THE PROBABILISTIC MINIMUM VERTEX COVERING PROBLEM

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Le problème de la couverture minimum de sommets probabiliste

Résumé
Une instance du problème de la couverture de sommets probabiliste est une paire \((G = (V, E), Pr)\) obtenue en associant à chaque sommet \(u_i \in V\) une probabilité « d'occurrence » \(p_i\). Nous considérons une stratégie de modification \(M\) transformant une couverture de sommets \(C\) de \(G\) en une couverture de sommets \(C_I\) pour le sous-graph de \(G\) induit par l'ensemble de sommets \(I \subseteq V\). L'objectif, pour la couverture de sommets probabiliste, est de déterminer une couverture de sommets de \(G\) minimisant la somme, sur tous les sous-ensembles \(I \subseteq V\), des produits : probabilité de \(I\) fois \(C_I\). Dans cet article, nous étudions la complexité de résolution optimale du problème de couverture de sommet probabiliste.

Mots-clé : problèmes combinatoires, complexité, couverture de sommets, graphe

The probabilistic minimum vertex covering problem

Abstract
An instance of the probabilistic vertex-covering problem is a pair \((G = (V, E), Pr)\) obtained by associating with each vertex \(u_i \in V\) an “occurrence” probability \(p_i\). We consider a modification strategy \(M\) transforming a vertex cover \(C\) for \(G\) into a vertex cover \(C_I\) for the subgraph of \(G\) induced by a vertex-set \(I \subseteq V\). The objective for the probabilistic vertex-covering problem is to determine a vertex cover of \(G\) minimizing the sum, over all \(I \subseteq V\), of the products of \(p_I\) and \(C_I\). In this article, we study the complexity of optimal solution.

Mots-clé : problèmes combinatoires, complexité, couverture de sommets, graphe
1 Introduction

Given a graph $G = (V, E)$, a vertex cover is a set $C \subseteq V$ such that for every $v_i v_j \in E$ either $v_i \in C$, or $v_j \in C$. The vertex-covering problem (VC) consists in finding a vertex cover of minimum size. A natural generalization of VC, denoted by WVC, in the sequel, is the one where positive weights are associated with the vertices of $V$. The objective for WVC is to compute a vertex cover for which the sum of the weights of its vertices is the smallest over all vertex covers of $G$.

Let $\mathcal{G}$ be the set of graphs. Given $G \in \mathcal{G}$, denote by $C(G)$ the set of vertex covers of $G$. For any subset $I \subseteq V$ we denote by $G[I]$ the subgraph of $G$ induced by $I$. The probabilistic vertex-covering, denoted by PVC in what follows, is a quintuple $(\mathcal{H}, C(G), \mathbb{P}, E(G^*_C), \text{min})$ such that:

- $\mathcal{H}$ is the set of instances of PVC defined as $\mathcal{H} = \{G = (G, \mathbb{P})\}$, where $G \in \mathcal{G}$ and $\mathbb{P}$ is an $n$-vector of vertex-probabilities;
- $C(G)$ is the set of vertex covers of $G$ ($C(G) = C(G)$).
vertex \( v_i \) (the degree of \( v_i \)). We will set \( \delta = \min_{v_i \in V} \{|\Gamma(v_i)|\} \) and \( \Delta = \max_{v_i \in V} \{|\Gamma(v_i)|\} \). Given a vertex cover \( C \) of \( G \) and a subset \( V' \subseteq V \), we will set \( C[V'] = C \cap V' \) and \( \overline{C}[V'] = (V \setminus C) \cap V' \). Finally, when dealing with WVC, we will denote by \( w_i \) the weight of \( v_i \in V \).

2 The strategies M1, M2 and M3 and a general preliminary result

2.1 Specification of M1, M2 and M3

Given a vertex cover \( C \) and a vertex-subset \( I \subseteq V \). M1 consists in simply moving vertices of \( C \setminus I \) (the absent vertices of \( C \)) out of \( C \) and in retaining \( C[I] = C \cap I \) as vertex cover of \( G[I] \).

BEGIN (*M1(C,G[I])* )

\[
\begin{align*}
C'[I] &= C \cap I; \\
\text{OUTPUT} &\quad C[I];
\end{align*}
\]

END. (*M1(C,G[I])* )

It is easy to see that \( C[I] \) constitutes a vertex cover for \( G[I] \). In fact, in the opposite case, there would be at least an edge \( v_i v_j \) of \( G[I] \) for which neither \( v_i \), nor \( v_j \) would belong to \( C[I] \). But, since \( \forall v_i, v_j \in E \) and \( I \) is a vertex cover for \( G \), at least one of \( v_i \) and \( v_j \), \( v_i \) and \( v_j \), would be in \( C[I] \).
Proposition 1. Consider a vertex cover $C$ of $G$ and strategies $M_1$, $M_2$ and $M_3$. With each vertex $v_i \in V$ associate a probability $p_i$ and a random variable $X_{v_i}^{M_k, C} \mid k = 1, 2, 3$ defined for every...
where in the last summation we count all the sub-graphs \( G[I] \) such that \(|C[I]| = i\), we add their probabilities and we take into account that \( \sum_{j=0}^{n-i} C_j^{n-i} p^j (1-p)^{n-j} = 1 \).

The above can be generalized in order to compute every moment of any order for \( \mathcal{M}1 \). For instance,

\[
E((C[I]_i)^2) = \sum_{i \in V} \Pr[I] (|C[I]|)^2 = \sum_{i=1}^{\ell} \ell C_i p^i (1-p)^{\ell-i} = pr(p + 1 + p) \quad \text{and, consequently,}
\]

\[
\text{Var}(C[I]) = E \left( (C[I]_i - (E(C[I]))^2 \right) = \tau p(1-p).
\]

So, for \( \mathcal{M}1 \) and for the case of identical vertex-probabilities, the random variable representing the size \( \tau \) follows a binomial law with parameters \( \tau \) and \( p \).

4 The complexity of \( (\mathcal{H}, C, \mathcal{M}2, E(G^m_C), \min) \)

We recall that strategy \( \mathcal{M}2 \) consists in removing the isolated vertices from \( C[I] \). Then, the following theorem holds.

**Theorem 2.**

\[
E(G^m_C) = \sum_{v_i \in C} p_i \left( 1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right)
\]

and \( \hat{C}_2 \) is a minimum-weight vertex cover of \( G_w \) where, for every \( v_i \in V \),

\[
w_i = p_i \left( 1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).
\]

Consequently, \( (\mathcal{H}, C, \mathcal{M}2, E(G^m_C), \min) \) is \( \text{NP} \)-hard.

**Proof.** Remark first that \( C[I] = C2[I] \in C[I] \setminus (\cup_{v_i \in C_2}(\Gamma(v_i) \cap I) = \emptyset) v_i \).

Starting from proposition 1 we get

\[
E(G^m_C) = \sum_{v_i \in C} \Pr \left[ X^m_{C2[I]} = 1 \right] = \sum_{v_i \in C} \sum_{I \subseteq V \setminus v_i} \Pr[I] \left[ 1_{\{v_i \in C2[I]\}} \right] = \sum_{v_i \in C} \sum_{I \subseteq V \setminus v_i} \Pr[I] \left[ 1 - 1_{\{v_i \in C2[I]\}} \right].
\]

But, for any \( v_i \in C \), \( v_i \notin C2[I] \leftrightarrow I \cap \Gamma(v_i) = \emptyset \); consequently,

\[
E(G^m_C) = \sum_{v_i \in C} \sum_{I \subseteq V \setminus v_i} \Pr[I] - \sum_{v_i \in C} \sum_{I \subseteq V \setminus v_i} \Pr[I] \left[ 1 - 1_{\{I \cap \Gamma(v_i) = \emptyset\}} \right] = \sum_{v_i \in C} p_i \left( 1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right).
\]

It is easy to see that \( E(G^m_C) \) can be computed in \( O(n^2) \); consequently, \( (\mathcal{H}, C, \mathcal{M}2, E(G^m_C), \min) \in \text{NP} \). With a reasoning completely similar to the one of theorem 1 one can immediately deduce that \( \hat{C}_2 = \text{argmin}_{C \in \mathcal{C}(G)} \{ \sum_{v_i \in C} p_i (1 - 1_{\{v_i \in \Gamma(v_i) = \emptyset\}} (1 - p_j)) \} \), i.e., a minimum-weight vertex cover of \( G_w \) where, for \( v_i \in V \), \( w_i = p_i (1 - 1_{\{v_j \in \Gamma(v_i) = \emptyset\}} (1 - p_j)) \).

5 The complexity of \( (\mathcal{H}, C, \mathcal{M}3, E(G^m_C), \min) \)

5.1 Building \( E(G^m_C) \)

Recall that \( \mathcal{M}3 \) consists in removing from \( C[I] \) both the isolated vertices and the vertices all the neighbors of which belong in \( C[I] \). Consequently, the solution \( C3[I] \) computed by \( \mathcal{M}3 \) is minimal.
Proposition 2.

\[ E(G^n_C) = \sum_{v_i \in C} p_i \left( 1 - \prod_{v_j \in \Gamma(v_i)} (1 - p_j) \right) - \sum_{v_i \in C} \sum_{I \subseteq V} \Pr[I] 1\{v_i \in C_I\} 1\{\Gamma(v_i) \cap \Gamma(I) \neq \emptyset\} 1\{\cap \Gamma[I] = \emptyset\} \]  \hspace{1cm} (1)

Proof. By proposition 1, \( E(G^n_C) = \sum_{v_i \in C} \Pr[X_i^{n, C} = 1] \). Moreover,

\[ \Pr[X_i^{n, C} = 1] = \sum_{I \subseteq V} \Pr[I] 1\{v_i \in C_I\}. \]

Note that \( \Gamma(v_i) = C[\Gamma(v_i)] \cup \bar{C}[\Gamma(v_i)] \) and that if \( v_i \in C \), then \( \bar{C}[\Gamma(v_i)] \neq \emptyset \) (because \( C \) is minimal).

Let now \( v_i \in C \) and \( I_0 = \{I \subseteq V : v_i \in I\} \). The set of graphs \( G[I], I \in I_0 \), can be partitioned
\[
+ \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}}
\]

\[
= p_i \left( 1 - \prod_{v_j \in \mathcal{C} \cap \Gamma(u_i)} (1 - p_j) \right) + \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}}
\]

\[
\Pr[X_i^{\text{NS, C}} = 0] = \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} + \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}}
\]

\[
= \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} \left( I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} + I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} \right) + 1 - p_i
\]

\[
= \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} + 1 - p_i
\]

\[
+ \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}}
\]

\[
= p_i \prod_{v_j \in \Gamma(u_i)} (1 - p_j^2) + 1 - p_i + \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}}
\]

It is easy to see, after some easy but tedious algebra, that \(\Pr[X_i^{\text{NS, C}} = 1] + \Pr[X_i^{\text{NS, C}} = 0] = 1\).

Then, the result of proposition 2 is get using

\[
E(G_C^{\text{NS}}) = \sum_{u_i \in \mathcal{C}} \Pr[X_i^{\text{NS, C}} = 1] = \sum_{u_i \in \mathcal{C}} \left( 1 - \Pr[X_i^{\text{NS, C}} = 0] \right).
\]

5.2 Bounds for \(E(G_C^{\text{NS}})\)

Let us first note that from theorems 1, 2 and proposition 2, we have \(E(G_C^{\text{NS}}) \leq E(G_C^{\text{WS}}) \leq E(G_C^{\text{HC}})\). Since the result of proposition 2 does not allow the achievement of a precise characterization of \(C_3\), we give in the following theorem bounds for \(E(G_C^{\text{NS}})\).

**Theorem 3.**

\[
\sum_{u_i \in \mathcal{C}} p_i \left( 1 - \prod_{v_j \in \Gamma(u_i)} (1 - p_j) \right) \leq E(G_C^{\text{NS}}) \leq \sum_{u_i \in \mathcal{C}} p_i \left( 1 - \prod_{v_j \in \Gamma(u_i)} (1 - p_j) \right).
\]

If \(p_i = p\), \(\forall u_i \in V\) and \(\delta_C = \min_{u_i \in \mathcal{C}} \{|\Gamma(u_i)|\}\), then

\[
p \left( \tau - \max \left\{ \tau (1 - p) \Delta, (1 - p)^{\delta_C} \right\} \right) \geq E(G_C^{\text{NS}}) \geq p^2 \tau.
\]

The upper bound for \(E(G_C^{\text{NS}})\) is attained for the bipartite graphs when considering the one of the color classes as a priori solution.

**Proof.** The upper bound is obvious since it is nothing else than the expression of theorem 2 for \(E(G_C^{\text{NS}})\).

Let

\[
A_i = \sum_{\mathcal{C} \in \mathcal{V}} \Pr[I_1] I_{\{v \in \mathcal{C} \cap \Gamma(u_i) \neq \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}} I_{\{\mathcal{C} \cap \Gamma(u_i) = \emptyset\}}
\]

(2)
\[(see \ the \ last \ term \ of \ the \ expression \ (1) \ in \ proposition \ 2). \ \text{Then,} \]
\[A_i \leq \sum_{I \in \mathcal{C}} \Pr[I \{I \cap C[I(V_i)] \neq \emptyset\} \{I \cap C[I(V_i)] = \emptyset\}] \leq \sum_{I \in \mathcal{C}} \Pr[I \{I \cap C[I(V_i)] = \emptyset\}] \{I \cap C[I(V_i)] = \emptyset\} \]
\[\leq p_i \left( \prod_{v_j \in C[I(V_i)]} (1 - p_j) - \prod_{v_j \in C[I(V_i)]} (1 - p_j) \prod_{v_j \in C[I(V_i)]} (1 - p_j) \right) \]
\[\leq p_i \left( \prod_{v_j \in C[I(V_i)]} (1 - p_j) - \prod_{v_j \in C[I(V_i)]} (1 - p_j) \right). \]

Combining the expression for \(A_i\) above with the expression for \(E(G^\text{NS}_C)\) of proposition 2, we easily get the lower bound claimed.

Let us now suppose that \(p_i = p, \forall v_i \in V\). Then the expression for \(E(G^\text{NS}_C)\) becomes
\[E \left( G^\text{NS}_C \right) = pr - p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|} - \sum_{v_i \in C} A_i \]
with \(0 \leq A_i \leq p((1 - p)^{|\Gamma(v_i)|} - (1 - p)^{|\Gamma(v_i)|}). \) Since \(A_i \geq 0\) we have
\[E \left( G^\text{NS}_C \right) \leq pr - p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|}. \]

Using, \(\forall i, |\Gamma(v_i)| \leq \Delta, \) we get
\[E \left( G^\text{NS}_C \right) \leq p \left( r - \tau (1 - p)^\Delta \right). \quad (3) \]

Remark that
\[p \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)|} = p(1 - p)^\Delta \sum_{v_i \in C} (1 - p)^{|\Gamma(v_i)| - \delta_C} \]
\[\geq 1 \]
because there exists at least a vertex \(v_0 \in C\) with \(|\Gamma(v_0)| = \delta_C\). Consequently,
\[E \left( G^\text{NS}_C \right) \leq p \left( r - (1 - p)^\Delta \right). \quad (4) \]

Combining expressions (3) and (4) one immediately obtains the upper bound claimed.

For identical vertex-probabilities, the lower bound for \(E(G^\text{NS}_C)\) becomes \(p r (1 - (1 - p)^{|\Gamma(v_i)|})\). Since \(C\) is minimal, \(|\Gamma(V_i)| \neq 0\), i.e., \(|\Gamma(V_i)| \geq 1\). Using the latter inequality in the former one we get \(E(G^\text{NS}_C) \geq p r (1 - (1 - p)) = p^2 r\), proving so the lower bound claimed.

Consider now a bipartite graph \(B = (V_1, V_2, E)\) and one of its color classes, say color class \(V_1\), as a priori solution. Remark that, \(\forall v_i \in V_1, \Gamma(v_i) = \tilde{V}_i[\Gamma(v_i)] \in V_2. \) Consequently, the upper and lower bounds for \(E(B^\text{NS}_S)\) coincide. \(\blacksquare\)

5.3 On the quality of the bounds obtained

We prove in this section that the bounds of \(E(G^\text{NS}_S)\) obtained in theorem 3 are quite tight.

Let \(G = (V, E)\) be a graph consisting of a clique \(K_\ell\) (on \(\ell\) vertices) and of an independent set \(S\) on \(\sigma\) vertices; moreover consider that any vertex of \(K_\ell\) is linked to any vertex of \(S\). Let
us suppose that all the vertices of $G$ have the same probability $p$ and consider $C = V(K_\ell)$, the vertex-set of $K_\ell$, as a priori solution. Finally set $n = \ell + \sigma$.

For $v \in C$ we have $C(\Gamma(v)) = V(K_\ell) \setminus \{v\}$ and $C(\Gamma(v)) = S$. Then, expression (1) becomes

$$E(G, G) = \sum_{v \in C} p - \sum_{v \in C} \prod_{v_j \in \Gamma(v)} (1 - p) - \sum_{v \in C} \sum_{I \subseteq V} \Pr[I]\{1_{\Gamma(v) \subseteq C} \cdot 1_{(I \cap V(K_\ell) \setminus \{v\}) \neq \emptyset} \cdot 1_{(I \cap S) = \emptyset}\} \tag{5}$$

Let $V(K_\ell) = \{v_1, v_2, v_3, \ldots, v_\ell\}$ (the degrees of the vertices of $V(K_\ell)$ are all equal) and let $S = \{v_{\ell+1}, v_{\ell+2}, v_{\ell+3}, \ldots, v_n\}$. In step $i$, strategy M3 tests if vertex $v_i$ can be removed.

Revisit now the third term of expression (5) (the double sum). This term deals with instances $G[I]$ such that $I \subseteq V(K_\ell) \setminus \{v_i\}$, in other words, instances where all present vertices are elements of $V(K_\ell)$, i.e., they are also part of $C$. Since $v_i \in V(K_\ell)$, $\Gamma(v_i) \cap I \subseteq C \cap I$ and vertex $v_i \in C$ can be removed only if, it is not isolated and for $j < i$, $v_j \notin I$ (because, in the opposite case, $v_j$ would be removed at step $j < i$, and in other words, before $v_i$ and, since $v_i v_j \in E(K_\ell)$, $v_i$ should not be removed). Consequently, $1_{\Gamma(v) \subseteq C} = 1_{(I \cap (v_1, v_2, v_3, \ldots, v_{\ell-1})) = \emptyset}$ and for a fixed index $i$, expression (2) becomes

$$A_i = \sum_{I \subseteq V} \Pr[I] \{1_{(I \cap (v_1, v_2, v_3, \ldots, v_{\ell-1})) = \emptyset} \cdot 1_{(I \cap V(K_\ell) \setminus \{v_i\}) \neq \emptyset} \cdot 1_{(I \cap S) = \emptyset}\}$$

$$= \sum_{I \subseteq V} \Pr[I] \{1_{(I \cap v_i = v_i)} \cdot 1_{(I \cap (v_1, v_2, v_3, \ldots, v_{\ell-1})) = \emptyset}\}$$

$$= \prod_{j=1}^{i-1} (1 - p) \prod_{j=i+1}^{\ell} (1 - p) \prod_{j=\ell+1}^{n} (1 - p)$$

$$= (1 - p)^{i-1} p \left(1 - (1 - p)^{\ell-(i+1)+1}\right) (1 - p)^{n-(\ell+1)+1}$$

$$= (1 - p)^{i-1} p \left(1 - (1 - p)^{\ell-i}\right) (1 - p)^{n-\ell}.$$ 

We so have,

$$\sum_{v_i \in C} A_i = p \left(1 - p\right)^{n+i-\ell-1} - (1 - p)^{n-1}$$

$$= p(1 - p)^{n-\ell-1} \sum_{i=1}^{\ell} (1 - p)^i - p\ell(1 - p)^{n-1}$$

$$= (1 - p)^{n-\ell-1} \left(1 - p - (1 - p)^{\ell+1}\right) - p\ell(1 - p)^{n-1}$$

$$= (1 - p)^{n-\ell-1} - p\ell(1 - p)^{n-1}$$
For reasons of facility, set
\begin{align*}
e(\ell, p, n) & \overset{\text{def}}{=} \ell p - (1 - p)^{n-\ell} \left(1 - (1 - p)^{\ell}\right) \\
b(\ell, p, n) & \overset{\text{def}}{=} \ell p \left(1 - (1 - p)^{n-\ell}\right) \\
B(\ell, p, n) & \overset{\text{def}}{=} \ell p (1 - (1 - p)^{n-1})
\end{align*}
(7) (8) (9)
and remark that expressions (8) and (9) are, respectively, the lower and upper bounds of theorem 3 for \(E(G^3_n)\).

We now study the difference \((B - e)(\ell, p, n)\) (expressions (7) and (9)). We have,
\[(B - e)(\ell, p, n) = (1 - p)^{n-\ell} - (1 - p)^n - \ell p (1 - p)^{n-1} \leq 1\]
(10)

Starting from expression (10), we prove in appendix that the following hold:
\[0 \leq D(\ell, n) \overset{\text{def}}{=} (B - e)(\ell, p, n) \leq \left(\frac{n - \ell}{n - 1}\right)^{\frac{n-\ell}{n-1}} - \left(\frac{n - \ell}{n - 1}\right)^{\frac{n-\ell+1}{n-1}}\]
(11)

Note that the upper bound of expression (11) is quite tight for \(D(\ell, n)\). Let us now estimate this bound for several values of \(\ell\):

- if \(\ell = 1\), then \(D(\ell, n) = 0\);
- if \(1 < \ell \ll n\) (e.g., \(\ell = o(n)\)), then \(\lim_{n \to \infty} D(\ell, n) = 0\);
- if \(\ell = \lambda n\), for a fixed \(\lambda < 1\), then
  \[\lim_{n \to \infty} D(\lambda n, n) = e^{\frac{1-\lambda}{\lambda} \ln(1-\lambda)} - e^{\frac{\ln(1-\lambda)}{\lambda}}\]
  and this limit's value is a fixed positive constant;
- if, finally, \(\ell = n - 1\), then \(\lim_{n \to \infty} D(n - 1, n) = 1\).

Let us now study the difference \((e - b)(\ell, p, n)\). We first have
\[(e - b)(\ell, p, n) = \ell p (1 - p)^{n-\ell} - (1 - p)^n + (1 - p)^n \leq (\ell - 1) \left((1 - p)^{n-\ell} - (1 - p)^{n-\ell+1}\right)\]
(12)

one can refine the bound of expression (12) obtaining (see in the appendix for the proof)
\[0 \leq d(\ell, n) \overset{\text{def}}{=} (e - b)(\ell, p, n) \leq (\ell - 1) \left(\frac{n - \ell}{n - \ell + 1}\right)^{n-\ell} \left(1 - \frac{n - \ell}{n - \ell + 1}\right)\]
\[\leq (\ell - 1) e^{(n-\ell) \ln(1-\frac{1}{n-\ell+1})} \frac{1}{n - \ell + 1}\]
(13)

As previously, we estimate this bound for several values of \(\ell\):

- if \(\ell = 1\), then \(d(\ell, n) = 0\);
- if \(\ell = o(n)\), then \(\lim_{n \to \infty} d(\ell, n) = 0\);
- if \(\ell = \lambda n\), for a fixed \(\lambda < 1\), then
  \[d(\lambda n, n) \leq \frac{\lambda n - 1}{(1 - \lambda)n} e^{\frac{n(1-\lambda)\ln(1-\frac{1}{(1-\lambda)n+1})}{\lambda(n+1)}} \sim \frac{\lambda}{1 - \lambda} e^{-1}\]
  and this last value is a fixed constant;
<table>
<thead>
<tr>
<th>n = 14</th>
<th>$E(G_{1/2}^{p})$, $E(G_{1/2}^{c})$</th>
<th>$E(G_{1/2}^{p})/E(G_{1/2}^{c})$</th>
<th>$E(G_{1/2}^{opt})/E(G_{1/2}^{c})$</th>
<th>$p = 0.1$</th>
<th>$p = 0.5$</th>
<th>$p = 0.9$</th>
<th>$p = 1/n$</th>
<th>$p = 1/\sqrt{n}$</th>
<th>$p = 1/\ln n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(G_{1/2}^{p})$</td>
<td>0.34</td>
<td>0.82</td>
<td>0.98</td>
<td>0.26</td>
<td>0.62</td>
<td>0.74</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(G_{1/2}^{c})$</td>
<td>0.73</td>
<td>0.88</td>
<td>0.98</td>
<td>0.73</td>
<td>0.79</td>
<td>0.84</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(G_{1/2}^{opt})$</td>
<td>0.97</td>
<td>0.96</td>
<td>0.99</td>
<td>0.98</td>
<td>0.95</td>
<td>0.95</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| n = 15 |
|---|---|---|---|---|---|---|---|---|---|
| $E(G_{1/2}^{p})/E(G_{1/2}^{c})$ | 0.34 | 0.83 | 0.97 | 0.25 | 0.62 | 0.73 |
| $E(G_{1/2}^{c})$ | 0.73 | 0.89 | 0.98 | 0.72 | 0.79 | 0.83 |
| $E(G_{1/2}^{opt})/E(G_{1/2}^{c})$ | 0.97 | 0.95 | 0.99 | 0.97 | 0.95 | 0.95 |

| n = 16 |
|---|---|---|---|---|---|---|---|---|
| $E(G_{1/2}^{p})/E(G_{1/2}^{c})$ | 0.35 | 0.83 | 0.98 | 0.25 | 0.62 | 0.73 |
| $E(G_{1/2}^{c})$ | 0.72 | 0.88 | 0.98 | 0.71 | 0.78 | 0.83 |
| $E(G_{1/2}^{opt})/E(G_{1/2}^{c})$ | 0.96 | 0.95 | 0.99 | 0.97 | 0.94 | 0.94 |

Table 1: Relations between functionals.

| n = 14 |
|---|---|---|---|---|---|---|---|---|
| $B_{1/2}^{p}/E(G_{1/2}^{c})$ | 0.34 | 0.62 | 0.92 | 0.32 | 0.45 | 0.53 |
| $B_{1/2}^{c}$ | 1.67 | 1.18 | 1.02 | 1.71 | 1.42 | 1.29 |

| n = 15 |
|---|---|---|---|---|---|---|---|---|
| $B_{1/2}^{p}/E(G_{1/2}^{c})$ | 0.33 | 0.61 | 0.92 | 0.31 | 0.44 | 0.52 |
| $B_{1/2}^{c}$ | 1.68 | 1.18 | 1.02 | 1.74 | 1.42 | 1.29 |

| n = 16 |
|---|---|---|---|---|---|---|---|---|
| $B_{1/2}^{p}/E(G_{1/2}^{c})$ | 0.32 | 0.61 | 0.92 | 0.29 | 0.43 | 0.51 |
| $B_{1/2}^{c}$ | 1.69 | 1.17 | 1.02 | 1.77 | 1.43 | 1.29 |

Table 2: On the quality of the bounds for $E(G_{1/2}^{c})$.

- if, finally, $\ell = n - 1$, then $d(n - 1, n) \sim n/4$.

One can remark that for $\ell = o(n)$, $E(G_{1/2}^{c})$ tends (for $n \to \infty$) to both $b(\ell, n, p)$ and $B(\ell, n, p)$. This is due to the fact that $o(n)/\lim_{n\to\infty}(B - b)(\ell, n, p) \sim 0$.

We see from the above discussion that for the graph and the a priori solution considered and for $n \to \infty$, the distance of $E(G_{1/2}^{c})$ from the bounds given in theorem 3 can take an infinity of values being either arbitrarily close to, or arbitrarily far from them. Consequently, it seems very unlikely that the bounds computed could be substantially improved.

6 Some simulation results

Limited computational experiments have been conducted in order to study the realistic performances of the three strategies proposed. We have randomly generated graphs of orders $n \in \{14, 15, 16\}$, forty graphs per order, in the following way: for each graph, a number $p_E \in [0, 1]$ is randomly drawn; next, for each edge $e$, a randomly chosen probability $p_e$ is associated with, and, if $p_e \leq p_E$, then $e$ is retained as an edge in the graph under construction. For all the generated instances, the minimum vertex-degree varies between 0 (graph non-connected) and 14, while the maximum one varies between 1 and 15. The ratio $|E|/n$ varied between 0.06 and 7.3 for all the graphs generated. The a priori solution considered was the minimum vertex cover of each graph, denoted by $C^*$. We have supposed identical vertex-probabilities and have tested six probability-values: 0.1, 0.5, 0.9, $1/\ln n$, $1/\sqrt{n}$ and $1/n$.

We have computed quantities $E(G_{1/2}^{p})$, $E(G_{1/2}^{c})$, $E(G_{1/2}^{opt})$ and $E(G^{opt})$. The last quantity is the mathematical expectation of a thought process consisting in finding the optimal vertex cover in any subinstance of $G$. This implies computation, by branch-and-bound, of the optimal solution.
for any subgraph of $G$, i.e., $2^n$ executions of branch-and-bound. This is the reason for having restricted our studies in small graphs. In the table 1 we present the mean ratios for the functionals of the different strategies. What we can see from this table is that the ratio $E(G_{opt})/E(G_{in})$ varies between 0.94 and 0.99. In other words, $C^*$ seems to be very close to $C_3$. In the opposite the

<table>
<thead>
<tr>
<th></th>
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<th>$p = 1/\sqrt{n}$</th>
<th>$p = 1/\ln n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 14$</td>
<td>$\sigma_{\alpha_1}/E(G_{\alpha_1})$</td>
<td>1.08</td>
<td>0.36</td>
<td>0.12</td>
<td>1.30</td>
<td>0.60</td>
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<td>$\sigma_{\alpha_2}/E(G_{\alpha_2})$</td>
<td>2.26</td>
<td>0.46</td>
<td>0.13</td>
<td>3.08</td>
<td>0.90</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{opt}/E(G_{opt})$</td>
<td>2.23</td>
<td>0.46</td>
<td>0.14</td>
<td>3.05</td>
<td>0.89</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>$\sigma_{\alpha_1}/E(G_{\alpha_1})$</td>
<td>1.03</td>
<td>0.34</td>
<td>0.11</td>
<td>1.29</td>
<td>0.58</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\alpha_2}/E(G_{\alpha_2})$</td>
<td>2.13</td>
<td>0.44</td>
<td>0.13</td>
<td>3.08</td>
<td>0.88</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{opt}/E(G_{opt})$</td>
<td>2.10</td>
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</tr>
<tr>
<td>$n = 16$</td>
<td>$\sigma_{\alpha_1}/E(G_{\alpha_1})$</td>
<td>0.99</td>
<td>0.33</td>
<td>0.11</td>
<td>1.28</td>
<td>0.57</td>
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<td></td>
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<td>2.03</td>
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<td>0.87</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{opt}/E(G_{opt})$</td>
<td>2.01</td>
<td>0.42</td>
<td>0.13</td>
<td>3.09</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Table 3: Standard deviations for $\alpha_1$, $\alpha_2$ and reoptimization.
Acknowledgement. Many thanks to Marc Demange for very helpful discussions.

References


A  Proof of expression (11)

Consider expression (10) and set $x = 1 - p$. Then, expression (10) becomes

$$
(B - e)(\ell, x, n) = x^{n-\ell} - x^n - \ell(1 - x)x^{n-1} = x^{n-\ell} + (\ell - 1)x^n - \ell x^{n-1} = x^{n-\ell}(1 + (\ell - 1)x^{\ell - 1}) \leq x^{n-\ell} - x^{n-1} \leq 1 \tag{A.1}
$$

Set $f(x) = 1 + (\ell - 1)x^{\ell - 1}$. Then, $f'(x) = \ell(\ell - 1)(x^{\ell - 1} - x^{\ell - 2}) < 0$. Consequently, $f(x)$ is decreasing in $x \in [0, 1]$, therefore $f(x) \geq f(1) = 0$ and the lower bound of expression (11) is proved.

Revisit expression (A.1) and set $g(x) = x^{n-\ell} - x^n$ in $[0, 1]$. First remark that $g(0) = g(1) = 0$. Moreover, $g'(x) = (n - \ell)x^{n-\ell - 1} - (n - 1)x^{n-2}$ and

$$
g'(x) = 0 \iff x_0 = \left(\frac{n - \ell}{n - 1}\right)^{\frac{1}{\ell - 1}}.
$$

So, $g(x)$ increases, with respect to $x$, in $[0, x_0)$ while it is decreasing in $x$ in $(x_0, 1]$. Hence,

$$
(B - e)(\ell, x, n) \leq g(x) \leq g(x_0) = \left(\frac{n - \ell}{n - 1}\right)^{\frac{1}{\ell - 1}} - \left(\frac{n - \ell}{n - 1}\right)^{\frac{1}{\ell - 1}}
$$

and the upper bound of expression (11) is also proved.

B  Proof of expression (13)

Revisit expression (12) and, as previously, set $x = 1 - p$. Then, expression (12) becomes

$$
(e - b)(\ell, x, n) = \ell(1 - x)x^{n-\ell} - x^n + x^n = (\ell - 1)x^{n-\ell} - \ell x^{n-\ell + 1} + x^n = x^{n-\ell}(x^{\ell - 1} - x_{\ell - 1}^{\ell - 1}) \leq (\ell - 1)(x^{n-\ell} - x^{n-\ell + 1}) \tag{B.1}
$$

Set $h(x) = x^{\ell - 1} - x_{\ell - 1}^{\ell - 1}$. Then, $h'(x) = \ell(x^{\ell - 2} - 1) < 0$, so, $h(x)$ is decreasing in $x \in [0, 1]$ and, consequently, $h(x) \geq h(1) = 0$, proving so the lower bound of expression (13).

Revisit expression (B.1) and set $\varphi(x) = x^{n-\ell} - x^{n-\ell + 1}$. Then, $\varphi'(x) = (n - \ell)x^{n-\ell - 1} - (n - \ell + 1)x^{n-\ell}$ and

$$
\varphi'(x) = 0 \iff x_0 = \frac{n - \ell}{n - \ell + 1}.
$$

So, $\varphi(x)$ increases in $[0, x_0)$ and decreases in $(x_0, 1]$. Hence, in $[0, 1]$,

$$
(e - b)(\ell, x, n) \leq (\ell - 1)\left(\frac{n - \ell}{n - \ell + 1}\right)^{n-\ell} \left(1 - \frac{n - \ell}{n - \ell + 1}\right) \leq (\ell - 1)e(n-\ell)n(1 - \frac{n - \ell}{n - \ell + 1}) \frac{1}{n - \ell + 1}
$$

and the lower bound of expression (13) is also proved.