PQI INTERVAL ORDERS

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PQI Ordres d'intervalle

Résumé

Nous présentons la réponse à un problème ouvert dans la modélisation des préférences à l'aide d'intervalles. Soit un ensemble fini $A$ et trois relations binaires $P, Q, I$, appelées "préférence stricte", "préférence faible" et "indifférence", respectivement. Nous présentons les conditions nécessaires et suffisantes pour pouvoir associer à chaque élément de $A$ un intervalle de façon à ce que si un intervalle est "complètement à droite" de l'autre on obtient la relation $P$, si un intervalle est inclus dans l'autre on obtient la relation $I$ et si un intervalle "est à droite" de l'autre, mais leur intersection n'est pas vide on obtient la relation $Q$ ($Q$ modélisant l'hésitation entre $P$ et $I$). Deux structures de préférences spécifiques sont caractérisées: le PQI ordre d'intervalle et le PQI quasi-ordre. Détecter l'existence d'un PQI quasi-ordre est immédiat. Par contre, la détection d'un PQI ordre d'intervalle est plus difficile parce que le théorème d'existence est une formule du deuxième ordre. Pour cette raison nous présentons un algorithme pour la détection de PQI ordres d'intervalle et nous démontrons qu'il est "backtracking free". Avec ce résultat nous pouvons présenter une implémentation matricielle de l'algorithme et montrer qu'il est polynomial.

Mots clés: Intervalle, Ordre d'intervalle, Indifférence, Préférence Faible, Préférence Stricte.

PQI Interval Orders

Abstract

We provide an answer to an open problem concerning the representation of preferences by intervals. Given a finite set $A$ with three relations $P, Q, I$ standing for "strict preference", "weak preference" and "indifference" respectively, necessary and sufficient conditions are provided for representing each element of $A$ by an interval in such a way that $P$ holds when one interval is completely to the right of the other, $I$ holds when one interval is included to the other and $Q$ holds when one interval is to the right of the other, but they do have a non empty intersection ($Q$ modeling the hesitation). Two specific preference structures: PQI semi orders and PQI interval orders, will be considered. While the detection of a PQI semi order is straightforward, the case of the PQI interval order is more difficult as the theorem of existence consists in a second-order formula. To this purpose, the paper also presents an algorithm for detecting a PQI interval order and demonstrates that it is backtracking free. This result leads to a matrix version of the algorithm which can be proved to be polynomial.

Keywords: Intervals, Interval Orders, Indifference, Weak Preference, Strict Preference.
1 Introduction

Comparing intervals, instead of discrete values, is a frequently encountered problem in preference modelling and decision aid. This is due to the fact that the comparison of alternatives (outcomes, objects, candidates, ...) gen-
specific relations in order to represent situations of hesitation in preference
modelling (see Turnley & Wilke 1995).
- reflexive: if $\forall x \ S(x, x)$
- symmetric: if $\forall x, y \ S(x, y) \rightarrow S^{-1}(y, x)$
- asymmetric: if $\forall x, y \ S(x, y) \rightarrow S(y, x)$
- complete: if $\forall x, y, x \neq y, \ S(x, y) \lor S^{-1}(y, x)$
- transitive: if $\forall x, y, z \ S(x, y) \land S(y, z) \rightarrow S(x, z)$
- negatively transitive: if $\forall x, y, z \ \neg S(x, y) \land \neg S(y, z) \rightarrow \neg S(x, z)$

**Definition 2.2** A binary relation $S$ is:
- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence iff it is reflexive, symmetric and transitive.

In this paper we will consider relations representing strict preference, weak preference and indifference situations. We will denote them $P, Q, I$ respectively. Moreover, such relations are expected to satisfy some “natural” properties of the type announced in the following two definitions.

**Definition 2.3** A $(P, I)$ preference structure on a set $A$ is a couple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ is asymmetric;
- $I \cup P$ is complete;
- $P$ and $I$ are mutually exclusive ($P \cap I = \emptyset$).

**Definition 2.4** A $(P, Q, I)$ preference structure on a set $A$ is a triple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ and $Q$ are asymmetric;
- $I \cup P \cup Q$ is complete;
- $P, Q$ and $I$ are mutually exclusive.

Finally we introduce an equivalence relation as follows:

**Definition 2.5** The equivalence relation associated to a $(P, Q, I)$ preference structure is the binary relation $E$, defined on the set $A$, such that, $\forall x, y \in A$:

\[
E(x, y) \iff \forall z \in A : \begin{cases} 
  P(x, z) \iff P(y, z) \\
  Q(x, z) \iff Q(y, z) \\
  I(x, z) \iff I(y, z) \\
  Q(z, x) \iff Q(z, y) \\
  P(z, x) \iff P(z, y) 
\end{cases}
\]
4 \( (P, Q, I) \) interval orders

As mentioned in the introduction, we are interested in situations where, comparing elements evaluated by intervals, one wants to distinguish three situations: indifference if one interval is included in the other, strict preference if one interval is completely "to the right" of the other and weak preference when one interval is "to the right" of the other, but they have a non-empty intersection. Definition 4.1 precisely states this kind of situation, \( l(x) \) and \( r(x) \) respectively representing the left and right extremities of the interval associated to any element \( x \in A \).

**Definition 4.1** A \( (P, Q, I) \) preference structure on a finite set \( A \) is a PQI interval order, iff there exist two real valued functions \( l \) and \( r \) such that, \( \forall x, y \in A, x \neq y \):
- \( r(x) > l(x) \);
- \( P(x,y) \Leftrightarrow r(x) > l(x) > r(y) > l(y) \);
- \( Q(x,y) \Leftrightarrow r(x) > r(y) > l(x) > l(y) \);
- \( I(x,y) \Leftrightarrow r(x) > r(y) > l(y) > l(x) \) or \( r(y) > r(x) > l(x) > l(y) \).

The reader will notice that the above definition immediately follows Definition 3.1 since a preference structure characterized a PI interval order can always be seen as PQI interval order also. We give now necessary and sufficient conditions under which such a preference structure exists.

**Theorem 4.1** A \( (P, Q, I) \) preference structure on a finite set \( A \) is a PQI interval order, iff there exists a partial order \( I_i \) such that:
- i) \( I = I_i \cup I_r \cup I_o \) where \( I_o = \{(x, x), x \in A\} \) and \( I_r = I_i^{-1} \);
- ii) \( (P \cup Q \cup I_i)P \subset P \);
- iii) \( P(P \cup Q \cup I_r) \subset P \);
- iv) \( (P \cup Q \cup I_i)Q \subset P \cup Q \cup I_i \);
- v) \( Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r \);

Proof

**Necessity.**

We first give an outline of necessity demonstration which is the easy part of the theorem. If \( (P, Q, I) \) is a PQI interval order, then defining
- \( I_i(x,y) \Leftrightarrow l(y) < l(x) < r(x) < r(y) \)
- \( I_r(x,y) \Leftrightarrow l(x) < l(y) < r(y) < r(x) \)
we obtain two partial orders satisfying the desired properties. As an example we demonstrate property (v):

\[ Q(x, y) \text{ and } (P \cup Q \cup I_r)(y, z) \text{ imply } r(x) > r(y) \text{ and } r(y) > r(z), \text{ hence } r(x) > r(z), \text{ so that } (P \cup Q \cup I_r)(x, z). \]

**Sufficiency.**

Conversely let us assume the existence of \( I_l \) satisfying the properties of the theorem. Define a set \( A' \) isomorphic to \( A \) (\( A' \) and \( A \) being disjoint) and denote by \( x' \) the image of \( x \in A \) in \( A' \). In the set \( A \cup A' \) let us define the relation \( S \) as follows: \( \forall x, y \in A, x \neq y \)

- \( S(x', x) \)
- \( S(x, y) \Leftrightarrow (P \cup Q \cup I_l)(x, y) \)
- \( S(x', y') \Leftrightarrow (P \cup Q \cup I_r)(x, y) \)
- \( S(x, y') \Leftrightarrow P(x, y) \)
- \( S(x', y) \Leftrightarrow \neg P(y, x) \)

We demonstrate now that \( S \) is a linear order (irreflexive, complete and transitive relation) in \( A \cup A' \).

Irreflexivity results from irreflexivity of \( P, Q, I_l \) and \( I_r \).

To demonstrate completeness of \( S \) remark that for \( x \neq y \):

\[ \neg S(x, y) \Leftrightarrow \neg (P \cup Q \cup I_l)(x, y) \]
\[ \Leftrightarrow (P \cup Q \cup I_l)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \]
\[ \Rightarrow S(y, x) \]

\[ \neg S(x', y') \Leftrightarrow \neg (P \cup Q \cup I_r)(x, y) \]
\[ \Leftrightarrow (P \cup Q \cup I_r)(y, x) \text{ since } P \cup Q \cup I \text{ is complete and } I = I_l \cup I_r \cup I_o \]
\[ \Rightarrow S(y', x') \]

\[ \neg S(x, y') \Leftrightarrow \neg P(x, y) \]
\[ \Leftrightarrow S(y', x) \]

\[ \neg S(x', y) \Leftrightarrow P(y, x) \]
\[ \Leftrightarrow S(y, x') \]

We demonstrate now that \( S \) is transitive.
• $S(x,y)$ and $S(y,z)$ imply $(P \cup Q \cup I)(x,y)$ and $(P \cup Q \cup I)(y,z)$. From conditions ii) and iv) of the theorem, we know that $(P \cup Q \cup I)(x,y)$ and $(P \cup Q)(y,z)$ imply $(P \cup Q \cup I)(x,z)$, hence $S(x,z)$. From transitivity of $I$, we have that $I(x,y)$ and $I(y,z)$ imply $I(x,z)$, hence $S(x,z)$. Finally, if $(P \cup Q)(x,y)$ and $I(y,z)$ then $(P \cup Q \cup I)(x,z)$ because, if not, we would have $(P \cup Q \cup I)(x,z)$ which with $I(y,z)$ would give $(P \cup Q \cup I)(y,z)$ (by conditions ii) and iv) and transitivity of $I$), contradiction. So we get $S(x,z)$.

• $S(x,y)$ and $S(y,z')$ imply $(P \cup Q \cup I)(x,y)$ and $P(y,z)$, which, by condition ii), give $P(x,z)$, hence $S(x,z')$.

• $S(x,y')$ and $S(y',z)$ imply $P(x,y)$ and $\neg P(x,y)$. If $\neg S(x,z)$, then $(P \cup Q \cup I)(x,z)$ which, with $P(x,y)$ and by condition ii) would give $P(x,y)$, a contradiction. Thus $S(x,z)$. This reasoning applies also in the case $y = z$.

• $S(x,y')$ and $S(y',z')$ imply $P(x,y)$ and $(P \cup Q \cup I)(y,z)$, which, by condition iii), give $P(x,z)$, hence $S(x',z')$.

• $S(x',y')$ and $S(y',z)$ imply $(P \cup Q \cup I)(x,y)$ and $(P \cup Q \cup I)(y,z)$. If $\neg S(x',z)$, then $P(x,z)$ which, with $(P \cup Q \cup I)(x,y)$ and by condition iii) would give $P(z,y)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y = z$.

• $S(x',y')$ and $S(y',z')$ imply $(P \cup Q \cup I)(x,y)$ and $(P \cup Q \cup I)(y,z)$. From conditions iii) and v) of the theorem, we know that $(P \cup Q)(x,y)$ and $(P \cup Q \cup I)(y,z)$ imply $(P \cup Q \cup I)(x,z)$, hence $S(x',z')$. From transitivity of $I$, we have that $I(x,y)$ and $I(y,z)$ imply $I(x,z)$, hence $S(x',z')$. Finally, if $I(x,y)$ and $(P \cup Q)(y,z)$ then $(P \cup Q \cup I)(x,z)$ because, if not, we would have $(P \cup Q \cup I)(x,z)$ which with $I(x,y)$ would give $(P \cup Q \cup I)(z,y)$ (by condition iii) and v) and transitivity of $I$), contradiction. So we get $S(x',z')$.

• $S(x',y)$ and $S(y,z)$ imply $\neg P(y,z)$ and $(P \cup Q \cup I)(y,z)$. If $\neg S(x',z)$, then $P(x,z)$ which, with $(P \cup Q \cup I)(y,z)$ and by condition ii) would give $P(y,z)$, a contradiction. Thus $S(x',z)$. This reasoning applies also in the case $y = z$.

• $S(x',y)$ and $S(y,z')$ imply $\neg P(y,z)$ and $P(y,z)$. If $\neg S(x',z')$, then $(P \cup Q \cup I)(z,x)$ which, with $P(y,z)$ and by condition iii) would give $P(y,z)$, a contradiction. Thus $S(x',z')$. This reasoning applies also in the case $y = z$. 

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Since $S$ is a linear order on $A \cup A'$, there exists a real valued function $u$ such that, $\forall x, y \in A$:
- $S(x, y) \iff u(x) > u(y)$;
- $S(x', y') \iff u(x') > u(y')$;
- $S(x, y') \iff u(x) > u(y')$;
- $S(x', y) \iff u(x') > u(y)$.

We define $\forall x \in A$, $l(x) = u(x)$ and $r(x) = u(x')$ and we obtain:

- $\forall x : r(x) > l(x)$, since $S(x', x)$.
- $\forall x, y : P(x, y) \iff S(x, y') \iff l(x) > r(y)$.
- $\forall x, y : Q(x, y) \iff S(x, y) \land S(x', y') \land \neg P(x, y) \iff l(x) > l(y)$ and $r(x) > r(y)$ and $r(y) > l(x)$, equivalent to:
  \[ r(x) > r(y) > l(y) \land r(y) > r(x) > l(x) \land l(x) > l(y) \]
  since $I(x, y)$ holds in all the remaining cases.

We can complete the investigation providing a characterization of PQI semi orders.

**Definition 4.2** A PQI semi order is a PQI interval order such that $\exists k > 0$ constant for which $\forall x : r(x) = l(x) + k$

In other words, a PQI semi order is a $(P, Q, I)$ preference structure for which there exists a real valued function $l : A \rightarrow \mathbb{R}$ and a positive constant $k$ such that $\forall x, y, x \neq y$:
- $P(x, y) \iff l(x) > l(y) + k$;
- $Q(x, y) \iff l(y) + k > l(x) > l(y)$;
- $I(x, y) \iff l(x) = l(y)$ (actually $I$ reduces to $I_o$).

For such preference structures the following theorem holds.

**Theorem 4.2** A $(P, Q, I)$ preference structure is a PQI semi order iff:

i) $I$ is transitive

ii) $PP \cup PQ \cup QP \subseteq P$;

iii) $QQ \subseteq P \cup Q$;
Proof

Necessity is trivial. We give only the sufficiency proof. Since \( I \) is an equivalence relation, we consider the relation \( \phi \cup \mathcal{L} \) on the set \( I \). Such a
On the one hand, if we consider the relation $\hat{F} = \Omega \Omega^{-1}$, it is easy to
ii. \( P.Q \cup Q.P \cup P.P \subset P \) and \( Q.Q \subset P \cup Q \);

iii. \((P,Q^{-1} \cap I) \subset I_t\);

iv. \((P^{-1}.Q \cap I) \subset I_t\);

v. \((I_1 \cap P) \subset I_t.I_r\);

vi. \((I_1 \cap (Q \cup Q^{-1})) \subset (I_t.I_r) \cup (I_r.I_t)\) \( \times \)

vii. \( I_t.I_t \subset I_t\);

**Proof**

We will now prove that the conditions i – vii are equivalent to i – v of theorem 4.1.

*Necessity - (i-v) of 4.1 \( \Rightarrow \) (i-vii) of 5.1*

ii. \( P.Q \cup Q.P \cup P.P \subset P \) and \( Q.Q \subset P \cup Q \);

\((P \cup Q \cup I_t)P \subset P \Rightarrow P.P \cup Q.P \subset P\)

\(P(P \cup Q \cup I_r) \subset P \Rightarrow P.Q \subset P\)

\((P \cup Q \cup I_1)Q \subset P \cup Q \cup I_t \Rightarrow Q.Q \subset (P \cup Q) \cup I_t\)

\(Q(P \cup Q \cup I_r) \subset P \cup Q \cup I_r \Rightarrow Q.Q \subset (P \cup Q) \cup I_r\)

\(I_1 \) asymmetric \( \Rightarrow I_t \cap I_r = \emptyset\)

We have then \( Q.Q \subset (P \cup Q) \cup (I_t \cap I_r) = (P \cup Q)\).

iii. \((P,Q^{-1} \cap I) \subset I_t\);

We will prove that \( P.Q^{-1} \cap (I_0 \cup I_r) = \emptyset\). Suppose that:

\( \exists x, y, z : P(x,y) \land Q(z,y) \land I_0(x,z) \) i.e. \( x = z \)
As $P, Q, I$ are mutually exclusive and the above result (vii), we have

$$(Q \cup Q^{-1}) \cap (I \cup I_lI_l \cup I_rI_r) = \emptyset$$

v. $(I_lI_l \cap P) \subseteq I_lI_l$;

Similarly to vi, we have $(I_lI_l \cap P) \subseteq (I_lI_r \cup (I_rI_l))$, we still have to prove that $P \cap I_rI_l = \emptyset$.

$\exists x, y, z : P(x, y) \land I_r(x, y) \land I_r(x, y) \Rightarrow P(x, y) \land I_r(x, y)$, impossible.

Sufficiency - (i-vii) of 5.1 $\Rightarrow$ (i-iv) of 4.1

ii. $(P \cup Q \cup I_l).P \subseteq P$.

$(P \cup Q \cup I_l).P \subseteq P$ by ii of 5.1

$I_lI_l \subseteq P$. Suppose that:

$\exists x, y, z : I_l(x, y) \land P(y, z) \land P(z, x)$.

Impossible since it implies $P(y, x)$ by ii of 5.1

$\exists x, y, z : I_l(x, y) \land P(y, z) \land Q(z, x)$.

Impossible since it implies $P(y, x)$ by ii of 5.1

$\exists x, y, z : I_l(x, y) \land P(y, z) \land I_l(x, x)$.

Impossible since it implies $I_l(x, y)$ by vii of 5.1

$\exists x, y, z : I_l(x, y) \land P(y, z) \land I_l(x, x)$.

Impossible since it implies $I_l(x, y)$ by vii of 5.1

$\exists x, y, z : I_l(x, y) \land P(y, z) \land Q(x, x)$.

Impossible since it implies $I_l(x, y)$ by iii of 5.1

$\exists x, y, z : I_l(x, y) \land P(y, z) \land Q(x, x)$.

Impossible since it implies $I_l(x, y)$ by iii of 5.1.

iii. $P(P \cup Q \cup I_l^{-1}) \subseteq P$.

$(P \cup Q \cup I_l^{-1}).P \subseteq P$ by ii of 5.1.
\[ \exists x, y, z : I_l(x, y) \land Q(y, z) \land P(z, x). \]

Impossible since it implies \( P(y, x) \) by \( ii \) of 5.1

\[ \exists x, y, z : I_l(x, y) \land Q(y, z) \land Q(z, x). \]

Impossible since it implies \( P(y, x) \lor Q(y, x) \) by \( ii \) of 5.1

\[ \exists x, y, z : I_l(x, y) \land Q(y, z) \land I_l(x, z). \]

Impossible since it implies \( I_l(x, y) \) by \( vii \) of 5.1.

v. \( Q.(P \cup Q \cup I_l^{-1}) \subset P \cup Q \cup I_l^{-1} \).

\[ Q.P \subset P \] by \( ii \) of 5.1;

\[ Q.Q \subset P \cup Q \] by \( ii \) of 5.1;

\[ Q.I_l^{-1} \subset P \cup Q \cup I_l^{-1} \]. Suppose that:

\[ \exists x, y, z : Q(x, y) \land I_l^{-1}(y, z) \land P(z, x). \]

Impossible since it implies \( P(x, y) \) by \( ii \) of 5.1

\[ \exists x, y, z : Q(x, y) \land I_l^{-1}(y, z) \land Q(z, x). \]

Impossible since it implies \( P(y, x) \lor Q(y, x) \) by \( ii \) of 5.1

\[ \exists x, y, z : Q(x, y) \land I_l^{-1}(y, z) \land I_l(x, z). \]

Impossible since it implies \( I_l(x, y) \) by \( vii \) of 5.1.

From this theorem, we have the following algorithm which constructs \( I_l \) by converting elements of \( I \) either to \( I_l \) or \( I_r \). By definition, when \( I_l(x, y) \) is established, \( I_r(y, z) \) is also established. The algorithm is a direct application of conditions \( i \) to \( vii \) of theorem 5.1. Therefore if it succeeds in transforming all elements of \( I \) in elements of \( I_l \) (or \( I_r \)) then the PQI preference structure under investigation is a PQI interval order. If on the other hand it fails then the PQI preference structure under investigation is not a PQI interval order. Failure of the algorithm can occur either because condition \( ii \) is not satisfied or because during the construction of \( I_l \) a contradiction occurs (in the sense that two elements of the set \( A \) are linked by two different relations).

Algorithm 5.1

**Step 1:** Verify \( P.Q \cup Q.P \cup P.P \subset P \) and \( Q.Q \subset P \cup Q \);

**Step 2:** \( \forall x, y, z I_l(x, y) \land P(x, z) \land Q(y, z) \rightarrow I_l(x, y) \);

**Step 3:** \( \forall x, y, z I_l(x, y) \land P(z, x) \land Q(y, z) \rightarrow I_l(x, y) \);

**Step 4:** \( \forall x, y, z I_l(x, y) \land I_l(y, z) \land P(x, z) \rightarrow I_l(x, y) \land I_l(y, z) \);

**Step 5:** \( \forall x, y, z I_l(x, y) \land I_l(y, z) \land (Q \cup Q^{-1})(x, z) \rightarrow I_l(x, y) \land I_l(y, z) \);

**Step 6:** \( \forall x, y, z I_l(x, y) \land I_l(y, z) \rightarrow I_l(x, y) \);

**Step 7:** If there is no \( I_l(x, y) \) or \( I_l(x, y) \) that is established in \( I_l \), then the proof is complete.
Steps 1 to 4, are deterministic, in the sense that each $I_i$ established is mandatory. If a contradiction occurs, i.e., a newly established $I_i(x, y)$ has been formerly established as $\Phi(x, y)$, $\Phi$ being any among $P, Q, P^{-1}, Q^{-1}$, the algorithm fails. Steps 5 and 6 however, use already established $I_i$ in order to establish further $I_i$. The problem arises from Step 7 where $I_i$ is arbitrarily chosen. When the algorithm goes back to Step 5 to continue with establishing $I_i$, if a contradiction occurs, intuitively, it should backtrack to the last $I_i(x, y)$ established, reverse it to $I_i(y, x)$ and try again. In other terms the algorithm appears to have to explore a "tree structure" defined by the branches created by each arbitrary choice. In such a case the risk is to have to make an exhaustive research of the whole "tree".

In the following we will demonstrate that the algorithm previously presented is "backtracking free". In other words, any contradiction, implies the non-existence of a PQI interval order on $A$ and the algorithm can stop immediately without backtracking.

**Theorem 5.2** The algorithm 5.1 is backtracking free.

**Proof** We elaborate the demonstration observing how the setting of $I_i(x, y)$ (steps 5, 6) is propagated and analyzing contradictory situations. The demonstration consists of decomposing the problem in smaller cases and showing for each of them that when a contradiction occurs there is no backtracking necessity and the algorithm fails (the PQI preference structure is not a PQI interval order).

Before reaching step 7 the first time, the process is deterministic, we can therefore construct the graph $G_0 = (A, V_0)$ where $A$ is the usual set of objects on which the PQI preference structure applies and $V_0 = PUQUU \cup I_i$, where $I$ consists of $(x, y)$ which are not yet set. $G_0$ is complete and all its arcs are directed except the ones in $I$. In the following we denote as a "triangle" a set of three elements in $A(x, y, z)$ such that $x \Phi y \Psi z \Theta x$, where $\Phi, \Psi, \Theta$ are any among $P, P^{-1}, Q, Q^{-1}, I_i, I_i^{-1}, I$.

**Proposition 5.3** In $G_0$, a triangle with at least an $I$-arc must be one of the following:

1 - $I.I.I$
2 - $I.I.I_i$
3 - $I.I.Q$
4 - $I.I_i.I_i^{-1}$
5 - $I.I_i^{-1}.I_i$

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6. $I,P,P^{-1}$
7. $I,P^{-1},P$
8. $I,Q,Q^{-1}$
9. $I,Q^{-1},Q$

Proof.
The application of steps 1-6 of the algorithm 5.1 excludes all other possibilities. For example, all triangles $I_i,I,Q$ have changed to $I_i,I_i^{-1},Q$ by step 5.

Denote as $I$-path a path where each of its arcs is an $I$-arc. Consider then the partial graph $G^*$ of $G_0$, $G^* = (A,V_1)$ where $V_1 = \{(x,y) | x \neq y, \exists I$-path from $x$ to $y\}$. The proofs of propositions 5.4 to 5.11 can be found in Appendix A.

**Proposition 5.4** $G^*$ consists of connected components which:

i. are complete;

ii. do not contain any $P$-arc;

iii. are closed under the propagation of the setting of $I_i$.

We have proved that $G^*$ consists of connected components in which the propagation of the setting of $I_i(x,y)$ is limited. Each component contains only $Q$ or $I$ or $I_i$ arcs, while $P$ arcs exist only among such components. Therefore, we can limit ourselves in analyzing only one connected component, denoted by $G_1 = (A_1,V_1)$.

Let $(x^*,y^*)$ be an $I$-arc arbitrarily chosen in step 7 to become an $I_i$-arc. Consider iteration $k$ of the algorithm. Denote as $I_i^k$ the set of $I$-arcs set in $I_i$ in the current step and as $I_i^K$ the cumulative set of $I$-arcs set in $I_i$ in all the former iterations of the algorithm. We have that $I_i^K = I_i^k \cup I_i^{k-1}$. Conventionally, in step 5, $(x^*,y^*)$ is added to $I_i^k$.

**Proposition 5.5** $I$-arcs set to $I_i$ by transitive closure (step 6) are never used in step 5 when the algorithm iterates.

Denote as a $Q$-path a path whose arcs are $Q$ or $Q^{-1}$ ones. In the set $A$, let us consider now the following equivalence relation: $\Theta(x,y) \iff \exists$ a $Q$-path from $x$ to $y$ and use $X,Y,Z$ to denote equivalence classes. Therefore we can see graph $G_1$ as composed by equivalence classes of nodes each of which contains only $Q$ and $I$ arcs. Further on among such equivalence classes only $I$-arcs do exist.

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Proposition 5.6 In step 5
i - the propagation of $I_1(x, y) \in X \times Y$ is limited to $X \times Y$.
ii - when $X \neq Y$, the propagation of $I_1$ covers the whole set $X \times Y$.
iii - If $(x^*, y^*) \in X \times X$ then $I_1^k \subset X \times X$
iv - If $(x^*, y^*) \in X \times Y$, $X \neq Y$ then $I_1^k = X \times Y$.
v - Whatever $(x, y)$ is chosen to be set in $I_1$ in Step 5 the result is the same.
vi - If $I_1(y^*, x^*)$ is chosen instead of $I_1(x^*, y^*)$ then all the settings in this step will be reversed.

Proposition 5.6 states that, during the $k$-th iteration of the algorithm, Step 5 sets to $I_1$ either some $I$-arcs included in one equivalence class (of relation $\Theta$) or all $I$-arcs among two equivalence classes.

Consider now Step 6. In each application of step 6, setting $I_1(x, z)$ from $I_1(x, y)$ and $I_1(y, z)$, implies that at least one arc, let's say $(x, y)$, has to be set during, either this step, or the two last steps 5,7. In a formal notation we have:

Proposition 5.7 In Step 6:
i - If $(x, y) \in X \times X$ then $z \in X$.
ii - If $(x^*, y^*) \in X \times X$ is set in Step 5 then $I_1^k \subset X \times X$.
iii - If it exists $I_1^k(x, z) \in X \times Z$, $X \neq Z$ then $X \times Z \subset I_1^k$.
iv - If $(x^*, y^*) \in X \times Y$, $X \neq Y$ is set in Step 5, only arcs connecting different classes are set in Step 6 (in other terms if $I_1(x, z) \in X \times Z$ is set in Step 6 then $Z \neq X \wedge Z \neq Y$).

These results show that if we choose an arc $(x^*, y^*)$ to set in $I_1$, if it is inside one equivalence class it does not propagate $I_1$ outside this class, while if it connects two different classes, it does not propagate $I_1$ into any class. Furthermore, as the algorithm has passed through steps 5, 6 before the establishment of $G_1$ at least once, all the arcs between two classes $X, Y$ are of the same type (either $I$-arcs or $I$-arcs). Therefore, the problem can be further decomposed into two sub-problems:
a) - Outside all the equivalent classes, we consider the same problem with $G_1$ replaced by $G_2 = (A_2, V_2)$ where $A_2$ is the quotient set $A^\Theta$ and $V_2 = \{(X, Y)|X, Y \in A_2 \wedge \exists(x, y) \in X \times Y \text{ such that } I_1 \text{ or } I_1 \text{ holds}\}$ according to the type of the arcs connecting $X, Y$.
b) - Inside each equivalent class, we consider the same problem with $G_1$ replaced by $G_3 = (A_3, V_3)$.

The sub-problem a) is trivial, as the graph $G_2$ contains only $I$ or $I$ arcs, furthermore, the part of $G_2$ covered by $I$-arcs is already $I$ transitively
closed since the algorithm has already gone through Step 6. The problem is reduced to the construction of a linear order. Therefore, we have to deal only with the sub-problem (b).

We have to demonstrate now that the algorithm is backtracking free on $G_2$ where the arcs are $Q, I, I$ and there is a $Q$-path connecting any two different nodes. We consider now the possible situations where a contradiction may occur.

**Proposition 5.8** In step 5

i. $I^k(x, y) \land I^k(y, z) \Rightarrow I^k(x, z)$ i.e. if $(x, y)$ and $(y, z)$ are set in this step, then so is $(x, z)$.

ii. $I^k(x, y) \land I_i^{K-1}(y, z) \land I_i^k(z, t) \Rightarrow I_i^k(x, t)$.

iii. $I_i^{K-1}(x, y) \land I_i^{K-1}(y, z) \Rightarrow I_i^{K-1}(x, z)$.

N.B. We may emphasize that, while in Step 5, $I^k(x, y) \land I_i^{K-1}(y, z)$ does not necessarily imply $I_i^k(x, z)$.

**Proposition 5.9** In step 5, an $I_i$-circuit occurs only with a contradiction.

**Proposition 5.10** If the first contradiction occurs at step 6, then there
Theorem 6.1 Algorithm 5.1 is in polynomial time \(O(n^5)\)

Proof The algorithm presented in the previous section can be represented in the following way (including some small variations discussed immediately after):

Algorithm 6.1

Step 1: \(p_{ij} + p_{jk} \leq 1 + p_{ik}, p_{ij} + q_{jk} \leq 1 + p_{ik}, q_{ij} + q_{jk} \leq 1 + p_{ik} + q_{ik} \forall i, j, k;

Step 2: \(i_{ij} = p_{ik} = q_{jk} = 1 \Rightarrow l_{ij} = 1 \forall i, j, k;

Step 3: \(i_{ij} = p_{ki} = q_{kj} = 1 \Rightarrow l_{ij} = 1 \forall i, j, k;

Step 4: \(i_{ij} = i_{ik} = i_{kj} = 1 \Rightarrow l_{ik} = l_{kj} = 1 \forall i, j, k;

Step 5: \(q_{ij} + q_{jk} = i_{ik} = i_{kj} = 1 \Rightarrow l_{ik} = l_{kj} \forall i, j, k;

Step 6: \(i_{ij} = l_{ik} = 1 \Rightarrow l_{ik} = 1 \forall i, j, k;

Step 7: For \(I(x, y)\) not yet established as \(I_l\) or \(I_r\), choose arbitrarily \(I_l(x, y)\). If the \(I_l\) established belongs to an equivalence class established in Step 5, put all the elements of the class equal to 1. Return to 6 (instead of 5).

A critical step in this algorithm is step 5 since it introduces implicitly a recursive establishment of \(I_l\). In order to avoid an infinite recursion and the associated contradictions it is necessary to “fix” \(I_l\) as soon as it is generated by step 5 so that only \(I(x, y)\) which are not yet established may still be considered in the recursive application of step 5. This is possible partition the set of non zero elements of the matrix \(I\) into classes which will have the same value of \(l_{ij}\) because of step 5. Then as soon as one element of one of these classes turns to 1, the whole class will turn to 1. Under such an adjustment the following positive consequences hold:
- if there is no solution then a contradiction in establishing an \(I_l\) will appear before step 6;
- after step 7 you just have to return to step 6.

We can now discuss complexity. Steps 1 to 4 are obviously in \(O(n^3)\) as step 6 (transitive closure) is. Step 5 is in \(O(n^5)\) as can be seen by the following implementation (remark that in the worst case \(n = |G_3|\)):

```java
function step5: boolean
    forall i, j, k
        if (l_{ik} = l_{kj} = 1 = Q_{ij} + Q_{ji})
            if (not setLabel(i, j, k))
                return true
```

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function setLabel(i,j,k: integer)
    if (Lik, Lkj no label)
        set new label to Lik and Lkj
    else if (Lik = L1, Lkj no label)
        set Lkj to L1
    else if (Lik no label, Lkj = L2)
        set Lik to L2
    else if (Lik = L1 et Lkj = -L1)
        return false (conflict)
    else if (Lik = L1 et Lkj = L2)
        unify these two labels
    endif
    return true
end

Furthermore it is easy to see that the decomposition of the PQI graph in $G_1$ and its connected components, the decomposition in $G_2$ and $G_3$ and the construction of the linear order in $G_2$ are all in polynomial time. Therefore the whole algorithm is in polynomial time.

\[ \Box \]

7 Conclusions

The paper has presented an answer for the problem concerning the representation of preferences by intervals by showing necessary as well as sufficient conditions to see if a preference structure is a PI interval order, PQI semi order or PQI interval order. For PQI interval order, it provided also an algorithm to verify whether a PQI preference structure on a finite set $A$ is a PQI interval order. In other words verify if it is possible to associate to each element of $A$ an interval such that if the interval associated to $x$ is completely to the right of the interval associated to $y$, then $x$ is strictly preferred to $y$, if one interval is included in the other, then $x$ is indifferent to $y$ and if the interval associated to $x$ is to the right of the interval associated to $y$, their intersection being not empty, then $x$ is weakly preferred to $y$. We first demonstrate that the algorithm, although it appears having to explore a tree generated by branches of arbitrary choices, is backtracking free and then we demonstrate that runs in polynomial time. We consider such a result very promising, since it enables an efficient check of the existence of PQI interval orders which are very common in many different cases, including preference modeling and temporal logic.
References


Appendix A

Proof of Proposition 5.4
i. If $x, y$ belongs to a connected component then there exists a path $a_0 = x, a_1, \ldots, a_k = y$. \( \forall i = 0 \ldots k - 1, \) if exists an $I$-path from $a_i$ to $a_{i+1}$ then there exists an $I$-path from $x$ to $y$ and therefore $(x, y) \in V_i$.

ii. If a $P$-arc exists, choose $P(x, y)$ such that the length $k$ of the $I$-path $a_0 = x, a_1, \ldots, a_k = y$ is minimal. Consider then the arc $a_1, a_k$ (it exists from the completeness of the component), then from proposition 5.3 we have $P(a_1, a_k)$ and therefore we have another $P$-arc with length of the $I$-path $< k$. Impossible.

iii. Immediate from conditions vi and vii of Theorem 5.1 (steps 5 and 6 of the algorithm).

\[ \square \]

Proof of Proposition 5.5
First consider $(x_1, x_2)$ such that $I^k(x_1, x_2)$ in step 6. Therefore it exists $I^k(x_1, x_2) \wedge I^k(x_3, x_2)$. If, for example, $(x_1, x_3)$ was also established in $I^k$ (in the current step 6) then it exists $I(x_1, x_3) \wedge I(x_4, x_3)$ and so on until an $I^{k-1}$-path is obtained. Therefore for all $(x, y)$ such that $I_i$ is established in the current step 6 exists an $I^{k-1}$-path from $x$ to $y$.

Let now $(x, y)$ to be an arc set to $I_i$ in the last step 6, participating to the setting of arc $(x, z)$ in step 5 through let's say $Q(x, y)$. Let us consider the situation in the last step 6:

$I^k(x, y) \Rightarrow \exists I^{k-1}$-path $t_0 = x, t_1, \ldots, t_k = y$.

Consider the triangle $x, t_{k-1}, y$ where $Q(x, y) \wedge I^{k-1}(t_{k-1}, y)$.

If $Q(t_{k-1}, z)$ then $Q(t_{k-1}, z) \wedge Q(x, y) \Rightarrow (P \cup Q)(t_{k-1}, y)$, conflict with $I_i(t_{k-1}, y)$.

If $I(t_{k-1}, y)$ then $I^{k-1}(t_{k-1}, y) \wedge I(t_{k-1}, z) \Rightarrow I^{k-1}(t_{k-1}, y)$ (at least in the last step 5). Therefore it exists an $I^{k-1}$-path from $x$ to $z$, that is $I_i(x, z)$ must be set at least at the same time as $(x, y)$. We conclude that $Q(x, t_{k-1})$. Repeat this procedure, and we get at last $Q(x, t_1)$, which together with $I^{k-1}(x, t_1)$ gives $I^{k-1}(x, z)$ i.e. $(x, z)$ must have been set before $(x, y)$.

\[ \square \]

Proof of Proposition 5.6
i - In each application of step 5, consider $(x, y) \in X \times Y$ such that $I_i(x, y)$. Relation $I_i$ will propagate to $(x', y)$ or $(x, y')$, $x'$, $y'$ arbitrary. From Theorem 5.1 and proposition 5.4 (no $P$-arcs in $G^2$) we know that there have to exist
Q-paths from $x$ to $x'$ and from $y$ to $y'$. Therefore $(x', y') \in X \times Y$.  
ii - $(x', y') \in X \times Y$ implies that there exist Q-paths $a_0 = x, a_1, \ldots, a_k = x'$, and $b_0 = y, b_1, \ldots, b_l = y'$. Applying consecutively step 5 on these two paths we obtain the setting in $I_t$ of $(x, y), (a_1, y), \ldots, (x', y)$ and then of $(x', b_1), (x', b_2), \ldots, (x', y')$.

iii and iv - Immediate from propositions (5.5), (5.6.i) and (5.6.ii).

v and vi - Immediate from Theorem 5.1.

Proof of Proposition 5.7

i - Otherwise, consider the first setting with $z \in Z \neq X$. It implies that $I_t(y, z) \in X \times Z$, $Z \neq X$ and since $(x, z)$ is the first such setting, $I^k_{t-1}(y, z)$ holds. We have $x, y \in X \land z \in Z \land I^k_{t-1}(y, z)$ which implies $I^k_{t-1}(x, z)$ as it must be set at least in the last step 5 (proposition (5.6.ii)). Contradiction.

ii - Immediate from (5.6.iii), (5.7.1).

iii - Otherwise it should exist $(x', x') \in X \times Z \setminus I^k_t$. In the next step 5 $(x, z)$, which is set in this step 6, will propagate $I_t$ to $(x', z')$, which is impossible because of (5.5).

iv - Suppose that $(x^*, y^*) \in X \times Y$, $X \neq Y$ is introduced in step 5. Then all the arcs of $X \times Y$ are set to $I_t$ and only these arcs. The setting in step 6 is the propagation of such arcs. Let $I_t(x, y) \land I_t(y, z) \Rightarrow I_t(x, z)$, $x \in X$, $y \in Y$, $z \in Z$ the first setting in step 6 with $I^k_{t-1}(x, y)$, $I^k_{t-1}(y, z)$ and $(x, y)$ set in the last steps 5, 7, i.e. $(x, y) \in X \times Y$ and $X \times Y \subset I^k_{t-1}$. If $Z = X$ then $(x, y)$ should have been set at the same time as $(x, y)$, which contradicts $I^k_{t-1}(y, z)$. We conclude that $Z \neq X$ (and similarly that $Z \neq Y$).

Proof of Proposition 5.8

i - Let $Q = Q \cup Q^{-1}$ and $\Psi = I_t \cup I_t^{-1} \cup Q$. In an equivalence class we have $\forall (x, z) \Psi (x, z)$.  

If in step 5 we had $Q(x, z)$ then $I^k_t(x, y) \land Q(x, z) \land I(x, y) \Rightarrow I^k_t(x, y)$, in contradiction with $I^k_t(y, z)$. Therefore we have $\neg Q(x, z)$ and $\Psi = I_t \cup I_t^{-1}$.

The transition from $I^k_t(x, y)$ to $I^k_t(y, z)$ in step 5 passes through 2 Q-paths $x_1 = x, x_2, \ldots, x_n = y$ and $y_1 = y, y_2, \ldots, y_n = z$ where $(x_i = x_{i+1}$ and $y_i \neq y_{i+1}$ and $I(x_i, y_i)$ or $(x_i \neq x_{i+1}$ and $y_i = y_{i+1}$ and $I(x_i, y_i)$). We consider the two different transitions from $(x, y)$ to $(y, z)$.

1. If $y_2 = y$ then $x \neq x_2$ and therefore $Q(x, x_2) \land I(x, y)$ We have then $Q(x, x_2) \land I^k_t(x, y) \land I(x, y) \Rightarrow I^k_t(x, y)$. But $Q(x_2, z)$ is in contradiction with $I^k_t(x_2, y)$. Therefore $I^k_t(y, z) \Rightarrow \neg Q(x_2, z)$.
We have $Q(x, x_2) \land I(x_2, z) \land \Psi(x, z) \Rightarrow \Psi(x_2, z)$. Therefore the situation is not changed ($x_2$ plays now the role of $x$).

2. If $y_2 \neq y$ then $x = x_2$. Therefore $Q(y, y_2) \land I(x, y_2) \Rightarrow I^k_I(x, y_2)$. If $Q(y_2, z)$ holds we have $Q(y_2, z) \land I^k_I(x, y_2) \land I(x, z) \Rightarrow I^k_I(x, z)$. Otherwise, $Q(y, y_2)$ and $I^k_I(y, z)$ give $I^k_I(y_2, z)$ and the situation is not changed ($y_2$ plays now the role of $y$).

In order to pass from $y$ to $z$, it must exist a $k$ such that $y_{k+1} = z$ and $y_k \neq z$, i.e. $Q(y_k, y_{k+1}) \Rightarrow Q(y_k, z) \Rightarrow I^k_I(x, z)$.

iii - Let $\Psi(x, z)$. Since $Q(y, t)$ or $Q(x, z)$ is in contradiction with $I^k_I(x, y)$, $I^{k-1}_I(y, z)$ and $I^k_I(x, t)$ we have $I(y, t)$ and $I(x, z)$.

If $\Psi = Q$ then $I^k_I(x, y) \land I(y, t) \land Q(x, z) \Rightarrow I^k_I(x, t)$. But $I^k_I(t, y) \land I^k_I(x, t) \Rightarrow I^k_I(y, y)$ (5.8.i) in contradiction with $I^{k-1}_I(y, z)$. Therefore $\Psi = I_I \cup I_{-I}$. The transition from $I^k_I(x, y)$ to $I^k_I(x, t)$ in step 5 passes through $2$ Q-paths $x_1 = x, x_2, \ldots x_n = z$ and $y_1 = y, y_2, \ldots y_n = z$ where $x_1 = x_{k+1}$ and $y_i \neq y_{i-1}$ and $I(x_i, y_i)$, $I(x_i, y_{i+1})$ or $(x_i = x_{k+1}$ and $y_i = y_1+1$ and $I(y_i, y_{i+1})$, $I(x_i, y_{i+1})$, $I(x_{i+1}, y_{i+1}))$.

We consider the two different transitions from $(x, y)$ to $(x, t)$.

1. If $y_2 = y$ then $x \neq x_2$ therefore $Q(x, x_2)$.

2. If $y_2 \neq y$ then $x_2 = x$, therefore $Q(y, y_2)$.

If $Q(y_2, t)$, we have $Q(y_2, t) \land I^k_I(x_2, y) \land \Psi(x, t) \Rightarrow I^k_I(x, t)$. Otherwise, if $Q(y_2, i)$ then $Q(y_2, z) \land I^k_I(x, y_2) \land I(x, z) \Rightarrow I^k_I(x, z)$.

Therefore, we have $I^k_I(x, z) \land I^k_I(x, t) \Rightarrow I^k_I(x, t)$ (5.8.i). If $I(y_2, z)$ then $Q(y_2, y_2) \land I^{k-1}_I(y, z) \land I(y_2, z) \Rightarrow I^{k-1}_I(y_2, z)$ and the situation is not changed ($y_2$ plays now the role of $y$).

So, we have either $I(x, t)$ when it exists $Q(y, z)$ or the only way to pass from $y$ to $t$ is through some $y_{k+1} = t$ and $y_k \neq t$ i.e. $Q(y_k, y_{k+1}) \Rightarrow Q(y_k, t) \Rightarrow I(x, t)$.

If $I^{k-1}_I(x, y)$ and $I^{k-1}_I(x, z)$ are not at least in the last step $k$ then...