformed. In this case, the importance parameters are said to be dependent on the encoding of the criterion.

Such terminology, when applied to the characterizations of the sets $\Delta^H_j(x_j, y_j)$ for the MCAP studied in section 3, leads to the following results:

- The coefficients $k_j$ of a weighted sum aggregation are dependent on the encoding of criteria.
- The ranking of criteria that specifies the relative importance of criteria in the lexicographic aggregation is intrinsic to the significance axis of criteria.
- The coefficients $k_j$ of the MCAP based on the majority rule are independent of the encoding of criteria.
- In the MCAP Electre I, the importance coefficients $k_j$ are intrinsic while the veto thresholds are dependent on the encoding of criteria.

Hence, the reader should be aware that it is inconsistent to assign values to importance parameters that are dependent on the encoding of the considered criterion without explicitly taking this encoding into account. Moreover, we should keep in mind that some MCAP are appropriate only if $\Phi_j$ is restricted to a limited type of transformation (for example, the weighted sum is convenient only if $\Phi_{j, \phi}, j \in F$, are restricted to $\phi$ such that $\phi(g_j) = \alpha g_j + \beta$ with $\alpha > 0$).

4.3. Importance invariant by translation

The formalism introduced in section 2.3 leads us to highlight an interesting property: the invariance of the importance of a criterion by translation.

**Definition 4.2:** the importance of criterion $g_j$ is invariant by translation if and only if:

$$\forall \alpha \in \mathbb{R}, \forall x_j, y_j \in X_j, \Delta^H_j(x_j, y_j) = \Delta^H_j(x_j + \alpha, y_j + \alpha), \quad \forall H \in \{P, S, V\}$$

When the importance of a criterion $g_j$ is invariant by translation, a given difference between the performances of two alternatives has the same influence on the comprehensive preferences whatever the value of the performances. This definition may be restricted to a subset of $X_j^2$. 

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Two particular cases of non-invariance of the importance of a criterion by translation should be emphasized:

- The condition $\forall \alpha \geq 0, \forall x_j, y_j \in X_j^2, \ \Delta^H_j(x_j, y_j) \subseteq \Delta^H_j(x_j + \alpha, y_j + \alpha), \ \forall H = \{p, s, v\}$ characterizes the situation in which the importance of criterion $g_j$ increases with its performance.
- Reversing the inclusions in the preceding condition leads to characterize the situation in which the importance of criterion $g_j$ decreases with its performance.

When the importance parameters of a criterion are intrinsic to its significance axis, if invariance by translation holds for a specific encoding, this property still holds for any other admissible encoding. On the other hand, if certain importance parameters are dependent on the encoding, this property of invariance may be verified for certain encodings but becomes false for others.

4.4. Link between the notions of importance and compensation

The following considerations and definitions will demonstrate the strong relationship which exists between the notions of relative importance of criteria and compensatoriness of preferences. This last notion is, in its current use as well as in multicriteria methods, linked to the possibility of making substitutions between performances on different criteria. These substitutions allow us to compensate for a disadvantage on one or several criteria by a sufficient advantage on other criteria.

When an MCAP uses such ideas in order to solve conflicts between criteria, this MCAP is usually said to be of a compensatory nature; in the opposite case, this MCAP is said to be non-compensatory. Consequently, a preference structure induced by an MCAP has a more or less compensatory nature.

More precisely, we will say that there is no possibility of compensation relative to criterion $g_j$ if, in each indifference situation $\mathbf{x}_j\mathbf{y}_j$ with $x_j^p, y_j^p$, when we destroy the indifference situation by substituting $u_j$ for $x_j$ and $v_j$ for $y_j$ (with $u_j^p, v_j^p$), it is not possible to re-establish it by modifying performances on other criteria ($g_i, i \neq j$).

The following definition formalizes this notion:

**Definition 4.3:** There is no possibility of compensation relative to criterion $g_j$ iff

$\forall x_j, y_j \in X_j^2$ verifying $x_j^p, y_j^p$, $\forall (x_j, y_j) \in E_j(x_j, y_j)$,

if there exist $u_j, v_j \in X_j^2$ verifying $u_j^p, v_j^p$ and $(x_j, y_j) \notin E_j(u_j, v_j)$,

then $E_j(u_j, v_j) = \emptyset$
On the contrary, there are some possibilities of compensation between the criteria \(g_i\) and \(g_k\) if there exist \(x\) and \(y\) such that \(x \not\sim y\) for which it is possible to destroy this indifferece situation by decreasing the difference of performances between their \(k^{th}\) component and to re-establish it by increasing the difference of performances on their \(j^{th}\) component.

The following definition formalizes this conception:

**Definition 4.4:** There are some possibilities of compensation between the criteria \(g_i\) and \(g_k\) iff:
\[ \exists x_j, y_j \text{ verifying } x_j P y_j, \exists (x^k_j, y^k_j) \in E_j(x_j, y_j), \]
\[ \exists (x^k_j, y^k_j) \in X^k_j \text{ deduced from } (x_j, y_j) \text{ by decreasing the } k^{th} \text{ performance of } x_j \]
\[ \text{and/or by increasing the } k^{th} \text{ performance of } y_j, \text{ such that } (x^k_j, y^k_j) \not\in E_j(x_j, y_j) \]
\[ \text{and} \]
\[ \exists u_j, v_j \in X^j_j \text{ verifying } u_j P y_j, \text{ such that } (x^k_j, y^k_j) \in E_j(u_j, v_j). \]

In the MCAP weighted sum (see 3.1), there are possibilities of compensation between any pair of criteria; in the three others (see 3.2 to 3.4), there is no possibility of compensation relatively to criterion \(g_i \) (\(\forall j \in F\)).

Definition 4.4 can easily be extended to possibilities of compensation between coalitions of criteria.

Let us recall that the first formal definition of a totally non-compensatory (I,P) preference structure was given by [Fishburn 76] (\(R=\emptyset\)):

An (I,P) preference structure \(\Psi\) is totally non-compensatory (TNC) iff:
\[ \forall x,y,u,v \in X^4, [P(x,y)=P(u,v) \text{ and } P(y,x)=P(v,u)] \Rightarrow [xP y \Leftrightarrow uP v] \]
with \(P(x,y)\{i \in F / x_i P y_i\}\)

An (I,P) preference structure is TNC if two pairs of alternatives that have the same preferential profile are linked with the same comprehensive preference relation. In other words, a TNC MCAP considers preferences restricted to a single criterion \(g_i \) (\(\forall j \in F\)) to be ordinal information when building comprehensive preferences\(^{11}\). The reader will easily verify that in a TNC (I,P) preference structure, there is no possibility of compensation relative to any criterion (according to definition 4.3).

[Bouyssou 86] proposes a definition of a minimally compensatory (I,P) preference structure: An (I,P) preference structure \(\Psi\) is minimally compensatory iff:

---

\(^{11}\) This idea has been generalized by [Bouyssou & Vansnick 86]. See also [Vansnick 86].

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\[ \exists x,y,u,v \in X^4 \text{ such that } \begin{align*} P(x,y) &= P(u,v) \\ P(y,x) &= P(v,u) \\ x_i &= y_i \text{ and } u_i = v_i \forall i \in I(x,y) \end{align*} \text{ with } [xSy \text{ and } vPu]\]

with

\[\begin{align*}
P(x,y) &= \{ i \in F / x_i \not\equiv P_i y_i \} \\
I(x,y) &= \{ i \in F / i \not\in P(x,y) \text{ and } i \not\in P(y,x) \}\end{align*}\]

This definition formally expresses phenomena similar to those taken into account in definition 4.4. However, it does not specify between which criteria the compensation occurs. Moreover, the reader will easily verify that, if there are some possibilities of compensation between two criteria \(g_j\) and \(g_k\) (definition 4.4), then the corresponding preference structure is minimally non-compensatory.

The analysis of the concept of compensatoriness of preference structures and its relations with the notion of RIC is too vast a subject to be developed fully here. Definitions 4.3 and 4.4 lead us to think that the theoretical framework proposed in this paper constitutes a suitable basis on which to ground further research.

5. OPERATIONAL INTEREST OF THE PROPOSED FORMALISM

So as to give meaning to importance parameters of any type, it is necessary to refer to a preference structure \(\Psi\) (see 1.2). In other words, assigning numerical values to such parameters requires questioning such a structure. The way to do this differs according to whether we adopt a descriptivist or a constructivist approach\(^{12}\).

Adopting the descriptivist approach involves acknowledging that way in which any two performance vectors \((x_p \neq x_j)\) and \((y_p \neq y_j)\) are compared on the comprehensive level is stable and well-defined in the mind of the DM prior to being questioned on his/her preference structure \(\Psi^*\). The preference model built for decision aid generates a preference structure \(\Psi\), which is intended to give an exact image as possible of \(\Psi^*\). Towards this end, the numerical values of importance parameters have to be chosen in such a way that the sets \(\Delta^H_i(x_p, y_p), H \in (P,S,V)\) induced from the adopted model match the pre-existing sets \(\Delta^H_i(x_p, y_j), H \in (P,S,V)\). Hence, adjustment techniques should be used to assign values to importance parameters. Several authors (see [Mousseau 92] for a review) talk in terms of estimating the numerical value of certain parameters, such as weight \(w_j\). This supposes that the sets \(\Delta^H_i(x_p, y)\) and \(\Delta^H_i(x_p, y_j)\), \(H \in (P,S,V)\) (or more simply their respective borderlines) admit of a common analytical representation.

\(^{12}\) For more details on this distinction, see [Roy 93].

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The constructivist approach\(^{13}\), on the contrary, acknowledges that preferences are not entirely pre-formed in the decision-maker’s mind and that the very nature of the work involved in the modelling process (and, a fortiori, in decision-aid) is to specify and even to modify pre-existing elements. The preference structure \(\Psi\) induced by the MCAP adopted for decision aid, cannot be viewed here as the faithful image of a pre-existing preference structure \(\Psi'\), as it is considered locally ill-defined and unstable. Thus, \(\Psi\) is here nothing other than the result of a set of rules deemed appropriate for comparing any performance vectors \((x_i, y_i)\) and \((x_j, y_j)\). The importance parameters appear as keys which allow us to differentiate the role played by each criterion in the preference model selected for use. The numerical values attributed to them correspond to a working hypothesis accepted for decision-aid. They must be seen as suitable values with which it seems reasonable and instructive to work. So as to judge if this is the case with a given set of values, it is important to analyse the shape and the variety of the sets \(\Delta_{ij}^H(x_i, y_j)\), \(H \in \{P,S,V\}\) which result from it. One should check if these values fit with the opinion of the DM (or of various stakeholders) regarding the fact that such and such \((x_i, y_i)\) belongs or does not belong to one of the preceding sets. The analysis of these sets does not aim at discovering a unique set of values for importance parameters that fits the DM’s opinion, but it can determine a domain containing an appropriate set of values for reasoning, investigating and communicating among the stakeholders in the decision-making process.

As shown in the examples of section 3, the shape and the variety of the sets \(\Delta_{ij}^H(x_i, y_j)\), \(H \in \{P,S,V\}\) vary significantly according to the type of MCAP considered. Whatever the approach, descriptivist or constructivist, the preceding remark highlights the fact that the meaning of importance parameters (and consequently the appropriate manner for giving a numerical value to them) depends on the MCAP. The discussions presented in section 4 show the danger of questioning the DM directly on values assigned to importance parameters. These discussions have proved that assigning values to certain importance parameters relying on the metaphor of weight results in serious misunderstandings (see 4.1 and 4.2). Even when this metaphor is pertinent, its evocative power is different when used with intrinsic weights (referring to voting power), or with contingent weights (such as scaling constants for example). Specifically, this metaphor is of no use in evaluating parameters such as veto thresholds in the Electre methods (see [Roy 91b]), or coefficients \(k_{ij}\) that account for certain forms of dependencies between criteria by adding terms of the form \(k_{ij}g_i(a)g_j(a)\) to an additive aggregation \(\sum_{k \in F} k_{ij}g_i(a)\) (see [Keeney & Raiffa 76]). In order to determine suitable values for such parameters, it is possible to base an argument on the fact a couple \((x_i, y_i)\) belongs (or not) to such and such \(\Delta_{ij}^H(x_i, y_j)\).

\(^{13}\) An overview of behavioral decision research within this approach can be found in [Paynes et al. 92].
Conclusion

The foregoing considerations show that the notion of relative importance of criteria is more complex than it is commonly assumed to be. The formal definition that we have proposed (see section 1.3) seems sufficiently general when criteria are true or semi-criteria. The examples from section 3 and the considerations developed in section 4 allow us to think that the sets $\Delta_i(x_i, y_j) \in \{P,S,V\}$ (certain results have been established in section 2) constitute useful keys in order to study the significance of an MCAP and to clarify other notions closely linked to the notion of importance.

Importance parameters of any type give an account of a value system. Consequently, the values assigned to such parameters will always have a subjective character. As is true for any other subjective data, these values can be grasped only through communicating with certain stakeholders in the decision process. The definitions, results and comments presented in this paper should enable us to develop better means of conceptualizing this type of communication in order to understand and decode it with greater insight within the framework of the MCAP selected for use.
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APPENDIX A: By definition\(^\text{14}\), a family F of n criteria \(g_1, g_2, \ldots, g_n\) is consistent iff F verifies the three following axioms (these axioms express natural requirements in a formal way):

- **Exhaustivity**: if \(\forall j \in F, g(a) = g(b)\), then \[
\begin{align*}
\text{cH}_a & \Rightarrow \text{cH}_b \\
\text{aH}_c & \Rightarrow \text{bH}_c & \forall H \in \{P, I, R\}
\end{align*}
\]

- **Cohesion**: if, \(\forall j \in F \setminus \{k\}\)
  \[
  \begin{align*}
  g(b^k) &= g_j(b), g_k(b^k) \geq g_k(b) \\
  g(a) &= g(a_j), g_k(a) \geq g_k(a_j)
  \end{align*}
\]
on one of the inequalities being strict,
then \(bPa \Rightarrow b^P a_k, bIa \Rightarrow b^I a_k\)
 moreover, if
\[
\begin{align*}
  g(a) &= g_k(b) \\
  a_k I_k b^k
  \end{align*}
\]
then \(bHa \Rightarrow b^H a_k, aHa \Rightarrow a^H b^k\)
\(\forall H \in \{P, I, R, S\}\)

- **Non-redundancy**: \(\forall j \in F, F \setminus \{j\}\) is either not exhaustive or not cohesive.

APPENDIX B: Proofs

**Proof of Result 2.1:**

The first two inclusions are direct consequences of:
\[
xPy \Rightarrow xSy \Rightarrow \text{non}[yPx]
\]
It follows from the exhaustivity and cohesion axioms that \(\forall i \in F, g_i(a) = g_i(b) \forall i \neq j\) and \(aPb \Rightarrow aSb\) (see [Roy & Bouyssou 93]). Hence, the last inclusion is true.

**Proof of result 2.2:**

This property is an immediate consequence of the cohesion axiom which formally expresses a certain form of monotony of preferences: the higher the performance, the better the alternative.

**Proof of result 2.3:**

The justification of this property comes from the cohesion axiom.
\(H = P\): if \((x_j, y_j) \in \Delta^H(x_j, y_j)\), i.e. \(xPy\),
then, \(\forall u_j, v_j \in X_j\) verifying \(u_j \Delta_j x_j, v_j \Delta_j y_j\), we have \(u \Delta x v \Delta y\) with \(x_j = u_j\) and \(y_j = v_j\)

then as a consequence of the cohesion axiom, it holds \(uPv\)
i.e., \((u_j, v_j) \in \Delta^H_j(x_j, y_j) (H \in \{P, S, V\})\)

**Proof of result 2.4:**

\[
\Rightarrow \text{Let us suppose that } O_j(x_j, y_j) = \emptyset, \text{ then } O_j(x_j, y_j) = \emptyset.
\]
i.e., \(\Delta^H_j(x_j, y_j) \Delta^H_j(x_j, y_j) = \emptyset\)
\[
\exists x_j, y_j \in X_j\) verifying:
\[
\text{not}[\langle x_j, y_j \rangle P(x_j, x_j)] \text{ and not}[\langle x_j, y_j \rangle S(x_j, y_j)]
\]
i.e., \(\text{not}[\langle x_j, y_j \rangle P(x_j, x_j)] \text{ and not}[\langle x_j, y_j \rangle P(x_j, y_j)] \text{ and not}[\langle x_j, x_j \rangle I(x_j, y_j)]
\]
Thus \(\exists x_j, y_j \in X_j\) verifying \(x_j, y_j \in R(x_j, y_j)\)

\(^{14}\) for more details, see [Roy & Bouyssou 93] chap. 2.
Let us suppose that $\exists x_i, y_i \in X_i$ such that $(x_i, y_i) \in R(x_i, y_i)$
then, there are only three preferential situations between $(x_i, y_i)$ and $(y_j, y_j)$:
$(x_i, y_i)P(y_j, y_j)$, $(x_i, y_i)I(y_j, y_j)$ and $(y_j, y_j)P(x_i, y_i)$
however, $(x_i, y_i)S(y_j, y_j) \iff (x_i, y_i)P(y_j, y_j)$ or $(x_i, y_i)I(y_j, y_j)$
then it holds $(x_i, y_i)S(y_j, y_j) \Rightarrow \neg[(y_j, y_j)P(x_i, y_i)]$
then $\Delta_i^B(x_i, y_i) = \Delta_i^V(x_i, y_i)$, i.e., $O_i(x_i, y_i) = \emptyset$
thus $O_i(x_i, y_i) = \emptyset$

APPENDIX C : Illustration of the concept of borderline

Let us illustrate the interpretation of the borderline through a bi-criteria example in which each criterion is evaluated on a discrete scale containing five levels ($A, B, C, D, E$, $i = 1, 2$).
Table 1 shows an example of the borderline of $\Delta_i^B(x_i, y_i)$ corresponding to $x_i = A$ and $y_i = B$.

<table>
<thead>
<tr>
<th></th>
<th>$x_2 = A$</th>
<th>$x_2 = B$</th>
<th>$x_2 = C$</th>
<th>$x_2 = D$</th>
<th>$x_2 = E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2 = A$</td>
<td>$\in$</td>
<td>$\in$</td>
<td>$\notin$</td>
<td>$\notin$</td>
<td>$\notin$</td>
</tr>
<tr>
<td>$y_2 = B$</td>
<td>$\in$</td>
<td>$\in$</td>
<td>$\notin$</td>
<td>$\notin$</td>
<td>$\notin$</td>
</tr>
<tr>
<td>$y_2 = C$</td>
<td>$\in$</td>
<td>$\in$</td>
<td>$\notin$</td>
<td>$\notin$</td>
<td>$\notin$</td>
</tr>
<tr>
<td>$y_2 = D$</td>
<td>$\in$</td>
<td>$\in$</td>
<td>$\notin$</td>
<td>$\notin$</td>
<td>$\notin$</td>
</tr>
<tr>
<td>$y_2 = E$</td>
<td>$\in$</td>
<td>$\in$</td>
<td>$\notin$</td>
<td>$\notin$</td>
<td>$\notin$</td>
</tr>
</tbody>
</table>

Table 1: Borderline of $\Delta_i^B(x_i = A, y_i = B)$.

$\in$ element of $\Delta_i^B(A, B)$.

$\notin$ non-element of $\Delta_i^B(A, B)$.

$\in$ element of the borderline of $C_i^B(A, B)$.

The reader will easily verify that if he knew all the shaded cells, he could deduce all the other elements of $\Delta_i^B(A, B)$.