APPROXIMATION OF WEIGHTED HEREDITARY
INDUCED-SUBGRAPH MAXIMIZATION PROBLEMS

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Approximation de problèmes héréditaires de maximization portant sur la recherche de sous-graphes induits pondérés

Résumé

Le but central de cet article est l'étude de l'approximation polynomiale de problèmes héréditaires de maximisation portant sur la recherche de sous-graphes induits pondérés. Après avoir donné un résultat général d'approximation pour cette classe de problèmes, nous montrons que celui-ci a plusieurs corollaires importants, par exemple, il améliore d'un facteur d'$O(\log \log n)$ le meilleur rapport d'approximation connu pour le stable maximum pondéré.

Mots-clé : problèmes combinatoires, complexité, algorithme polynomial d'approximation, NP-complétude, stable, problème héréditaire.

Approximation of weighted hereditary induced-subgraph maximization problems

Abstract

The main purpose of this paper is to study polynomial approximation of weighted hereditary induced subgraph problems. After giving a general approximation result for this class of problems, we show that the result has many important corollaries since, for instance, it allows improvement of the best-known approximation ratio for weighted independent set by a factor of $O(\log \log n)$.

Keywords: combinatorial problems, computational complexity, polynomial-time approximation algorithm, NP-completeness, independent set, hereditary problem.
1 Preliminaries

We consider NP-complete optimization graph-problems $\Pi$ where the objective is to find a maximum-order induced subgraph $^1$ satisfying a non-trivial hereditary property $\pi$. Let $\mathcal{G}$ be the class of all the graphs; a graph-property $\pi$ is a mapping from $\mathcal{G}$ to \{0, 1\}, i.e., for a $G \in \mathcal{G}$, $\pi(G) = 1$ iff $G$ satisfies $\pi$ and $\pi(G) = 0$, otherwise; $\pi$ is hereditary if whenever it is satisfied by a graph it is also satisfied by every one of its induced subgraphs; it is non-trivial if it is true for infinitely many graphs and false for infinitely many ones ([2]). The property "$\pi$: is a clique", or "$\pi$: is an independent set" are typical examples of such non-trivial hereditary properties and the maximum clique-problem or the maximum independent set are in their turn typical examples of hereditary induced subgraph problems. A generalisation of the class of problems just introduced is the one where we consider that positive weights are associated with the vertices of the input graphs. Given a graph $G$, the objective of a weighted induced subgraph problem is to determine an induced subgraph $G^*$ of $G$ such that $G^*$ satisfies $\pi$ and, moreover, the sum of the weights of the vertices of $G^*$ must be the largest possible.

Remark 1. By definition of $\pi$ (as a mapping from the class of graphs to \{0, 1\}), it is obvious that $\pi$ is context-free; in other words, if a graph $G$ satisfies $\pi$, then $G$ satisfies $\pi$ into every graph whose $G$ is an induced subgraph.

Given a polynomial-time approximation algorithm (PTAA) $\mathcal{A}$ that solves a maximization NP-complete problem $\Pi$ (we denote by WPI the weighted version of $\Pi$), the quality of its approximation behaviour is expressed by the ratio $\rho$ of the value (size, or weight, or whatever) of the solution found by the algorithm to the optimal (objective) value of $\Pi$; the smallest such ratio
\[ M > 2 \text{ be an integer and let } V^{(i)} = \{ v_j \in V : w_{\text{max}}(G)/M^i < w_j \leq w_{\text{max}}(G)/M^{i-1} \}, i = 1, \ldots \]

Finally, let \( x = \sup \{ \ell : \beta_w(G[U_1 \leq \ell V^{(i)})] < \beta_w(G)/2, \beta_w(G[U_1 \leq \ell V^{(i)})] \geq \beta_w(G)/2 \}. \) Then, \[
\rho_{\Pi}(n) \geq \max \left\{ \frac{M^x}{2n}, \min \left\{ \frac{M - 2}{2M^2x} \rho_{\Pi}(n), \frac{1}{M^2} \rho_{\Pi}(n) \right\} \right\} .
\]

**Proof:** For the sequel, we set \( G_x = G[U_1 \leq x V^{(i)}] \), \( G_{x+1} = G[U_1 \leq x+1 V^{(i)}] \) and \( G_d = G[V \setminus U_1 \leq x V^{(i)}] \) (of course, \( \beta_w(G_d) \geq \beta_w(G)/2 \)).

We first remove vertices \( v_k \) such that the graph \( \{v_k\} \emptyset \) does not satisfy \( \pi \). Then, the following lemma holds.

**Lemma 1.** There exists a PTAA for WII achieving approximation ratio \( M^x/(2n) \).

**Proof:** The algorithm claimed consists of simply taking \( \nu^* = \arg\max_{v \in V} \{\nu_i\} \) as WII-solution and then adding vertices in such a way that the solution finally obtained remains feasible\(^3\) for \( G \).

Moreover, note that, since vertices such that the graph \( \{v_k\} \emptyset \) does not satisfy \( \pi \) have been removed, the graph \( \{\nu^*\} \emptyset \) is already a feasible solution. Consequently, \( \beta_w(G) \geq w_{\text{max}}(G), \beta_w(G) \leq 2\beta_w(G_d) \leq 2|S^*(G_d)|w_{\text{max}}(G_d) \leq 2|S^*(G_d)||w_{\text{max}}(G)/M^x \). Consequently,

\[
\rho(n) = \frac{\beta_w(G)}{\beta_w(G)} \geq \frac{M^x}{2|S^*(G_d)|} \geq \frac{M^x}{2n} \tag{1}
\]

q.e.d. \( \blacksquare \)

**Remark 2.** For every \( i \geq 1 \), the weight of any II-solution \( S^{(i)} \) of \( G[V^{(i)}] \) lies in the interval \([|S^{(i)}|w_{\text{max}}(G)/M^i, |S^{(i)}|w_{\text{max}}(G)/M^{i-1}]\). It is at least \(|S^{(i)}|w_{\text{min}}(G[V^{(i)}]) \geq |S^{(i)}|w_{\text{max}}(G)/M^i \) and at most \(|S^{(i)}|w_{\text{max}}(G[V^{(i)}]) \leq |S^{(i)}|w_{\text{max}}(G)/M^{i-1} \). \( \blacksquare \)

For the rest of the proof, we distinguish two cases with respect to \( \beta_w(G_x) \), namely \( \beta_w(G)/2 \geq \beta_w(G_x) \geq ((M - 2)/2M)\beta_w(G) \) and \( \beta_w(G_x) \leq ((M - 2)/2M)\beta_w(G) \), respectively.

We first consider the case \( \beta_w(G)/2 > \beta_w(G_x) \geq ((M - 2)/2M)\beta_w(G) \). Then the following lemma holds.

**Lemma 2.** Let \( \beta_w(G)/2 \geq \beta_w(G_x) \geq ((M - 2)/2M)\beta_w(G) \), \( p = \arg\max_{1 \leq i \leq x} \{\beta_w(G[V^{(i)}])\}, |V^{(p)}| = n^{(p)} \) and \( S^{(p)}(G[V^{(p)}]) \) be the maximum-size II-solution in \( G[V^{(p)}] \). If there exists a PTAA for II providing a maximal solution \( S^{(p)} \) such that \( |S^{(p)}| \geq \rho_{\Pi}(n^{(p)})|S^*(G[V^{(p)}])| \), then one can solve WII in \( G \), in polynomial time, within ratio \((1 - \rho_{\Pi}(n^{(p)})/2M^2)^2\rho_{\Pi}(n)\).

**Proof:** Obviously, \( \beta_w(G_x) \leq x\beta_w(G[V^{(p)}]) \) and \( \beta_w(G_x) \leq x|S^*(G[V^{(p)}])|w_{\text{max}}(G)/M^{p-1} \) (by Remark 2). So, \( \beta_w(G) \leq 2(M/2M - x)|S^*(G[V^{(p)}])|(w_{\text{max}}(G)/M^{p-1}) \).

On the other hand, application of a PTAA guaranteeing approximation ratio \( \rho_{\Pi}(n) < 1 \) for II in \( G[V^{(p)}] \) constructs a solution \( S^{(p)} \) of WII of weight at least \(|S^{(p)}|w_{\text{min}}(G[V^{(p)}]) \geq |S^{(p)}|w_{\text{max}}(G)/M^p \). Note that (by Remark 1) \( S^{(p)} \) is II-feasible for \( G \). Moreover, starting from this solution, one can greedily augment it in order to finally produce a maximal WII-solution for \( G \). This final solution verifies \( \beta_w(G) \geq |S^{(p)}|w_{\text{max}}(G)/M^p \).

Combination of the above expressions for \( \beta_w(G) \) and \( \beta_w(G) \) yields

\[
\rho(n) = \frac{\beta_w(G)}{\beta_w(G)} \geq \left( \frac{M - 2}{2xM^2} \right) \beta_w(G) \geq \frac{M - 2}{2xM^2} \rho_{\Pi}(n^{(p)}) \geq \frac{M - 2}{2xM^2} \rho_{\Pi}(n) \tag{2}
\]

and this concludes the proof of the case \( \beta_w(G)/2 > \beta_w(G_x) \geq ((M - 2)/2M)\beta_w(G) \) and of the lemma. \( \blacksquare \)

We now suppose that \( \beta_w(G_x) \leq ((M - 2)/2M)\beta_w(G) \) and prove the following lemma.

\(^3\)In other words, this solution satisfies \( \pi \).
Lemma 3. Let $\beta_w(G_x) \leq (M - 2)/2M$ be a maximum-size II-solution of $G[V^{(x+1)}]$ be a maximum-size II-solution of $G[V^{(x+1)}]$. If there exists a PTAA for $\Pi$ providing a maximal II-solution $S^{(x+1)}$ of cardinality at least $p_{\Pi}(n^{(x+1)})$-times the cardinality of $S^*(S^{(x+1)})$, then one can solve WII in $G$, in polynomial time, within ratio $(1/M^2)p_{\Pi}(n)$.

Proof: Note that since $x$ is the largest $\ell$ for which $\beta_w(G[U_{1 \leq i \leq V^{(x)}}) < \beta_w(G)/2$, set $V^{(x+1)}$ is not empty.

Let $S^{opt}(G_{x+1})$ be an optimal WII-solution in $G_{x+1}$ (i.e., $\beta_w(G_{x+1}) = w(S^{opt}(G_{x+1}))$; let $S(G_{x+1}) = S^{opt}(G_{x+1}) \cap V(G_{x+1})$ (where $V(G_{x+1})$ denotes the vertex-set of $G_{x+1}$) and $S(G[V^{(x+1)}]) = S^{opt}(G_{x+1}) \cap V^{(x+1)}$ (in other words, $\{S(G_{x+1}), S(G[V^{(x+1)}])\}$ is a partition of $S^{opt}(G_{x+1})$). Since $\pi$ is hereditary, sets $S(G_{x+1})$ and $S(G[V^{(x+1)}])$, being subsets of $S^{opt}(G_{x+1})$, also verify $\pi$ (and, consequently they are feasible WII-solutions for $G_x$ and $G[V^{(x+1)}]$, respectively). We then have:

\[
\begin{align*}
\beta_w(G_{x+1}) & \leq \beta_w(G_x) \\
\beta_w(G_{x+1}) & = w(S(G_{x+1})) + w(S(G[V^{(x+1)}])) \leq \beta_w(G_{x+1}) + \beta_w(G[V^{(x+1)}]) \\
& \leq \frac{M - 2}{2M} \beta_w(G) + \beta_w(G[V^{(x+1)}]) \\
\beta_w(G_{x+1}) & \geq \frac{\beta_w(G)}{2}.
\end{align*}
\]

It follows from the above expressions that $\beta_w(G[V^{(x+1)}]) \geq \beta_w(G)/M$ and this together with Remark 2 yield, after some easy algebra, $\beta_w(G) \leq |S^*(G[V^{(x+1)}])|w_{max}(G)/M^{x+1}$.

As previously, suppose that a PTAA provides, in polynomial time, a maximal solution $S^{(x+1)}$ for $\Pi$ in $G[V^{(x+1)}]$, the cardinality of which is at least $p_{\Pi}(n^{(x+1)})S^*(G[V^{(x+1)}])$. Then, one can greedily augment this solution in order to produce a maximal II-solution for $G$ of total weight $\beta_w(G) \geq S^{(x+1)}|w_{max}(G)/M^{x+1}$.

Combination of expressions for $S^*(G)$ and $\beta_w(G)$ yields...
and this concludes the proof of Theorem 1 which, obviously, works also in the case where weights are exponential in \( n \).

The proof of Theorem 1 can be seen as a reduction transforming every PTAA A achieving ratio \( \rho_A(n) \) for \( \Pi \) into a PTAA for \( \Pi \Pi \) achieving, at worst, ratio \( \varepsilon = \Omega(\rho_A(n)/\pi) \). Quantity \( 1/\pi \) is usually called expansion of the reduction (see [7]). By expression (1) and by the fact that the approximation ratio of any PTAA for \( \Pi \Pi \) must be less than 1 (\( \Pi \Pi \) being a maximization problem), \( \pi = O(\log_M n) \) (in fact, as we will see in the next section, it can be much smaller depending on the form of \( \rho_{\Pi}(n) \)). But even such an expansion is a meaningful improvement with respect to expansions induced by reductions between weighted – no-weighted versions of problems which do not admit constant-ratio PTAA’s. For example, for the case of IS, if we take into account that no PTAA can guarantee fixed constant ratio for it, reductions of [7] have expansion \( 1/n^2 \).

3 Important corollaries of Theorem 1

3.1 Maximum-weight independent set and maximum-weight clique

Consider now one of the most classical NP-complete problems, the maximum independent set. Given a graph \( G = (V, E) \), an independent set is a subset \( V' \subseteq V \) such that whenever \( \{v_i, v_j\} \subseteq V' \), \( v_i v_j \notin E \), and the maximum independent set problem (IS) is to find an independent set of maximum size. As we have already noted in section 1, property “is an independent set” is hereditary (the subset of an independent set is an independent set); moreover, it is non-trivial (not all sets \( V' \subseteq V \) are independent). A very well-known and largely studied generalization of IS is its weighted version (denoted by WIS); here, we associate positive weights with the vertices of \( G \) and the objective becomes to maximize the sum of the weights of an independent set.

In terms of \( n \), the best known approximation ratio for IS is, to our knowledge, achieved by the IS-PTAA of Boppana and Halldórsson ([1]): \( \rho_{\text{IS}}(n) = \Theta(\log^2 n/n) \). Embedding it in expression (2), we get

\[
\rho(n) \geq \left( \frac{M - 2}{2\pi M^2} \right) \Theta \left( \frac{\log^2 n}{n} \right) = \Theta \left( \frac{\log^2 n}{nx} \right). \tag{5}
\]

For instance, if \( x \leq \log \log n \), expression (5) induces \( \rho(n) \geq \Theta((\log^2 n/(n \log \log n)) \), while, if \( x \geq \log \log n \), then expression (1) guarantees \( \rho(n) \geq \Theta((\log n)\log M/n) \); on the other hand, expression (3) always guarantees \( \rho(n) \geq \Theta((\log n)^2/n) \). So, by expression (4), the approximation ratio guaranteed by algorithm WA (supposing \( M > 8 \)) called when used to solve WIS is, at worst,

\[
\rho_{\text{WA}}(n) \geq \Theta \left( \frac{\log^2 n}{n \log \log n} \right). \tag{6}
\]

Expression (6) and the discussion above introduce the following corollary.

**Corollary 1.** \( \rho_{\text{WIS}}(n) = \Omega(\log^2 n/(n \log \log n)) \).

The above result improves by a factor \( O(\log \log n) \) the best-known approximation ratio (function of \( n \)) for WIS (\( \Theta(\log^2 n/(n(\log \log n)^2)) \), due to Halldórsson ([5])).

Finally, let us note that the same approximation result holds for another famous NP-complete problem, the maximum weighted clique (see [2] for the statement of this problem).
3.2 Maximum-weight \( \ell \)-colorable induced subgraph

Given a graph \( G = (V, E) \) and a positive constant \( \ell \), the problem of the maximum \( \ell \)-colorable induced subgraph (denoted by \( Cl \ell \)) is to find a maximum-order induced subgraph \( G' = G[V'] \) of \( G \) (\( V' \subseteq V \)) such that \( G' \) is \( \ell \)-colorable (i.e., there exists a coloring for \( G' \) of cardinality at most \( \ell \)). In the weighted version of \( Cl \ell \), denoted by \( WC\ell \), positive weights are associated with the vertices of \( G \) and the objective becomes to find the maximum-weight \( \ell \)-colorable induced subgraph. Once more, property “is \( \ell \)-colorable” is hereditary (if the vertices of a graph \( G \) can be feasibly colored by at most \( \ell \) colors, then every subgraph of \( G \) induced by a subset of its vertices can be colored by at most \( \ell \) colors) and non-trivial (given a number \( \ell \) every graph is not \( \ell \)-colorable).

The best-known ratios for \( Cl \ell \) are the ones of [4]: \( \Theta((\log^{2} n)/n) \) (as function of \( n \)) and of [3]: \( \min\{\kappa/\mu, \Theta((\log \log \Delta)/\Delta)\} \), for every fixed \( \kappa \in \mathbb{N}^+ \), where \( \Delta, \mu \) are the maximum and the average degrees of the graph, respectively (as degree-function). For \( WC\ell \), no approximation result is, to our knowledge, mentioned until now.

Application of Theorem 1, using the algorithm of [4] as approximation algorithm for \( Cl \ell \) (the algorithm \( A \) in algorithm \( WA \) of section 2), leads to the following corollary.

**Corollary 2.** \( \rho_{WC}\ell = \Omega((\log^{2} n)/(n \log \log n)) \).

4 About continuous reductions between weighted and unweighted versions of independent set

In what follows, we use the term “continuous reduction”, introduced by Simon ([7]), to denote the approximation-ratio-preserving reductions. Informally, given two maximization problems \( \Pi \) and \( \Pi' \), a reduction \( \Pi \rightarrow \Pi' \) is called continuous with an expansion of \( \epsilon \) if each polynomial-time \( \rho' \)-approximation algorithm \( A' \) for \( \Pi' \) can be converted into a polynomial-time \( \rho \)-approximation algorithm \( A \) for \( \Pi \) such that \( \rho \geq \epsilon \rho' \). The corresponding notation is \( \Pi \rightarrow \Pi' \). It is very well-known and largely noted until now by many authors that the concept of continuous reductions is strongly based upon the very restrictive hypothesis that the approximation ratio \( \rho' \) of \( A' \) has to be a fixed constant. Since IS does not admit such algorithms (in 1996 it has been proved that it is hard to approximate IS within \( 1/n^{1-\epsilon} \), for any \( \epsilon > 0 \), [6]), continuous reductions proposed until now between IS and WIS fail to produce WIS-ratios as good as the ones known for IS. In fact, the best known ratios for IS are \( \Theta((\log^{2} n)/n) \) (in terms of \( n \), [1]) and \( \min\{\kappa/\mu, \Theta((\log \log \Delta)/\Delta)\} \), for every fixed \( \kappa \in \mathbb{N}^+ \), where \( \Delta, \mu \) are the maximum and the average degrees of the graph, respectively ([3]). On the other hand, the best corresponding ratios for WIS are \( \Theta((\log^{2} n)/(n \log \log n)) \) (proved in this paper) and \( 3/((\Delta + 2)) \) (see [5] for this last ratio), respectively. We think that devising a continuous reduction of constant expansion between IS and WIS, or proving that such a reduction does not exist, is a very interesting open problem, the solution of which is far from being obvious. In what follows, we give some easy cases of such reductions for restrictive WIS-classes recognizable in polynomial time. We denote by \( \alpha_{w}(G) \) and by \( \alpha'_{w}(G) \) the weight of a maximum-weight independent set and the weight of an approximately maximum-weight independent set of \( G \), respectively.

**Theorem 2.** Consider a fixed constant \( \ell \) and graphs \( G = (V, E) \) of order \( n \) where at least one of the following conditions is verified:

1. there exist at most \( \ell \) distinct weights,
2. \( w_{\text{max}}(G)/w_{\text{min}}(G) \leq \ell \),
3. weights are rational and fixed constants,
4. $w_{\text{max}}(G) \geq nw'$, where by $w'$ we denote the second largest vertex weight.

For all these graph-families, there exist continuous reductions of constant expansion between IS and WIS. Consequently, in all the above families of graphs, WIS can be approximated in polynomial time within

$$\rho_{\text{WIS}} = \rho_{\text{IS}} \geq \max \left\{ \Theta \left( \frac{\log^2 n}{n} \right), \min \left\{ \frac{\kappa}{\mu}, \Theta \left( \frac{\log \log \Delta}{\Delta} \right) \right\} \right.$$ for every fixed $\kappa \in \mathbb{N}^+$.

**Proof:** We consider the $\ell$ subgraphs, $G^{(1)}, \ldots, G^{(\ell)}$, of $G$, induced by the vertices of the same weight. Let $p = \arg\max_{1 \leq i \leq \ell} \{\alpha_{w_i}(G^{(i)})\}$ and $w_p$ be the weight of the vertices of $G^{(p)}$; then, $\alpha_{w_i}(G) \leq \ell \alpha_{w_i}(G^{(p)}) = \ell w_p |S^*(G^{(p)})|$. On the other hand, one can easily take $\alpha'_{w_i}(G) = \max_{1 \leq i \leq \ell} \{w_i |S^{(i)}|\}$, where $S^{(i)}$ is the largest among the independent sets provided by the algorithms of [1] and of [3] for IS. It is easy to see that, in this case, $\rho_{\text{WIS}} = \alpha'_{w_i}(G)/\alpha_{w_i}(G) \geq \cdots \cdots$.
being a fixed constant, it can be computed in polynomial time by exhaustive search. In this case, \( \alpha_w(G)/\alpha_w(G) \geq |S^*(G_m)|/(|S^*(G_m)| + 1) \geq 1/2 \) (since \( |S^*(G_m)| \geq 1 \)). Consequently, once more, the final statement of the theorem is verified and this concludes the proof of item 4 and of the theorem.

References


