PQI INTERVAL ORDERS

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<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Résumé</td>
<td>i</td>
</tr>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Notations and definition</td>
<td>2</td>
</tr>
<tr>
<td>3. Interval orders</td>
<td>4</td>
</tr>
<tr>
<td>4. P, Q, I interval orders</td>
<td>5</td>
</tr>
<tr>
<td>5. Detection of a PQI interval order</td>
<td>9</td>
</tr>
<tr>
<td>6. Matrix version of the algorithm</td>
<td>17</td>
</tr>
<tr>
<td>7. Conclusions</td>
<td>19</td>
</tr>
<tr>
<td>References</td>
<td>20</td>
</tr>
<tr>
<td>Appendix A</td>
<td>21</td>
</tr>
</tbody>
</table>
PQI Ordres d'intervalles

Résumé

Nous présentons la réponse à un problème ouvert dans la modélisation des préférences à l'aide d'intervalles. Soit un ensemble fini $A$ et trois relations binaires $P, Q, I$, appelées "préférence stricte", "préférence faible" et "indifférence", respectivement. Nous présentons les conditions nécessaires et suffisantes pour pouvoir associer à chaque élément de $A$ un intervalle de façon à ce que si un intervalle est "complètement à droite" de l'autre on obtient la relation $P$, si un intervalle est inclus dans l'autre on obtient la relation $I$ et si un intervalle "est à droite" de l'autre, mais leur intersection n'est pas vide on obtient la relation $Q$ ($Q$ modélisant l'hésitation entre $P$ et $I$). Deux structures de préférences spécifiques sont caractérisées: le PQI ordre d'intervalles et le PQI quasi-ordre. Déterminer l'existence d'un PQI quasi-ordre est immédiat. Par contre, la détection d'un PQI ordre d'intervalles est plus difficile parce que le théorème d'existence est une formule du deuxième ordre. Pour cette raison nous présentons un algorithme pour la détection
1 Introduction

Comparing intervals, instead of discrete values, is a frequently encountered problem in preference modelling and decision aid. This is due to the fact that the comparison of alternatives (outcomes, objects, candidates, ...) generally is realized through their evaluations on numerical scales, while such evaluations are often imprecise or uncertain. A well known preference structure, in this context, is the semi order (see Luce 1956) (for a comprehensive presentation see Firkot & Vincke 1997) and more generally the interval order (see also Fishburn 1985). An interval order is obtained when one considers that an alternative is preferred to another iff its interval is “completely to the right” of the other (hereafter we assume that the larger an evaluation of an alternative is on a numerical scale the better the alternative is), while any two alternatives, the intervals of which have a non empty intersection, are considered indifferent. Such a model has a strict probabilistic interpretation, since the interval associated to each alternative can be viewed as the extremes of the probability distributions of the evaluations of the alternatives. Under such an interpretation a “sure preference” occurs only if the distributions have an empty intersection. A second implicit assumption in this frame is that if there is no preference of an alternative over the other then they are indifferent.

It is easy however to notice that if, in the previous frame, we want to establish a “sure indifference”, it is much more natural to consider that two alternatives are indifferent if their associated intervals (or distributions) are embedded. In such a case we obtain a preference relation which is known to be a partial order of dimension 2, that is a partial order obtained from the intersection of exactly two linear orders; (see Roubens & Vincke 1985).

Practically we observe that we have three situations:

- a “sure indifference”: when the intervals associated to two alternatives are embedded;
- a “sure preference”: when the interval associated to one alternative is “more to the right” with respect to the interval associated to the other alternative and the two intervals have an empty intersection;
- an “hesitation between indifference and preference” which we denote as weak preference: when the interval associated to one alternative is “more to the right” with respect to the interval associated to the other alternative and the two intervals have a non empty intersection.

The preference structure having three relations $P$, $Q$, $I$ defined as such is called hereafter $PQI$ interval order. It fits better in the case we have
specific relations in order to represent situations of hesitation in preference modeling (see Tsoukiás & Vincke 1997). The problem is to give the necessary and sufficient conditions for which a preference structure characterized by the presence of the relations $P$, $Q$ and $I$ may admit a representation by intervals as the one previously discussed, and then to detect if a given $PQI$ preference structure satisfies these conditions. Such a problem was considered open for a long time (see Vincke 1988).

In this paper we present an answer for this problem. Section 2 provides the basic notations and definitions. In section 3 we recall some results concerning conventional interval orders. The main theoretical results of
- irreflexive: iff $\forall x \neg S(x, x)$
- symmetric: iff $\forall x, y \; S(x, y) \to S(y, x)$
- asymmetric: iff $\forall x, y \; S(x, y) \to \neg S(y, x)$
- complete: iff $\forall x, y, \; x \neq y \Rightarrow S(x, y) \vee S(y, x)$
- transitive: iff $\forall x, y, z \; S(x, y) \land S(y, z) \to S(x, z)$
- negatively transitive: iff $\forall x, y, z \; \neg S(x, y) \land \neg S(y, z) \to \neg S(x, z)$

Definition 2.2 A binary relation $S$ is:
- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence iff it is reflexive, symmetric and transitive.

In this paper we will consider relations representing strict preference, weak preference and indifference situations. We will denote them $P, Q, I$ respectively. Moreover, such relations are expected to satisfy some “natural” properties of the type announced in the following two definitions.

Definition 2.3 A $(P, I)$ preference structure on a set $A$ is a couple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ is asymmetric;
- $I \cup P$ is complete;
- $P$ and $I$ are mutually exclusive ($P \cap I = \emptyset$).

Definition 2.4 A $(P, Q, I)$ preference structure on a set $A$ is a triple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ and $Q$ are asymmetric;
- $I \cup P \cup Q$ is complete;
- $P$, $Q$ and $I$ are mutually exclusive.

Finally we introduce an equivalence relation as follows:

Definition 2.5 The equivalence relation associated to a $(P, Q, I)$ preference structure is the binary relation $E$, defined on the set $A$, such that, $\forall x, y \in A$:

\[
E(x, y) \text{ iff } \forall z \in A:\ \\
P(x, z) \iff P(y, z) \\
Q(x, z) \iff Q(y, z) \\
I(x, z) \iff I(y, z) \\
Q(x, z) \iff Q(z, y) \\
P(z, x) \iff P(z, y)
\]
Remark 2.1 In this paper we consider that two different elements of \( A \) are never equivalent for the given \( \langle P, Q, I \rangle \) preference structure. This is not restrictive as it suffices to consider the quotient of \( A \) by \( E \) to satisfy the assumption. Under such an assumption we will use in the numerical representation of the preference relations only strict inequalities without any loss of generality.

3 Interval Orders

In this section we recall some definitions and theorems concerning conventional interval orders and semi orders.

**Definition 3.1** A \( \langle P, I \rangle \) preference structure on a finite set \( A \) is a PI interval order iff \( \exists l, r : A \to \mathbb{R}^+ \) such that:

\[ \forall x : r(x) > l(x) \]
\[ \forall x, y : P(x, y) \iff l(x) > r(y) \]
\[ \forall x, y : I(x, y) \iff l(x) < r(y) \text{ and } l(y) < r(x) \]

In conventional interval orders when comparing two intervals two situations are considered:
- one interval is completely to the right of the other (strict preference);
- there is a non empty intersection of the intervals (indifference).

**Definition 3.2** A \( \langle P, I \rangle \) preference structure on a set \( A \) is a PI semi order iff \( \exists l : A \to \mathbb{R}^+ \) and a positive constant \( k \) such that:

\[ \forall x, y : P(x, y) \iff l(x) > l(y) + k \]
\[ \forall x, y : I(x, y) \iff |l(x) - l(y)| < k \]

Such structures have been extensively studied in the literature (see for example Fishburn 1985). We recall here below the two fundamental results which characterize interval orders and semi orders.

**Theorem 3.1** A \( \langle P, I \rangle \) preference structure on a finite set \( A \) is a PI interval order iff \( P.I.P \subseteq P \).

**Proof.** See Fishburn (1985).

**Theorem 3.2** A \( \langle P, I \rangle \) preference structure on a finite set \( A \) is a PI semi order iff \( P.I.P \subseteq P \) and \( I.P.P \subseteq P \).

**Proof.** See Fishburn (1985).
4 \langle P, Q, I \rangle interval orders

As mentioned in the introduction, we are interested in situations where, comparing elements evaluated by intervals, one wants to distinguish three situations: indifference if one interval is included in the other, strict preference if one interval is completely "to the right" of the other and weak preference when one interval is "to the right" of the other, but they have a non empty intersection. Definition 4.1 precisely states this kind of situation, \( l(x) \) and \( r(x) \) respectively representing the left and right extremities of the interval associated to any element \( x \in A \).

**Definition 4.1** A \( \langle P, Q, I \rangle \) preference structure on a finite set \( A \) is a PQI interval order, iff there exist two real valued functions \( l \) and \( r \) such that, \( \forall x, y \in A, x \neq y \):
- \( r(x) > l(x) \);
- \( P(x, y) \Leftrightarrow r(x) > l(x) > r(y) > l(y) \);
- \( Q(x, y) \Leftrightarrow r(x) > r(y) > l(x) > l(y) \);
- \( I(x, y) \Leftrightarrow r(x) > r(y) > l(y) > l(x) \) or \( r(y) > r(x) > l(x) > l(y) \).

The reader will notice that the above definition immediately follows Definition 3.1 since a preference structure characterized a PI interval order can always be seen as PQI interval order also. We give now necessary and sufficient conditions under which such a preference structure exists.

**Theorem 4.1** A \( \langle P, Q, I \rangle \) preference structure on a finite set \( A \) is a PQI interval order, iff there exists a partial order \( I_1 \) such that:

1. \( I = I_1 \cup I_r \cup I_l \) where \( I_l = \{(x, x), x \in A\} \) and \( I_r = I_1^{-1} \);
2. \( P \cup Q \subseteq P \subseteq P \cup Q \cup I_l \);
3. \( Q \subseteq P \cup Q \cup I_r \);
4. \( \emptyset \subset P \cup Q \cup I_l \);
5. \( P \cup Q \cup I_r \subseteq P \cup Q \cup I_l \);
6. \( \emptyset \subset P \cup Q \cup I_r \).

**Proof**

**Necessity.**

We first give an outline of necessity demonstration which is the easy part of the theorem. If \( \langle P, Q, I \rangle \) is a PQI interval order, then defining
- \( I_l(x, y) \Leftrightarrow l(y) < l(x) < r(x) < r(y) \)
- \( I_r(x, y) \Leftrightarrow l(x) < l(y) < r(y) < r(x) \)
we obtain two partial orders satisfying the desired properties. As an example we demonstrate property (v):

\[ Q(x, y) \text{ and } (P \cup Q \cup I_r)(y, z) \text{ imply } r(x) > r(y) \text{ and } r(y) > r(z), \text{ hence } r(x) > r(z), \text{ so that } (P \cup Q \cup I_r)(x, z). \]

Sufficiency.

Conversely let us assume the existence of \( I_t \) satisfying the properties of the theorem. Define a set \( A' \) isomorphic to \( A \) (\( A' \) and \( A \) being disjoint) and denote by \( x' \) the image of \( x \in A \) in \( A' \). In the set \( A \cup A' \) let us define the relation \( S \) as follows: \( \forall x, y \in A, x \neq y \)
- \( S(x', x) \)
- \( S(x, y) \leftrightarrow (P \cup Q \cup I_t)(x, y) \)
- \( S(x', y') \leftrightarrow (P \cup Q \cup I_r)(x, y) \)
- \( S(x, y') \leftrightarrow P(x, y) \)
- \( S(x', y) \leftrightarrow \neg P(y, x) \)

We demonstrate now that \( S \) is a linear order (irreflexive, complete and transitive relation) in \( A \cup A' \).

Irreflexivity results from irreflexivity of \( P, Q, I_t \) and \( I_r \).

To demonstrate completeness of \( S \) remark that for \( x \neq y \):

\[ \neg S(x, y) \leftrightarrow \neg (P \cup Q \cup I_t)(x, y) \]
\[ \leftrightarrow (P \cup Q \cup I_t)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_t \cup I_r \cup I_o \]
\[ \leftrightarrow S(y, x) \]

\[ \neg S(x', y') \leftrightarrow \neg (P \cup Q \cup I_r)(x, y) \]
\[ \leftrightarrow (P \cup Q \cup I_r)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_t \cup I_r \cup I_o \]
\[ \leftrightarrow S(y', x') \]

\[ \neg S(x, y') \leftrightarrow \neg P(x, y) \]
\[ \leftrightarrow S(y', x) \]

\[ \neg S(x', y) \leftrightarrow P(y, x) \]
\[ \leftrightarrow S(y, x') \]

We demonstrate now that \( S \) is transitive.
\[ S(x, y) \text{ and } S(y, z) \text{ imply } (P \cup Q \cup I)(x, y) \text{ and } (P \cup Q \cup I)(y, z). \] From conditions ii) and iv) of the theorem, we know that \((P \cup Q \cup I)(x, y)\) and \((P \cup Q)(y, z)\) imply \((P \cup Q \cup I)(x, z)\), hence \(S(x, z)\). From transitivity of \(I_i\) we have that \(I_i(x, y)\) and \(I_i(y, z)\) imply \(I_i(x, z)\), hence \(S(x, z)\). Finally, if \((P \cup Q)(x, y)\) and \(I_i(y, z)\) then \((P \cup Q \cup I)(x, z)\) because, if \(\not\exists \ x \text{ we would have } (P \cup Q \cup I)(x, z)\) which with \(I_i(x, z)\) would give
Since $S$ is a linear order on $A \cup A'$, there exists a real valued function $u$ such that, $\forall x, y \in A$:
- $S(x, y) \Leftrightarrow u(x) > u(y)$;
- $S(x', y') \Leftrightarrow u(x') > u(y')$;
- $S(x, y') \Leftrightarrow u(x) > u(y')$;
- $S(x', y) \Leftrightarrow u(x') > u(y)$.

We define $\forall x \in A$, $l(x) = u(x)$ and $r(x) = u(x')$ and we obtain:

- $\forall x : r(x) > l(x)$, since $S(x', x)$.
- $\forall x, y : P(x, y) \Leftrightarrow S(x, y') \Leftrightarrow l(x) > r(y)$.
- $\forall x, y : Q(x, y) \Leftrightarrow S(x, y) \land S(x', y') \land \neg P(x, y) \Leftrightarrow$
  $l(x) > l(y)$ and $r(x) > r(y)$ and $r(y) > l(x)$, equivalent to:
  $r(x) > r(y) > l(x) > l(y)$.
- $\forall x, y : I(x, y) \Leftrightarrow$
  
  $r(x) > r(y) > l(y) > l(x)$ or $r(y) > r(x) > l(x) > l(y)$
  since $I(x, y)$ holds in all the remaining cases.

We can complete the investigation providing a characterization of $PQI$ semi orders.

**Definition 4.2** A $PQI$ semi order is a $PQI$ interval order such that $\exists k > 0$ constant for which $\forall x : r(x) = l(x) + k$.

In other words, a $PQI$ semi order is a $(P, Q, I)$ preference structure for which there exists a real valued function $l : A \rightarrow \mathbb{R}$ and a positive constant $k$ such that $\forall x, y, x \neq y$:
- $P(x, y) \Leftrightarrow l(x) > l(y) + k$;
- $Q(x, y) \Leftrightarrow l(y) + k > l(x) > l(y)$;
- $I(x, y) \Leftrightarrow l(x) = l(y)$ (actually $I$ reduces to $I_0$).

For such preference structures the following theorem holds.

**Theorem 4.2** A $(P, Q, I)$ preference structure is a $PQI$ semi order iff:

i) $I$ is transitive

ii) $PP \cup PQ \cupQP \subseteq P$;

iii) $QQ \subseteq P \cup Q$;
Proof

Necessity is trivial. We give only the sufficiency proof. Since $I$ is an equivalence relation, we consider the relation $P \cup Q$ on the set $A/I$. Such a relation is clearly a linear order (irreflexivity and completeness result from definition 2.4 and transitivity from conditions ii) and iii) of the theorem). Therefore we can index the elements of $A/I$ by $i = 1, 2, \cdots, n$ in such a way that $\forall x_i, x_{i+1} \in A/I: (P \cup Q)(x_{i+1}, x_i)$.

Choosing an arbitrary positive value $k$, we define function $l$ as follows:

- $l(x_1) = 0$ and for $i = 2, 3, \cdots, n$
- $l(x_{i+1}) > l(x_i)$
- $l(x_i) > l(x_j) + k \ \forall j < i$ such that $P(x_i, x_j)$
- $l(x_i) < l(x_m) + k \ \forall m < i$ such that $Q(x_i, x_m)$.

This is always possible because $P(x_i, x_j)$ and $Q(x_i, x_m)$ imply $(P \cup Q)(x_m, x_j)$ (if not, we would have $(P \cup Q)(x_j, x_m)$ which, with $P(x_i, x_j)$ and by condition ii) would give $P(x_i, x_m)$, hence $m > j$ and $l(x_m) > l(x_j)$). By construction the function $l$ satisfies the numerical representation of a $PQI$ semi order.

5 Detection of a PQI interval order

The problem is the following:

Given a set $A$ and a $(P, Q, I)$ preference structure on it, verify whether it is a PQI interval order. The difficulty resides in the fact that the theorem previously announced contains a second order condition which is the existence of the partial order $I$. For this purpose we give two propositions which show the difficulties in detecting such a structure.

Proposition 5.1 There exist $(P, Q, I)$ preference structures which are $P\tilde{I}$ interval orders (where $\tilde{I} = Q \cup I \cup Q^{-1}$), but are not PQI interval orders.

Proof Consider the following case.

- $A = \{a, b, c, d, e\}$;
- $P = \{(a, c), (d, e), (a, e)\}$;
- $Q = \{(d, c), (a, b), (b, e)\}$;
- $I = \{(a, d), (c, e), (b, d), (b, c), (d, a), (e, c), (d, b), (c, b)\} \cup I_o$
On the one hand if we consider the relation \( \hat{I} = Q \cup I \cup Q^{-1} \) it is easy to observe that the \( (P, \hat{I}) \) preference structure is a \( PI \) interval order (\( PI \subset P \) holds). On the other hand if we accept that the given \( (P, Q, I) \) preference structure is a \( PQI \) interval order then we have (by definition 4.1 and theorem 4.1) that:
- \( I(a, d) \) has to be \( I_i(a, d) \) because of \( c_i \);
- \( I(d, b) \) has to be \( I_i(d, b) \) because of \( c_i \);
therefore by transitivity we should have \( I_i(a, b) \), while we have \( Q(a, b) \) which is impossible. Therefore we can conclude that for this particular case the \( PQI \) interval order representation is impossible.

Proposition 5.2 There exist \( (P, Q, I) \) preference structures which have more than one \( PQI \) interval order representation.

Proof Consider the following case.
- \( A = \{a, b, c\} \);
- \( P = \emptyset \);
- \( I = \{(a, c), (b, c), (c, a), (c, b)\} \cup I_0 \);
- \( Q = \{(a, b)\} \)

It is easy to observe that both \( I_i(a, c), I_i(b, c) \) and \( I_i(c, a), I_i(c, b) \) are possible, thus allowing two different \( PQI \) interval orders: one in which the interval of \( c \) is included in the intervals of both \( a \) and \( b \) and the other where the intervals of \( b \) and \( a \) are included in the interval \( c \). Both representations are correct, although incompatible with each other.

The basic theorem 4.1, which gives necessary and sufficient conditions to see if a \( PQI \) preference structure is a \( PQI \) interval order, is unfortunately a formula in a second order logic (a formula where predicates can be variables). Generally the satisfaction of second order formula can be undecidable. Moreover, the theorem does not give a constructive procedure for verifying its satisfaction. In the following we give a second theorem, equivalent to theorem 4.1, which enables to define an algorithm detecting if a \( PQI \) preference structure is a \( PQI \) interval order.

Theorem 5.1 A \( PQI \) preference structure on a finite set \( A \) is a \( PQI \) interval order iff there exists a partial order \( I_t \) such that:
i. \( I = I_t \cup I_r \cup I_o \) where \( I_o = \{(x, x), \ x \in A\} \) and \( I_r = I^{-1}_t \);
ii. \( P \cup Q \subseteq P \cup P \subseteq P \) and \( Q \subseteq P \cup Q \);

iii. \((P \cap Q)^{-1} \subseteq I_i\);

iv. \((P^{-1} \cap Q) \subseteq I_i\);

v. \((I_i \cap P) \subseteq I_i \cap I_i\);

vi. \((I_i \cap (Q \cup Q^{-1})) \subseteq (I_i \cap I_i) \cup (I_i \cap I_i)\)

vii. \(I_i \subseteq I_i\);

**Proof**

We will now prove that the conditions i – vii are equivalent to i – v of theorem 4.1.

**Necessity** - (i-v) of 4.1 \(\Rightarrow\) (i-vii) of 5.1

ii. \(P \cup Q \subseteq P \cup P \subseteq P\) and \(Q \subseteq P \cup Q\);

\[ (P \cup Q \subseteq I_i) \subseteq P \Rightarrow P \cup Q \subseteq P \]

\[ P \cup Q \subseteq I_i \subseteq P \Rightarrow P \subseteq P \]
As $P, Q, I$ are mutually exclusive and the above result (vii), we have
$(Q \cup Q^{-1}) \cap (I \cup I_I \cup I_r, I_r) = \emptyset$

v. $(I, I \cap P) \subset I_I, I_r$

Similarly to vi, we have $(I, I \cap P) \subset (I_I, I_r) \cup (I_r, I_I)$, we still have to prove that $P \cap I_I, I_r = \emptyset$

$\exists x, y, z : P(x, y) \land I_r(x, y) \land I_r(x, z) \Rightarrow P(x, y) \land I_r(x, y)$, impossible.

Sufficiency - (i-vii) of 5.1 ⇒ (i-iv) of 4.1

ii. $(P \cup Q \cup I_I).P \subset P$

$(P.P \cup Q.P \cup P.\emptyset.P) \subset P$ by ii of 5.1

$I_I, P \subset P$. Suppose that:

$\exists x, y, z : I_I(x, y) \land P(y, z) \land P(z, x)$.

Impossible since it implies $P(y, x)$ by ii of 5.1

$\exists x, y, z : I_I(x, y) \land P(y, z) \land Q(z, x)$.

Impossible since it implies $P(y, x)$ by ii of 5.1

$\exists x, y, z : I_I(x, y) \land P(y, z) \land I_I(x, x)$.

Impossible since it implies $I_I(x, y)$ by vii of 5.1

$\exists x, y, z : I_I(x, y) \land P(y, z) \land I_I(x, x)$.

Impossible since it implies $I_I(x, y)$ by vii of 5.1

$\exists x, y, z : I_I(x, y) \land P(y, z) \land Q(x, x)$.

Impossible since it implies $I_I(x, y)$ by iii of 5.1.

iii. $P.(P \cup Q \cup I_I^{-1}) \subset P$

$(P.P \cup Q.P \cup P.\emptyset.P) \subset P$ by ii of 5.1;

$I_I^{-1}, P \subset P$. Suppose that:

$\exists x, y, z : P(x, y) \land I_I^{-1}(y, z) \land P(z, x)$.

Impossible since it implies $P(x, y)$ by ii of 5.1

$\exists x, y, z : P(x, y) \land I_I^{-1}(y, z) \land Q(z, x)$.

Impossible since it implies $P(y, x)$ by ii of 5.1

$\exists x, y, z : P(x, y) \land I_I^{-1}(y, z) \land I_I(x, z)$.

Impossible since it implies $I_I(x, z) \land I_I(y, z)$ by v of 5.1

$\exists x, y, z : P(x, y) \land I_I^{-1}(y, z) \land I_I(x, z)$.

Impossible since it implies $I_I(x, y)$ by vii of 5.1

$\exists x, y, z : P(x, y) \land I_I^{-1}(y, z) \land Q(z, x)$.

Impossible since it implies $I_I(x, y)$ by iv of 5.1.

iv. $(P \cup Q \cup I_I).Q \subset P \cup Q \cup I_I$

$(P.Q \cup Q.Q \subset P$ by ii of 5.1;

$I_I.Q \subset P \cup Q \cup I_I$. Suppose that:
\[ \exists x, y, z : I_t(x, y) \land Q(y, z) \land P(z, x). \]
Impossible since it implies \( P(y, x) \) by \( ii \) of 5.1
\[ \exists x, y, z : I_t(x, y) \land Q(y, z) \land Q(z, x). \]
Impossible since it implies \( P(y, x) \lor Q(y, x) \) by \( ii \) of 5.1
\[ \exists x, y, z : I_t(x, y) \land Q(y, z) \land I_t(z, x). \]
Impossible since it implies \( I_t(z, y) \) by \( vii \) of 5.1.

v. \( Q(P \cup Q \cup I_t^{-1}) \subseteq P \cup Q \cup I_t^{-1}. \)
\( Q \subseteq P \) by \( ii \) of 5.1;
\( Q \subseteq P \cup Q \) by \( ii \) of 5.1;
\( Q \subseteq P \cup Q \cup I_t^{-1} \). Suppose that:
\[ \exists x, y, z : Q(x, y) \land I_t^{-1}(y, x) \land P(z, x). \]
Impossible since it implies \( P(x, y) \) by \( ii \) of 5.1
\[ \exists x, y, z : Q(x, y) \land I_t^{-1}(y, z) \land Q(z, x). \]
Impossible since it implies \( P(y, x) \lor Q(y, x) \) by \( ii \) of 5.1
\[ \exists x, y, z : Q(x, y) \land I_t^{-1}(y, z) \land I_t(x, z). \]
Impossible since it implies \( I_t(x, y) \) by \( vii \) of 5.1.

From this theorem, we have the following algorithm which constructs \( I_t \)
by converting elements of \( I \) either to \( I_t \) or \( I_r \). By definition, when \( I_t(x, y) \) is
established, \( I_r(y, x) \) is also established. The algorithm is a direct application
of conditions \( i \) to \( vii \) of theorem 5.1. Therefore if it succeeds in transforming
all elements of \( I \) in elements of \( I_t \) (or \( I_r \)) then the PQI preference structure
under investigation is a PQI interval order. If on the other hand it fails then
the PQI preference structure under investigation is not a PQI interval order.
Failure of the algorithm can occur either because condition \( ii \) is not
satisfied or because during the construction of \( I_t \) a contradiction occurs (in
the sense that two elements of the set \( A \) are linked by two different relations).

**Algorithm 5.1**

1. **Step 1:** Verify \( P \cup Q \cup P \cup P \cup P \subseteq P \) and \( Q \subseteq P \cup Q; \)
2. **Step 2:** \( \forall x, y, z : I_t(x, y) \land P(x, z) \land Q(y, z) \rightarrow I_t(x, y); \)
3. **Step 3:** \( \forall x, y, z : I_t(x, y) \land P(z, x) \land Q(z, y) \rightarrow I_t(x, y); \)
4. **Step 4:** \( \forall x, y, z : I_t(x, y) \land I_t(y, z) \land P(x, z) \rightarrow I_t(x, y) \land I_t(z, y); \)
5. **Step 5:** \( \forall x, y, z : I_t(x, y) \land I_t(y, z) \land (Q \cup Q^{-1})(x, z) \rightarrow I_t(x, y); \)
   \( \forall x, y, z : I_t(x, y) \land I_t(y, z) \land (Q \cup Q^{-1})(x, z) \rightarrow I_t(y, x); \)
6. **Step 6:** \( \forall x, y, z : I_t(x, y) \land I_t(y, z) \rightarrow I_t(x, z); \)
7. **Step 7:** If there is one \( I_t(x, y) \) not yet established as \( I_t \) or \( I_r \), choose one of
   them and set it as \( I_t(x, y) \). Then return to 5. Otherwise stop.
Steps 1 to 4, are deterministic, in the sense that each $I_i$ established is mandatory. If a contradiction occurs, i.e. a newly established $I_i(x, y)$ has been formerly established as $\Phi(x, y)$, $\Phi$ being any among $P$, $Q$, $P^{-1}$, $Q^{-1}$, the algorithm fails. Steps 5 and 6 however, use already established $I_i$ in order to establish further $I_i$. The problem arises from Step 7 where $I_i$ is arbitrarily chosen. When the algorithm goes back to Step 5 to continue with establishing $I_i$, if a contradiction occurs, intuitively, it should backtrack to the last $I_i(x, y)$ established, reverse it to $I_i(y, x)$ and try again. In other terms the algorithm appears to have to explore a “tree structure” defined by the branches created by each arbitrary choice. In such a case the risk is to have to make an exhaustive research of the whole “tree”.

In the following we will demonstrate that the algorithm previously presented is “backtracking free”. In other words, any contradiction, implies the non-existence of a PQI interval order on $A$ and the algorithm can stop immediately without backtracking.

**Theorem 5.2** The algorithm 5.1 is backtracking free.

**Proof** We elaborate the demonstration observing how the setting of $I_i(x, y)$ (steps 5, 6) is propagated and analyzing contradictory situations. The demonstration consists of decomposing the problem in smaller cases and showing for each of them that when a contradiction occurs there is no backtracking necessity and the algorithm fails (the PQI preference structure is not a PQI interval order).

Before reaching step 7 the first time, the process is deterministic, we can therefore construct the graph $G_0 = (A, V_0)$ where $A$ is the usual set of objects on which the PQI preference structure applies and $V_0 = P \cup Q \cup I \cup I_i$ where $I$ consists of $(x, y)$ which are not yet set. $G_0$ is complete and all its arcs are directed except the ones in $I$. In the following we denote as a “triangle” a set of three elements in $A (x, y, z)$ such that $x\Phi y \Psi z \Theta x$, where $\Phi, \Psi, \Theta$ are any among $P, P^{-1}, Q, Q^{-1}, I_i, I_i^{-1}, I$.

**Proposition 5.3** In $G_0$, a triangle with at least an $I$-arc must be one of the following:

1. $I, I, I$
2. $I, I, I_i$
3. $I, I, Q$
4. $I, I_i, I_i^{-1}$
5. $I, I_i^{-1}, I_i$

14
6 - \( I.P.P^{-1} \)
7 - \( I.P^{-1}.P \)
8 - \( I.Q.Q^{-1} \)
9 - \( I.Q^{-1}.Q \)

Proof.
The application of steps 1-6 of the algorithm 5.1 excludes all other possibilities. For example, all triangles \( I_i.I.Q \) have changed to \( I_i.I_i^{-1}.Q \) by step 5.

Denote as \( I \)-path a path where each of its arcs is an \( I \)-arc. Consider then the partial graph \( G^* \) of \( G_1 \). \( G^* = (A, V) \) where \( V = \{ (x, y) | x, y \) \( y, \exists \) I-path from \( x \) to \( y \} \). The proofs of propositions 5.4 to 5.11 can be found in Appendix A.

**Proposition 5.4** \( G^* \) consists of connected components which:
1. are complete;
2. do not contain any \( P \)-arc;
3. are closed under the propagation of the setting of \( I_i \).

We have proved that \( G^* \) consists of connected components in which the propagation of the setting of \( I_i(x, y) \) is limited. Each component contains only \( Q \) or \( I \) or \( I_i \) arcs, while \( P \) arcs exist only among such components. Therefore, we can limit ourselves in analyzing only one connected component, denoted by \( G_1 = (A_1, V_1) \).

Let \((x^*, y^*)\) be an \( I \)-arc arbitrarily chosen in step 7 to become an \( I_i \)-arc. Consider iteration \( k \) of the algorithm. Denote as \( I_i^K \) the set of \( I \)-arcs set in \( I_i \) in the current step and as \( I_i^K \) the cumulative set of \( I \)-arcs set in \( I_i \) in all the former iterations of the algorithm. We have that \( I_i^K = I_i^K \cup I_i^{K-1} \).

Conventionally, in step 5, \((x^*, y^*)\) is added to \( I_i^K \).

**Proposition 5.5** \( I \)-arcs set to \( I_i \) by transitive closure (step 6) are never used in step 5 when the algorithms iterates.

Denote as a \( Q \)-path a path whose arcs are \( Q \) or \( Q^{-1} \) ones. In the set \( A \), let us consider now the following equivalence relation: \( \Theta(x, y) \Leftrightarrow \exists \) a \( Q \)-path from \( x \) to \( y \) and use \( X, Y, Z \) to denote equivalence classes. Therefore we can see graph \( G_1 \) as composed by equivalence classes of nodes each of which contains only \( Q \) and \( I \) arcs. Further on among such equivalence classes only \( I \)-arcs do exist.
Proposition 5.6 In step 5
i - the propagation of \( I_1(x,y) \in X \times Y \) is limited to \( X \times Y \).
ii - when \( X \neq Y \), the propagation of \( I_1 \) covers the whole set \( X \times Y \).
iii - if \( (x^*,y^*) \in X \times X \) then \( I_1^k \subseteq X \times X \)
iv - if \( (x^*,y^*) \in X \times Y, X \neq Y \) then \( I_1^k = X \times Y \).
v - whatever \((x, y)\) is chosen to be set in \( I_1 \) in step 5 the result is the same.
vi - if \( I_1(y^*,x^*) \) is chosen instead of \( I_1(x^*,y^*) \) then all the settings in this step will be reversed.

Proposition 5.6 states that, during the \( k \)-th iteration of the algorithm, step 5 sets to \( I_k \) either some \( I \)-arcs included in one equivalence class (of relation \( \Theta \)) or all \( I \)-arcs among two equivalence classes.

Consider now step 6. In each application of step 6, setting \( I_1(x, z) \) from \( I_1(x, y) \) and \( I_1(y, z) \), implies that at least one arc, let's say \((x, y)\), has to be set during, either this step, or the two last steps 5,7. In a formal notation we have:

Proposition 5.7 In Step 6:
i - if \((x, y) \in X \times X \) then \( z \in X \).
ii - if \((x^*, y^*) \in X \times X \) is set in step 5 then \( I_1^k \subseteq X \times X \).
iii - if it exists \( I_1^k(x, z) \in X \times Z, X \neq Z \) then \( X \times Z \subseteq I_1^k \).
iv - if \((x^*, y^*) \in X \times Y, X \neq Y \) is set in step 5, only arcs connecting different classes are set in step 6 (in other terms if \( I_1(x, z) \in X \times Z \) is set in step 6 then \( Z \neq X \land Z \neq Y \)).

These results show that if we choose an arc \((x^*, y^*)\) to set in \( I_1 \), if it is inside one equivalence class it does not propagate \( I_1 \) outside this class, while if it connects two different classes, it does not propagate \( I_1 \) into any class. Furthermore, as the algorithm has passed through steps 5, 6 before the establishment of \( G_1 \) at least once, all the arcs between two classes \( X, Y \) are of the same type (either \( I \)-arcs or \( I \)-arcs). Therefore, the problem can be further decomposed into two sub-problems:
a) - outside all the equivalent classes, we consider the same problem with \( G_1 \) replaced by \( G_2 = (A_2, V_2) \) where \( A_2 \) is the quotient set \( A^\Theta \) and \( V_2 = \{(X,Y) | X \in A_2 \land \exists (x,y) \in X \times Y \text{ such that } I_1 \text{ or } I_1 \text{ holds}\} \) according to the type of the arc for \( A_2 \), \( X, Y \).
b) - inside each equivalent class, we consider the same problem with \( G_1 \) replaced by \( G_3 = (A_3, V_3) \).

The sub-problem a) is trivial, as the graph \( G_2 \) contains only \( I \) or \( I \) arcs, furthermore, the part of \( G_2 \) covered by \( I \)-arcs is already \( I \) transitively
closed since the algorithm has already gone through Step 6. The problem is reduced to the construction of a linear order. Therefore, we have to deal only with the sub-problem (b).

We have to demonstrate now that the algorithm is backtracking free on \( G_2 \) where the arcs are \( Q, I, I \) and there is a \( Q \)-path connecting any two different nodes. We consider now the possible situations where a contradiction may occur.
**Theorem 6.1** Algorithm 5.1 is in polynomial time \(O(n^5)\)

**Proof** The algorithm presented in the previous section can be represented in the following way (including some small variations discussed immediately after):

**Algorithm 6.1**

*Step 1:* \(p_{ij} + p_{jk} \leq 1 + p_{ik},\) \(p_{ij} + q_{jk} \leq 1 + p_{ik},\) \(q_{ij} + q_{jk} \leq 1 + p_{ik} + q_{ik} \quad \forall \ i, j, k;\)

*Step 2:* \(i_{ij} = p_{ik} = q_{jk} = 1 \Rightarrow l_{ij} = 1 \forall i, j, k;\)

...
function setLabel(i,j,k: integer)
    if (Lik, Lkj no label)
        set new label to Lik and Lkj
    else if (Lik = L1, Lkj no label)
        set Lkj to L1
    else if (Lik no label, Lkj = L2)
        set Lik to L2
    else if (Lik = L1 et Lkj = -L1)
        return false (conflict)
    else if (Lik = L1 et Lkj = L2)
        unify these two labels
    endif
    return true

Furthermore it is easy to see that the decomposition of the PQI graph in $G_1$ and its connected components, the decomposition in $G_2$ and $G_3$ and the construction of the linear order in $G_2$ are all in polynomial time. Therefore the whole algorithm is in polynomial time.

7 Conclusions

The paper has presented an answer for the problem concerning the representation of preferences by intervals by showing necessary as well as sufficient conditions to see if a preference structure is a PI interval order, PQI semi order or PQI interval order. For PQI interval order, it provided also an algorithm to verify whether a PQI preference structure on a finite set $A$ is a PQI interval order. In other words verify if it is possible to associate to each element of $A$ an interval such that if the interval associated to $x$ is completely to the right of the interval associated to $y$, then $x$ is strictly preferred to $y$, if one interval is included in the other, then $x$ is indifferent to $y$ and if the interval associated to $x$ is to the right of the interval associated to $y$, their intersection being not empty, then $x$ is weakly preferred to $y$. We first demonstrate that the algorithm, although it appears having to explore a tree generated by branches of arbitrary choices, is backtracking free and then we demonstrate that runs in polynomial time. We consider such a result very promising, since it enables an efficient check of the existence of PQI interval orders which are very common in many different cases, including preference modeling and temporal logic.
References


Appendix A

Proof of Proposition 5.4
i. If $x, y$ belongs to a connected component then there exists a path $a_0 = x, a_1, \ldots, a_k = y$. \forall i = 0...k-1, if exists an $I$-path from $a_i$ to $a_{i+1}$ then there exists an $I$-path from $x$ to $y$ and therefore $(x, y) \in V_I$.

ii. If a $P$-arc exists, choose $P(x, y)$ such that the length $k$ of the $I$-path $a_0 = x, a_1, \ldots, a_k = y$ is minimal. Consider then the arc $a_1, a_k$ (it exists from the completeness of the component), then from proposition 5.3 we have $P(a_1, a_k)$ and therefore we have another $P$-arc with length of the $I$-path $< k$. Impossible.

iii. Immediate from conditions vi and vii of Theorem 5.1 (steps 5 and 6 of the algorithm).

Proof of Proposition 5.5
First consider $(x_1, x_2)$ such that $I^k_i(x_1, x_2)$ in step 6. Therefore it exists $I^k_i(x_1, x_2) \cap I^k_i(x_3, x_2)$. If, for example, $(x_1, x_3)$ was also established in $I^k_i$ (in the current step 6) then it exists $I_i(x_1, x_4) \cap I_i(x_4, x_3)$ and so on until an $I_i^{K-1}$-path is obtained. Therefore for all $(x, y)$ such that $I_i$ is established in the current step 6 exists an $I_i^{K-1}$-path from $x$ to $y$.

Let now $(x, y)$ to be an arc set to $I_i$ in the last step 6, participating to the setting of arc $(x, z)$ in step 5 through let’s say $Q(z, y)$. Let us consider the situation in the last step 6:

$I^k_i(x, y) \Rightarrow \exists \ I_i^{K-1}$-path $t_0 = x, t_1, \ldots, t_k = y$.

Consider the triangle $x, t_{k-1}, y$ where $Q(x, y) \cap I_i^{K-1}(t_{k-1}, y)$.

If $Q(t_{k-1}, y)$ then $Q(t_{k-1}, y) \cap Q(x, y) \Rightarrow (P \cup Q)(t_{k-1}, y)$, conflict with $I_i(t_{k-1}, y)$.

If $I_i(t_{k-1}, y)$ then $I_i^{K-1}(t_{k-1}, y) \cap Q(x, y) \Rightarrow I_i^{K-1}(t_{k-1}, y)$ (at least in the last step 5). Therefore it exists an $I_i^{K-1}$-path from $x$ to $z$, that is $I_i(x, z)$ must be set at least at the same time as $(x, y)$. We conclude that $Q(x, t_{k-1})$. Repeat this procedure, and we get at last $Q(x, t_1)$, which together with $I_i^{K-1}(x, t_1)$ gives $I_i^{K-1}(x, y)$ i.e. $(x, z)$ must have been set before $(x, y)$.

Proof of Proposition 5.6
i - In each application of step 5, consider $(x, y) \in X \times Y$ such that $I_i(x, y)$. Relation $I_i$ will propagate to $(x', y)$ or $(x, y')$, $x', y'$ arbitrary. From Theorem 5.1 and proposition 5.4 (no $P$-arcs in $G^1$) we know that there have to exist
Q-paths from $x$ to $x'$ and from $y$ to $y'$. Therefore $(x', y') \in X \times Y$.

ii - $(x', y') \in X \times Y$ implies that there exist $Q$-paths $a_0 = x, a_1, ..., a_k = x'$, and

$\mathbf{b}_0 = y, b_1, ..., b_l = y'$. Applying consecutively step 5 on these two paths we obtain the setting in $I_t$ of $(x, y), (a_1, y), ..., (x', y)$ and then of $(x', b_1), (x', b_2), ..., (x', y')$.

iii and iv - Immediate from propositions (5.5), (5.6.i) and (5.6.ii).

v and vi - Immediate from Theorem 5.1.

\textbf{Proof of Proposition 5.7}

i - Otherwise, consider the first setting with $x \in Z \neq X$. It implies that $I_t(y, x) \in X \times Z, Z \neq X$ and since $(x, z)$ is the first such setting, $I_{t'}^{k-1}(y, z)$ holds. We have $x, y \in X \land z \in Z \land I_{t'}^{k-1}(y, z)$ which implies $I_{t'}^{k-1}(x, z)$ as it must be set at least in the last step 5 (proposition (5.6.ii)). Contradiction.

ii - Immediate from (5.6.iii), (5.7.i).

iii - Otherwise it should exist $(x', y') \in X \times Z \setminus I_t^{k}$. In the next step 5 $(x, z)$, which is set in this step 6, will propagate $I_t$ to $(x', y')$, which is impossible because of (5.5).

iv - Suppose that $(x^*, y^*) \in X \times Y, X \neq Y$ is introduced in step 5. Then all the arcs of $X \times Y$ are set to $I_t$ and only these arcs. The setting in step 6 is the propagation of such arcs. Let $I_t(x, y) \land I_t(y, z) \Rightarrow I_t(x, z), x \in X, y \in Y, z \in Z$ the first setting in step 6 with $I_t^{k-1}(x, y), I_t^{k-1}(y, z)$ and $(x, y)$ set in the last steps 5.7, i.e. $(x, y) \in X \times Y$ and $X \times Y \subset I_t^{k-1}$. If $Z = X$ then $(x, y)$ should have been set at the same time as $(x, y)$; which contradicts $I_t^{k}(y, z)$. We conclude that $Z \neq X$ (and similarly that $Z \neq Y$).

\textbf{Proof of Proposition 5.8}

i - Let $Q = Q \cup Q^{-1}$ and $\Psi = I_t \cup I_t^{-1} \cup Q$. In an equivalence class we have $\forall(x, z) \Psi(x, z)$.

If in Step 5 we had $Q(x, z)$ then $I_t^{k}(x, y) \land Q(x, z) \land I_t(x, y) \Rightarrow I_t^{k}(x, y)$, in contradiction with $I_t^{k}(y, z)$. Therefore we have $\neg Q(x, z)$ and $\Psi = I_t \cup I_t^{-1}$.

The transition from $I_t^{k}(x, y)$ to $I_t^{k}(y, z)$ in step 5 passes through 2 $Q$-paths $x_1 = x, x_2, ..., x_n = y$ and $y_1 = y, y_2, ..., y_m = z$ where $(x_i = x_{i+1}$ and $y_i = y_{i+1}$ and $\Psi(x, y), \Psi(y, x)) \iff (i \neq m)$.
We have \( Q(x, x_2) \land I(x_2, x) \land \Psi(x, z) \Rightarrow \Psi(x_2, z) \). Therefore the situation is not changed \((x_2 \text{ plays now the role of } x)\).

2. If \( y_2 \neq y \) then \( x = x_2 \). Therefore \( Q(y, y_2) \land I(x, y_2) \Rightarrow I^k_f(x, y_2) \). If \( Q(y_2, z) \) holds we have \( Q(y_2, z) \land I^k_f(x, y_2) \land I(x, z) \Rightarrow I^k_f(x, z) \). Otherwise, \( Q(y, y_2) \) and \( I^k_f(y, z) \) give \( I^k_f(y_2, z) \) and the situation is not changed \((y_2 \text{ plays now the role of } y)\).

In order to pass from \( y \) to \( z \), it must exist a \( k \) such that \( y_{k+1} = z \) and \( y_k \neq y \), i.e. \( Q(y_k, y_{k+1}) \Rightarrow Q(y_k, z) \Rightarrow I^k_f(x, z) \).

ii - Let \( \Psi(x, t) \). Since \( Q(y, t) \) or \( Q(x, x) \) is in contradiction with \( I^k_f(x, y) \), \( I^{k-1}_f(y, z) \) and \( I^k_f(x, t) \), we have \( I(y, t) \) and \( I(x, z) \).

If \( \Psi = Q \) then \( I^k_f(x, y) \land I(y, t) \land Q(x, t) \Rightarrow I^k_f(t, y) \). But \( I^k_f(t, y) \land I^k_f(x, t) \Rightarrow I^k_f(x, y, z) \) \((5.8.i)\) in contradiction with \( I^{k-1}_f(y, z) \). Therefore \( \Psi = I^k_f \). The transition from \( I^k_f(x, y) \) to \( I^k_f(x, t) \) in step 5 passes through 2 \( Q \)-paths \( x_1 = x, x_2, \ldots, x_n = z \) and \( y_1 = y, y_2, \ldots, y_n = z \) where \( x_i = x_{i+1} \) and \( y_i \neq y_{i+1} \) and \( I(x_i, y_i), I(x_{i+1}, y_{i+1}) \) or \( x_i \neq x_{i+1} \) and \( y_i = y_{i+1} \) and \( I(x_i, y_i), I(x_{i+1}, y_{i+1}) \).

1. If \( y_2 = y \) then \( x \neq x_2 \) therefore \( Q(x, x_2) \).

\( Q(x, x_2) \land I^k_f(x, y) \land I(x_2, y) \Rightarrow I^k_f(x_2, y) \). If \( Q(x_2, t) \) we have \( Q(x_2, t) \land I^k_f(x_2, y) \land I(x, t) \Rightarrow I^k_f(x, t) \). But \( I^k_f(x, t) \land I^k_f(t, y) \Rightarrow I^k_f(x, y, z) \) \((5.8.i)\), in contradiction with \( I^{k-1}_f(y, z) \).

We have then \( \neg Q(x_2, t) \Rightarrow I(x_2, t) \). \( Q(x_2, x) \land I(x_2, t) \land \Psi(x, t) \Rightarrow \Psi(x_2, t) \) and the situation is not changed \((x_2 \text{ plays now the role of } x)\).

2. If \( y_2 \neq y \) then \( x_2 = x \), therefore \( Q(y, y_2) \).

\( Q(y, y_2) \land I^k_f(x, y) \land I(x, y_2) \Rightarrow I^k_f(x, y_2) \).

If \( Q(y_2, t) \), we have \( Q(y_2, t) \land I^k_f(x, y_2) \land \Psi(x, t) \Rightarrow I^k_f(x, t) \). Otherwise, if \( Q(y_2, z) \) then \( Q(y_2, z) \land I^k_f(x, y_2) \land I(x, z) \Rightarrow I^k_f(x, z) \).

Therefore, we have \( I^k_f(x, z) \land I^{k-1}_f(x, z) \Rightarrow I^k_f(x, z) \) \((5.8.i)\).

If \( I(y_2, z) \) then \( Q(y_2, y) \land I^{k-1}_f(y, z) \land I(y_2, z) \Rightarrow I^{k-1}_f(y_2, z) \) and the situation is not changed \((y_2 \text{ plays now the role of } y)\).

So, we have either \( I(x, t) \) when it exists \( Q(y, t) \) or the only way to pass from \( y \) to \( t \) is through some \( y_{k+1} = t \) and \( y_k \neq t \) i.e. \( Q(y_k, y_{k+1}) \Rightarrow Q(y_k, t) \Rightarrow I(x, t) \).

iii - If \( I^{k-1}_f(x, y) \) and \( I^{k-1}_f(y, x) \) are set at least in the last step 6 then \( I^{k-1}_f(x, z) \) is also set at least in the last step 6.

**Proof of Proposition 5.9**

Let an \( I_k \)-circuit with arcs \( I^k_f \) or \( I^{k-1}_f \). With \((5.8.i)\), we can replace all \( I^k_f \)-paths with \( I^k_f \)-arcs. With \((5.8.iii)\), we can replace all \( I^{k-1}_f \)-paths with \( I^{k-1}_f \)-arcs. We get at last an \( I_k \)-circuit with alternative \( I^k_f \)-arcs and \( I^{k-1}_f \)-arcs.