DIFFERENTIAL APPROXIMATION RESULTS
FOR TRAVELING SALESMAN PROBLEM

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Résultats d'approximation différentielle pour le problème du voyageur du commerce

Résumé

Nous commençons par démontrer que les versions maximisation et minimisation du problème du voyageur de commerce sont approximables à rapport différentiel 1/2. Nous présentons ensuite une 3/4-approximation polynomiale du cas particulier à distances 1 et 2; ce résultat nous permet notamment de ramener le rapport standard connu pour la version maximisation de ce sous-problème de 5/7 à 7/8. Nous proposons enfin un résultat négatif: approximer le voyageur de commerce, à coût minimum comme maximum, à mieux que $3475/3476 + \varepsilon$ est NP-difficile pour tout $\varepsilon > 0$.

Mots-clé : algorithme d'approximation, rapport d'approximation, problème NP-complet, complexité, réduction, voyageur du commerce.

Differential approximation results for traveling salesman problem

Abstract

We prove that both minimum and maximum traveling salesman problems can be approximately solved, in polynomial time within approximation ratio bounded above by 1/2. We next prove that, when dealing with edge-distances 1 and 2, both versions are approximable within 3/4. Based upon this result, we then improve the standard approximation ratio known for maximum traveling salesman with distances 1 and 2 from 5/7 to 7/8. Finally, we
1 Introduction

Given a complete graph on \( n \) vertices, denoted by \( K_n \), with positive distances on its edges, the minimum traveling salesman problem (min\_TSP) consists in minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges. The maximum traveling salesman problem (max\_TSP) consists in maximizing the cost of a Hamiltonian cycle. Further special but very natural cases of TSP are the ones where edge-distances are defined using the \( \ell_2 \) norm (Euclidean TSP), or where edge-distances verify triangle inequalities (metric TSP); an interesting sub-case of the metric TSP is the one in which edge-distances are only 1 or 2.
and tries to compute a worst solution for its instance. In order to remove ambiguities about the concept of the worst solution, the following definition, proposed in [9], will be used here.

**Definition 2.** Given a typical instance \( I \) of an NPO problem \( \Pi \), the worst solution of \( I \) is the optimal solution of a new NPO problem \( \tilde{\Pi} \) where items 1 to 3 of definition 1 are identical for both \( \Pi \) and \( \tilde{\Pi} \), and

\[
\text{opt}(\tilde{\Pi}) = \begin{cases} 
\max & \text{opt}(\Pi) = \min \\
\min & \text{opt}(\Pi) = \max \end{cases}
\]

One of the features of the differential ratio is to be stable under affine transformation of the objective function of a problem and so it does not create a dissymmetry between minimization and maximization problems. This is very clear in the case of TSP. Dealing with \( \min_{-}\text{TSP} \) it is very well-known that its general version is not approximable in polynomial time within better than \( 2^p(n) \) for a polynomial \( p \). On the other hand, its maximization version, \( \max_{-}\text{TSP} \), the NP-hardness of which is immediately proved if one replaces distance \( d(i,j) \) for \( \min_{-}\text{TSP} \) by \( M - d(i,j) \) in \( \max_{-}\text{TSP} \), for an \( M \) greater than the largest edge distance in the input graph of \( \min_{-}\text{TSP} \), can be approximated in polynomial time within \( 5/7 \) ([20]).

Let us recall some standard terminology from the theory of the polynomial approximation of the NP-hard problems (for the standard approximation framework). Given an NP minimization (resp., maximization) problem \( \Pi \), a constant-ratio approximation algorithm for \( \Pi \) is a polynomial time approximation algorithm (PTAA) guaranteeing approximation ratio bounded above (resp., below) by a fixed constant, i.e., by a constant that does not depend on any input-parameter of \( \Pi \). \( \text{APX} \) is the class of the NP optimization problems solved by constant-ratio PTAAAs. A polynomial time approximation scheme (PTAS) for \( \Pi \) is a sequence of PTAAAs (receiving as inputs any instance of \( \Pi \) and a fixed constant \( c \)) guaranteeing approximation ratio bounded above (resp., below) by \( 1 + \epsilon \) (resp., \( 1 - \epsilon \)), for every \( \epsilon > 0 \). If a PTAS is polynomial in both \( n \) and \( 1/\epsilon \), then it is called fully polynomial time approximation scheme (FPTAS). For the differential approximation, the ratio achieved by polynomial time approximation schemata is \( 1 - \epsilon \) for both minimization and maximization. Finally, \( \text{APX} \)-complete is the class of problems in \( \text{APX} \), which, in addition, are complete with respect to the existence of a PTAS solving them, in other words, if any \( \text{APX} \)-complete problem could be solved by a PTAS, then any other \( \text{APX} \)-complete problem could be so.

As it is shown in [9, 8], many problems behave in completely different ways regarding traditional or differential approximation. This is, for example, the case of minimum graph-coloring or, even, of minimum vertex-covering. This paper deals with another example of the diversity in the nature of approximation results achieved within the two frameworks, the TSP. For this problem and its versions mentioned above, a bunch of standard-approximation results (positive or negative) have been obtained until nowadays. The first inapproximability result is the one of [21] (see also [13]) affirming that it is NP-hard to approximate \( \min_{-}\text{TSP} \) within any constant factor; with the same proof, one can easily refine the result of [21] to deduce the inapproximability of \( \min_{-}\text{TSP} \) within any ratio of the form \( 2^p(n) \) for any polynomial \( p \). On the other hand, the metric \( \min_{-}\text{TSP} \) is approximable within \( 3/2 \) ([5]), the symmetric \( \min_{-}\text{TSP}12 \) within \( 7/6 \) ([18]) (recall that the original proof of the NP-completeness of the \( \min_{-}\text{TSP} \) is done by reduction to \( \min_{-}\text{TSP}12 \), while the asymmetric version of \( \min_{-}\text{TSP}12 \) is approximable within \( 17/12 \) ([22]). Moreover, \( \min_{-}\text{TSP}12 \) is \( \text{APX} \)-complete ([18]), consequently, given the result of [2], it cannot be solved by a PTAS unless \( \text{P}=\text{NP} \); in other words, \( \exists \epsilon > 0 \) for which approximation of \( \min_{-}\text{TSP}12 \) within ratio smaller than \( 1 + \epsilon \) is NP-hard. Furthermore, even in graphs where the density of the subgraph spanned by the edges of length 1 is bounded below by a constant \( c \in [0,1/2] \), \( \min_{-}\text{TSP}12 \) cannot be solved by a polynomial time approximation schema ([11]). The works of [10] and more recently of [4] refine the result of [18] specifying
for $\epsilon$. In [4] is proved that for any $\epsilon > 0$, it is NP-hard to approximate $\text{min\_TSP12}$ within ratio smaller than, or equal to, $3475/3476 - \epsilon$; in other words, the result of [10] gives a value - equal to $1/3476 - \epsilon'$, $\forall \epsilon' > 0$ - for the hardness threshold $\epsilon$ of $\text{min\_TSP12}$ refining so the negative results of [18, 19]. Finally, another restrictive version of the metric $\text{min\_TSP}$, the Euclidean,
Consequently, every PTAA for metric min\_TSP can simultaneously solve general min\_TSP within the same differential approximation ratio. \[\]

Let \(d_{\text{min}} = \min \{d(i, j) : v_i v_j \in E\}\). Then, if one transforms every distance \(d(i, j)\) into \(d(i, j) - d_{\text{min}} + 1\), one obtains a complete graph where \(d_{\text{min}} = 1\) and with arguments completely analogous to the ones of proposition 1, the following holds.

**Proposition 2.** General min\_TSP and min\_TSP with \(d_{\text{min}} = 1\) are differentially equi-approximable.

We next consider another class of instances, the one where the edge-distances are either \(a\), or \(b\) (notorious member of this class of min\_TSP-problems, denoted by min\_TSPb, is the min\_TSP12). Suppose, without loss of generality, that \(a < b\). Then, by proposition 2, min\_TSPb is equi-approximable with min\_TSP1b. Consider now an instance of the latter problem. If one sets \(b = 2\) for all the \(b\)-edges (edges of distance \(b\)), then by arguments completely similar to the ones of the proof of proposition 1 (and since for a tour \(T\) containing \(k_b\) \(b\)-edges, \(d(T) = n + (b - 1)k_b\)), the following result holds.

**Proposition 3.** min\_TSPab and min\_TSP12 are differentially equi-approximable.

Note that results analogous to the ones of propositions 1, 2 and 3 do not hold in the standard approximation framework.

3 2\_OPT and differential approximation for the general minimum traveling salesman

In what follows, we denote by D-APX the analogous of the class APX, the class of NPO problems solved by a constant-ratio PTAA, for the differential approximation framework.

**Theorem 1.** min\_TSP is differentially approximable within approximation ratio \(1/2\) and this ratio is tight.

**Proof.** In what follows, suppose that a tour is listed as the set of its edges and consider the following algorithm of [7].

BEGIN /2\_OPT/

(1) start from any feasible tour \(T\);

(2) \text{REPEAT}

(3) \quad \text{pick a new set } \{v_i v_j, v_i v_j\} \subseteq T;

(4) \quad \text{IF } d(i, j) + d(i', j') > d(i, i') + d(j, j') \text{ THEN } T \leftarrow (T \setminus \{v_i v_j, v_i v_j\}) \cup \{v_i v_i', v_j v_j'\} \text{ FI}

(5) \quad \text{UNTIL no improvement of } d(T) \text{ is possible;}

(6) \text{OUTPUT } T;

END. /2\_OPT/

Suppose now that, starting from a vertex denoted by \(v_1\), the rest of the vertices is ordered following the tour \(T\) finally computed by 2\_OPT (so, given a vertex \(v_i, i = 1, \ldots, n - 1, v_{i+1}\) is its successor with respect to \(T\); \(v_{n+1} = v_1\)). Let us fix one optimal tour and denote it by \(T^*\). Given a vertex \(v_i\), denote by \(v_{s^*}(i)\) its successor in \(T^*\) (remark that \(v_{s^*}(i+1)\) is the successor of \(v_{s^*}(i)\) in \(T\); in other words, edge \(v_{s^*}(i) v_{s^*}(i+1) \in T\)). Finally let us fix one (of the eventually many) worst-case (maximum total-distance) tour \(T_0\).

The tour \(T\) computed by 2\_OPT is a local optimum for the 2-exchange of edges in the sense that every interchange between two non-intersecting edges of \(T\) and two non-intersecting edges of \(E \setminus T\) will produce a tour of total distance at least equal to \(d(T)\). This implies in particular that, \(\forall i \in \{1, \ldots, n\},\)

\[d(i, i + 1) + d(s^*(i), s^*(i) + 1) \leq d(i, s^*(i)) + d(i + 1, s^*(i) + 1);\]
so, writing the expression above for all \( i \in \{1, \ldots, n\} \), we get

\[
\sum_{i=1}^{n} (d(i, i + 1) + d(s^*(i), s^*(i) + 1)) \leq \sum_{i=1}^{n} (d(i, s^*(i)) + d(i + 1, s^*(i) + 1)) \tag{1}
\]

Moreover, it is easy to see that the following holds:

\[
\bigcup_{i=1, \ldots, n} \{v_{i}v_{i+1}\} = \bigcup_{i=1, \ldots, n} \{v_{s^*(i)}v_{s^*(i)+1}\} = T \tag{2}
\]

\[
\bigcup_{i=1, \ldots, n} \{v_{i}v_{s^*(i)}\} = T^* \tag{3}
\]

\[
\bigcup_{i=1, \ldots, n} \{v_{i+1}v_{s^*(i)+1}\} = \text{s some feasible tour } T' \tag{4}
\]

Let us show that \( T' = \bigcup_{i=1, \ldots, n} \{v_{i+1}v_{s^*(i)+1}\} \) is feasible. Recall that an acyclic permutation is a bijective function \( f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) such that, \( \forall i \in \{1, \ldots, n\} \):

\[
\begin{cases}
  f^{(k)}(i) \neq i & k < n \\
  f^{(n)}(i) = i
\end{cases}
\]

Every feasible tour \( T \), oriented as mentioned above, can be seen as an acyclic permutation. Consider now the following mappings

\[
s^* : i \mapsto s^*(i) \\
f : i \mapsto i + 1 \\
h : i \mapsto s^*(i - 1) + 1.
\]

It is easy to see that if \( s^* \) is an acyclic permutation and \( f \) is a permutation, then \( h = f \circ s^* \circ f^{-1} \) is an acyclic permutation. Moreover, it is not hard to see that pairs \( (i, h(i)) \) correspond \( \text{(mod n)} \) to the edges of \( T' \).

Combining expression (1) with expressions (2), (3) and (4), one gets:

\[
(2) \quad \sum_{i=1}^{n} d(i, i + 1) + \sum_{i=1}^{n} d(s^*(i), s^*(i) + 1) = 2\lambda_{2-\text{OPT}}(K_n) \\
(3) \quad \sum_{i=1}^{n} d(i, s^*(i)) = \beta(K_n) \\
(4) \quad \sum_{i=1}^{n} d(i + 1, s^*(i) + 1) = d(T') \leq \omega(K_n) \tag{5}
\]

and expressions (1) and (5) lead to

\[
2\lambda_{2-\text{OPT}}(K_n) \leq \beta(K_n) + \omega(K_n) \iff \frac{\omega(K_n) - 2\lambda_{2-\text{OPT}}(K_n)}{\omega(K_n) - \beta(K_n)} \geq \frac{1}{2}
\]

Consequently, \( \delta_{2-\text{OPT}} \geq 1/2 \).

Consider now a \( K_{4n+8} \) with \( n \geq 0 \), set \( V = \{v_i : i = 1, \ldots, 4n + 8\} \), let

\[
d(2k + 1, 2k + 2) = 1 \quad k = 0, 1, \ldots, 2n + 3 \\
d(4k + 2, 4k + 4) = 1 \quad k = 0, 1, \ldots, n + 1 \\
d(4k + 3, 4k + 5) = 1 \quad k = 0, 1, \ldots, n \\
d(4n + 7, 1) = 1
\]
and set the distances of all the remaining edges to 2. Then,

\[ T = \{v_tv_{i-1} : i = 1, \ldots, 4n + 7\} \cup \{v_{4n+8}v_1\} \]

\[ T^* = \{v_{2k+1}v_{2k+2} : k = 0, \ldots, 2n + 3\} \cup \{v_{2k+2}v_{4k+4} : k = 0, \ldots, n + 1\} \]
\[ \cup \{v_{4k+3}v_{4k+5} : k = 0, \ldots, n\} \cup \{v_{4n+7}v_1\} \]

\[ T_w = \{v_{2k+1}v_{2k+3} : k = 0, \ldots, 2n + 2\} \cup \{v_{2k+1}v_{2k+4} : k = 0, \ldots, 2n + 1\} \]
\[ \cup \{v_{2k+2}v_{2k+5} : k = 0, \ldots, 2n + 1\} \cup \{v_{4n+8}v_1\} \].

In figure 1, \( T^* \) and \( T_w \) are shown for \( n = 1 \) \( (T = \{1, \ldots, 11, 12, 1\}) \). Hence, \( \delta_{2 \_ \text{OPT}}(K_{4n+8}) = 1/2 \) and this completes the proof of the theorem. \( \blacksquare \)

![Figure 1. Tightness of the 2\_OPT approximation ratio for \( n = 1 \).](image)

From the proof of the tightness of the ratio of 2\_OPT, the following corollary is immediately deduced.

**Corollary 1.** \( \delta_{2 \_ \text{OPT}} = 1/2 \) is tight even for min\_TSP12.

A first case of polynomial complexity for algorithm 2\_OPT (even if edge-distances of the graph are exponential in \( n \)) is for graphs where the number of (feasible) tour-values, denoted by \( \sigma(K_n) \), is polynomial in \( n \). Here, since there exists a polynomial number of different min\_TSP solution-values, achievement of a locally minimal solution (starting, at worst for the worst-value solution) will need a polynomial number of steps (at most \( \sigma(K_n) \)) for 2\_OPT.

Theorem 1 obviously works in polynomial time when \( d_{\max} \) is bounded above by a polynomial of \( n \). However, even when this condition is not satisfied, there exist restrictive cases of min\_TSP for which 2\_OPT remains polynomial.

Consider now complete graphs with a fixed number \( k \in \mathbb{N} \) of distinct edge-distances, \( d_1, d_2, \ldots, d_k \). Then, any tour-value can be seen as \( k \)-tuple \( (n_1, n_2, \ldots, n_k) \) with \( n_1 + n_2 + \ldots + n_k = n \), where \( n_1 \) edges of the tour are of distance \( d_1 \), \( \ldots, n_k \) edges are of distance \( d_k \) \( (\sum_{i=1}^{k} n_id_i = \sigma(T)) \). Consequently, the consecutive solutions retained by 2\_OPT (in line (4)) before attaining a local minimum are, at most, as many as the number of the arrangements with repetitions of \( k \) distinct items between \( n \) items (in other words, the number of all the distinct \( k \)-tuples formed by all the numbers in \( \{1, \ldots, n\} \)), i.e., bounded above by \( O(n^k) \).
4.1 Specification of the min_TSP12-algorithm and evaluation of $\lambda(K_n)$

In the sequel, we will first specify a PTAA min_TSP12 and estimate the value $\lambda_{\text{TSP12}}(K_n) = d(T(K_n))$ of the Hamiltonian tour computed. Next, we will compute a lower bound for $\omega(K_n)$. As for theorem 1, we will exhibit a feasible tour of a certain value. Since worst solution's value is larger than the value of every other Hamiltonian tour of $K_n$, the value of the tour exhibited will be the bound claimed.

Let $\tilde{M}$ be an optimal triangle-free 2-matching of $K_n$ (recall that, as we have mentioned, such a matching is maximal, i.e., it does only contain cycles). Starting from $\tilde{M}$, one can easily construct an optimal 2-matching $M^*$ where every matching of two cycles consists in an edge.
PR ← PR ∪ \{C_s, C_t\};
M* ← M* \{C_s, C_t\};

OD
OUTPUT PR;
END. /PREPROCESS/

Suppose the WHILE loop of PREPROCESS executed \(q\) times and denote by \(\{C_1, C_2\}\) the cycles considered during the \(s\)th execution of the loop, \(s = 1, \ldots, q\). Then \(PR = \bigcup_{s=1}^{q} \{C_1, C_2\}\). Set \(r = p - 2q\) and denote by \(D_t\), \(t = 1, \ldots, r\), the cycles in \(\{C_1, \ldots, C_p\}\) \(\bigcup_{s=1}^{q} \{C_1, C_2\}\). Under all this,

\[
M^* = \left( \bigcup_{s=1}^{q} \{C_s, C_t\} \right) \bigcup \left( \bigcup_{t=1}^{r} D_t \right) \bigcup C_{p+1}.
\]

The following facts hold and complete the above discussion.

**Fact 4.** \(2q + r \geq 1\); if \(2q + r = 1\) then \(C_{p+1} \neq \emptyset\).

**Fact 5.** \(\forall s \in \{1, \ldots, q\}, \forall t \in \{1, 2\}, \forall e \in C_s, d(e) = 1\).

**Fact 6.** \(\forall t \in \{1, \ldots, r\}, \forall e \in D_t, d(e) = 1\).

**Fact 7.** \(\forall s \in \{1, \ldots, q\}, \exists i^s \in V(C_s), \exists I^s \in V(C_2)\) such that \(d(i^s, I^s) = 1\).

**Fact 8.** \(\forall (t, t') \in \{1, \ldots, r\} \times \{1, \ldots, r\}, t \neq t', \forall (u, v) \in V(D_t) \times V(D_{t'}), d(u, v) = 2\).

In the sequel, for \(s = 1, \ldots, q\), we denote by \(\alpha^s\) and \(\beta^s\) (resp., \(A^s\) and \(B^s\)) the vertices adjacent to \(i^s\) (resp., \(I^s\)) in \(C_s\) (resp., \(C_2\)). We set \(c = \sum_{s=1}^{q} (|C_s| + |C_2|), d = \sum_{t=1}^{r} |D_t|, E2 = \{e \in C_{p+1} : d(e) = 2\}\). Following these notations, \(n = c + d + |C_{p+1}|\) and, denoting by \(|E2|\) cardinality of the set \(E2\),

\[
d(M^*) = n + |E2|
\]  \hspace{2cm} (6)

We are well-prepared now to describe the algorithm proposed. Informally, it first patches cycles \(C_1, C_2\) into a single cycle \(C^*\), \(s = 1, \ldots, q\). Next, it patches cycle \(C^*\) with \(C^2\) to produce a cycle \(C\) which will be patched with \(C^3\), and so on, finally producing a single cycle \(C\). It does so for the cycles \(D_t, t = 1, \ldots, r\), producing a single cycle \(D\). Then it patches \(C\) and \(D\) in order to produce a partial tour \(T\) and finally it patches \(T\) and \(C_{p+1}\) obtaining so the final TSP-tour \(T(K_n)\).

### 4.1.1 Construction and evaluation of \(C\)

Construction of \(C\) is performed by means of the following procedure.

BEGIN /C/

FOR \(s \leftarrow 1\) to \(q\) DO using edge \(i^s I^s\) \(C^* \leftarrow CYCLE\_PATCH(C_s, C_2); OD\)

\(C^* \leftarrow C^1;\)

FOR \(s \leftarrow 1\) TO \(q - 1\) DO

replacing as many 2-edges as possible \(C \leftarrow CYCLE\_PATCH(C, C^s+1);\)

OD

OUTPUT \(C;\)

END. /C/

The call of algorithm \(CYCLE\_PATCH\) in the first FOR-loop of \(C\) is a very slightly different variant of the corresponding procedure presented above where one imposes to the 1-edge \(i^s I^s\) (fact 7) to be one of the cross-edges entering cycle \(C^s\).
Lemma 1. The 2-matching \((C^1, C^2, \ldots, C^q)\) produced during the \(q\) executions of the first FOR-loop of algorithm \(C\) has value \(d(C^1, C^2, \ldots, C^q) = c + q\).

Proof of lemma 1. Patching of \(C^s\) and \(C^s\) into \(C^s\) is done using 1-edge \(i^sI^s\) (fact 7), \(s = 1, \ldots, q\). Consequently, only one 2-edge has been included in \(C^s\) (the one used with \(i^sI^s\) to patch \(C^s\) and \(C^s\)). Such an edge always exists because of fact 1. So, for \(s = 1, \ldots, q\), execution of \(\text{CYCLE\_PATCH}(C^s, C^s)\) in the first FOR-loop of \(C\) will produce in all exactly \(q\) 2-edges replacing and \(q\) 1-edges replacing \(2q\) 1-edges. Consequently, \(d(C^1, C^2, \ldots, C^q) = \sum_{s=1}^{q} |C^s| + |C^s|\) + \(q = c + q\) and this completes the proof of lemma 1.

During the executions of \(\text{CYCLE\_PATCH}\) in the second FOR-loop of \(C\), we try that the total distance of the resulting cycle is no longer than the sum of the total distances of the cycles patched. In other words, we try to not produce additional 2-edges in the resulting cycle. Here the following lemma holds.

Lemma 2. The cycle \(C\) produced during the second FOR-loop of algorithm \(C\) does not increase \(d(C^1, C^2, \ldots, C^q)\).

Proof of lemma 2. The proof is done by induction on \(q\).

4.1.1.1 \(q = 1\)

The proof of this case is an immediate application of lemma 1 with \(q = 1\).

4.1.1.2 \(q = k\)

Suppose that during the \(k\) first executions of the FOR-loop, the number of 2-edges is at most \(k\).

4.1.1.3 \(q = k + 1\)

Suppose now that there exists at least one 2-edge in \(C\) (note also that \(C^{k+1}\), since it has been not processed yet, always contains the 2-edge produced by the execution of the first FOR-loop). Since the patching with \(C^{k+1}\) is done by algorithm \(C\) using two 2-edges, there is no additional 2-edge created. On the other hand, if no 2-edge exists in \(C\), the patching of \(C\) with \(C^{k+1}\) will
4.1.3 Construction and evaluation of $\hat{T}$

BEGIN /$\hat{T}$/
replacing as many 2-edges as possible $\hat{T} \leftarrow$ CYCLE_PATCH($C, D$);
OUTPUT $\hat{T}$;
END. /$\hat{T}$/

With the same arguments as in lemma 2, the following holds for $|\hat{T}|$:

\[
\begin{align*}
\{ \quad & d(\hat{T}) \leq c + d + q + r & 2q + r \geq 2 \\
\{ \quad & d(T) = d & (q, r) = (0, 1)
\end{align*}
\]  
(9)

4.1.4 Overall specification of the min_TSP12-algorithm, construction and evaluation of $T$

Once $\hat{T}$ constructed, call of CYCLE_PATCH($\hat{T}, C_{p+1}$), changing as many as 2-edges (at most 2) as possible, constructs the final TSP-solution $T(K_n)$ and the whole min_TSP12-PTAA is the following. The 2-matching $M$ produced in the first line of the algorithm below is supposed to be optimal and without cycles on less than, or equal to, four edges.

BEGIN /TSP12/
call the algorithm of [14] to produce $\hat{\mathcal{R}}$;
$\mathcal{R} = (C_1, \ldots, C_{p+1}) \leftarrow 2\_MIN(\hat{\mathcal{R}})$;
$M^* = PREPROCESS(\mathcal{R}) \cup \bigcup_{t=1}^{p} \{D_t\} \cup C_{p+1}$;
$C = C(M^*)$;
$D = D(M^*)$;
$\hat{T} \leftarrow$ CYCLE_PATCH($C, D$);
OUTPUT $T(K_n) \leftarrow$ CYCLE_PATCH($\hat{T}, C_{p+1}$);
END. /TSP12/

It is easy to see that, since all the algorithms called are polynomial, TSP12 works in polynomial time.

If $(q, r) = (0, 1)$ (in this case, by fact 4, $C_{p+1} \neq \emptyset$), then patching of $D_1$ with $C_{p+1}$ constructs a tour with $d(T(K_n)) = d(D_1) + d(C_{p+1}) + 1 = d(M^*) + 1 = d(M^*) + q + r$.

Suppose $2q + r \geq 2$. Then, by expression (9), $d(T) \leq c + d + q + r$. If $d(T) < c + d + q + r$, i.e., $d(T) \leq c + d + q + r - 1$, even if $\hat{T}$ does not contain any 2-edge, patching of $\hat{T}$ with $C_{p+1}$ will create only one additional 2-edge so, finally, $d(T(K_n)) \leq d(T) + d(C_{p+1}) \leq c + d + q + r + |C_{p+1}| + |E_2|$ and, by expression (6), $d(T(K_n)) \leq d(M^*) + q + r$. If $d(T) = c + d + q + r$, we simply exchange two 2-edges and the same expression for $d(T(K_n))$ always holds.

The discussion above leads to the following concluding expression for the quantity $d(T(K_n))$:

\[
d(T(K_n)) = \lambda_{TSP12}(K_n) \leq v(M^*) + q + r
\]  
(10)

4.2 A bound for $\omega(K_n)$

In what follows, we will exhibit a TSP12-solution, the objective value of which will provide us with a lower bound for the value $\omega(K_n)$ of the worst TSP12-solution on $K_n$. For this we define a set $W$ of disjoint elementary paths (d.e.p.), any one of them containing only 2-edges. Obviously, if $W = \{w_1, \ldots, w_{|W|}\}$, one, by properly linking $w_i$'s, can easily construct a tour $T'$ verifying $d(T') \geq n + \sum_{w_i \in W} |w_i|$ which is a lower bound for $\omega(K_n)$.
4.2.1 Disjoint elementary paths on $V(C)$

Recall that, for $q \neq 0$, $d(i^s, I^s) = 1$; hence, by fact 1, $d(a^s, A^s) = d(a^s, B^s) = d(b^s, A^s) = d(b^s, B^s) = 2$, $s = 1, \ldots, q$. Always by fact 1, either $d(i^s, B^s) = 2$, or $d(I^s, b^s) = 2$. Without loss of generality, we suppose all over the rest of the proof of theorem 2 that $d(i^s, B^s) = 2$.

Consequently, for $s = 1, \ldots, q$, set $W_{C,s} = \{b^s A^s, A^s a^s, a^s B^s, B^s i^s\}$ and the set of d.e.p. on the vertices of $C$ is $W_C = \bigcup_{s=1}^{q} W_{C,s}$ with

$$|W_C| = 4q \quad (11)$$

4.2.2 Disjoint elementary paths on $V(D)$

If $r > 1$, we choose, for $t = 1, \ldots, r$, a sequence $\{w_t, x_t, y_t, z_t\} \in V(D_t)$. Then, the set of d.e.p. and its cardinality on $V(D)$ is

$$W_D = \{w_1, w_2, \ldots, w_r, x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_r\} \quad (12)$$

$$|W_D| = 4(r - 1) + 3 = 4r - 1 \quad (13)$$

If $r \leq 1$, then we set $W_D = \emptyset$.

4.2.3 Disjoint elementary paths on $V(\hat{T})$

4.2.3.1 $q > 0$

Suppose first $r \neq 1$. If $r = 0$, then $W_T = W_C$. Suppose now $r > 1$. Then, by fact 1, there exists vertex $v_1 \in V(D_1)$ such that either $d(v_1, B^1) = 2$, or $d(v_1, I^1) = 2$. Let $e$ be this 2-edge. Without loss of generality, we can suppose $v_1 = u_1$ (see the paragraph just above). Then the set of d.e.p. on $V(\hat{T})$ is $W_T = W_C \cup W_D \cup \{e\}$ with (see expressions (11) and (13))

$$|W_T| = 4q + 4r - 1 + 1 = 4(q + r) \quad (14)$$

Suppose now that there exist $x \in v(D_1)$ and $v \in \{a^1, b^1, A^1, B^1\}$ with $d(x, v) = 1$; assume $v = a^1$ (so, $d(xa^1) = 1$). Let $w, y, z$ be three vertices in $V(D_1)$ such that $w, x, y, z$ are distinct in $D_1$. Then, by fact 1, $d(w, a^1) = d(x, a^1) = d(y, a^1) = d(y, I^1) = 2$.

If $d(y, I^1) = d(x, A^1) = 2$, then $W_1 = \{I^1 y, y a^1, a^1 w, w i^1, a^1 B^1, B^1 b^1, b^1 A^1, A^1 x\}$. If not, we can suppose (up to renaming of cycles $C_1^1, C_2^1$ and $D_1$ in the discussion that follows) $d(y, I^1) = 1$. Then, by fact 1 one of the edges $i^1 x$ and $a^1 y$ is a 2-edge; let us denote it by $e$. Set $f = a^1 y$ if $e = i^1 x$, or $f = i^1 w$ if $e = a^1 y$. Then, $W_1 = \{a^1 w, i^1 y, a^1 A^1, A^1 b^1, b^1 B^1, B^1 x\} \cup \{e, f\}$.

Figure 2 illustrates this case. In all the above cases set $W_T = (W_C \setminus W_{C_1}) \cup W_1$ is a set of d.e.p. (remark that the hypothesis $d(i^s, B^s) = 2$ does not intervene in the specification of the set $W_T$) of cardinality

$$|W_T| = 4q + 4 + 8 = 4(q + r) \quad (15)$$

From expressions (14) and (15) we conclude for the case $q > 0$:

$$|W_T| = 4(q + r) \quad (16)$$
4.2.3.2 \( q = 0 \)

Consider first \( |E2| = |C_{p+1}| \). Remark that \( r \geq 1 \) (if not \( T(K_n) = C_{p+1} \) is a minimum-distance Hamiltonian tour); note also that \( C_{p+1} \) can eventually be empty. From facts 2, 3 and 8, for any cycle \( D_t \), \( t = 1, \ldots, r \) the only 1-edges (other than the ones of \( D_t \)) incident to vertices of \( V(D_t) \) are pairs of \( V(D_t) \times V(D_t) \) not included in \( D_t \). However, in any feasible Hamiltonian tour, one cannot use more than \( \sum_{t=1}^{r} |E_t| \) of these edges in the construction. \[ 2 \]
4.2.4.3 $q = 0, |E2| < |C_{p+1}|

Let us first suppose $r = 1$. Then, let $H = \{e_1, e_2, e_3, e_4\}$ be an elementary path on four edges in $C_{p+1}$ with endpoints $u$ and $v$ and such that $d(e_1) = 1$ and $d(e_2) = 2$; let $M = \{w, x, y, z\}$ be a sequence of four successive vertices in $V(D_1)$ and set $H2 = \{e \in H : d(e) = 2\}$. Then, using facts 1 and 2, we can construct (see figure 3), between paths $H$ and $M$, a path $P$ containing at least $4 + |H2|$ 2-edges where $|P_{2}(v)| \leq 1$. We set $W = P \cup (E2 \setminus H2)$ that constitutes a d.e.p with $|W| = (4 + |H2|) + (|E2| - |H2|) = |E2| + 4$.

![Figure 3](image)

Figure 3. Construction of $W$ supposing $vz = \text{argmax}\{d(v', y), d(v, z)\}$.

Let us now suppose $r > 1$ and let $uv$ be an 1-edge of $C_{p+1}$. Moreover, from the previous paragraph, for the case we deal with, $W = W_D$, where $W_D$ is given by expression (12). By fact 1 we have that either $x_1 u$, or $y_1 v$ is a 2-edge; let us suppose $d(x_1 u) = 2$. Then, the set $W = W_D \cup E2 \cup \{x_1 u\}$ forms a d.e.p. composed of $|W| = (4r - 1) + |E2| + 1 = 4r + |E2| = 4(q + r) + |E2|$ 2-edges.

Consequently, dealing with $W$, we always have $|W| \geq 4(q + r) + |E2|$. One can obtain a tour $T_w(K_n)$ by properly linking d.e.ps by edges (at worst by 1-edges) in order that they form a Hamiltonian cycle on $K_n$. The so obtained $T_w(K_n)$ has objective value $d(T_w(K_n)) \geq n + 4(q + r) + |E2| = d(M^*) + 4(q + r)$.  

$$\omega(K_n) \geq d(T_w(K_n)) \geq n + 4(q + r) + |E2| = d(M^*) + 4(q + r). \quad (17)$$
4.3 The differential approximation ratio of TSP12

We have already seen that if \( q = 0 \) and \(|E2| = |C_{p+1}|\), then \( \delta(\text{min-TSP12}) = 1 \). So, for \( q > 0 \) or \( q = 0 \) and \( |E2| < |C_{p+1}| \) expressions (10), (17) and the fact that \( \beta(K_n) \geq d(M^*) \), we get

\[
\delta_{TSP12}(K_n) = \frac{\omega(K_n) - \lambda_{TSP12}(K_n)}{\omega(K_n) - \beta(K_n)} \geq \frac{d(M^*) + 4(q + r) - (d(M^*) + (q + r))}{d(M^*) + 4(q + r) - d(M^*)} = \frac{3(q + r)}{4(q + r)} = \frac{3}{4}.
\]

4.4 Ratio 3/4 is tight for TSP12

Consider two cliques and number their vertices by \{1,...,4\} and by \{5,6,...,n+8\}, respectively. Edges of both cliques have all distance 1. Cross-edges \( ij, i=1,3, j=5,\ldots,n+8, \) are all of distance 2, while every other cross-edge is of distance 1.

Unraveling of TSP12 will produce:

\[
\begin{align*}
T &= \{1, 2, 3, 4, 5, 6, \ldots, n+7, n+8, 1\} & \text{cycle-pathing on edges } (1, 4) \text{ and } (5, n+8) \\
T_w &= \{1, 2, 3, 4, 5, 6, 7, 8, \ldots, n+7, n+8, 1\} & \text{using 2-edges } (1, 5), (6, 3), (3, 7) \text{ and } (n+8, 1) \\
T^* &= \{1, 2, n+8, n+7, \ldots, 5, 4, 3, 1\} & \text{using 1-edges } (4, 5), (2, n+8) 
\end{align*}
\]

i.e., \( \lambda(K_{n+8}) = n + 9 \), \( \beta(K_{n+8}) = n + 8 \) and \( \omega(K_{n+8}) = n + 12 \) (in figure 4, \( T^* \) and \( T_w \) are shown for \( n = 2; T = \{1, \ldots, 10, 1\} \)). Consequently, \( \delta_{TSP12}(K_{n+8}) = 3/4 \) and this completes the proof of theorem 2.

Let us note that the differential approximation ratio of the 7/6-algorithm of [18], when running on \( K_{n+8} \), is also 3/4. The authors of [18] bring also to the fore a family of worst-case instances for their algorithm: one has \( k \) cycles of length four arranged around a cycle of length \( 2k \). We refer to [18] for more details on this matter.
<table>
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<th>$k$</th>
<th>TSP12</th>
<th>The algorithm of [18]</th>
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<td>3</td>
<td>0.931100364</td>
<td>0.846702091</td>
</tr>
<tr>
<td>4</td>
<td>0.9000002</td>
<td>0.833333</td>
</tr>
<tr>
<td>5</td>
<td>0.920289696</td>
<td>0.833333</td>
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<tr>
<td>6</td>
<td>0.9222222</td>
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<table>
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<tr>
<th>$\text{Differential ratio}$</th>
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<tr>
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<table>
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<th>$\text{Standard ratio}$</th>
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<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

Table 1: A limited comparison between TSP12 and the algorithm of [18] on some worst-case instances of the latter.

approximation measure. In what follows, we denote by $R[\Pi](I, S)$ the value of the approximation measure $R$ relative to a solution $S$ of an instance $I$ of $\Pi$. We suppose that $R$ has values in $[0, 1]$ (for the standard approximation, we inverse the approximation ratio in the case of minimization problems).

**Definition 3.** A $G$-reduction of the pair $(\Pi_1, R_1)$ to $(\Pi_2, R_2)$, denoted by $(\Pi_1, R_1) \leq^G (\Pi_2, R_2)$, is a triplet $(\alpha, g, c)$ such that:

- $\alpha: \mathcal{I}_1 \rightarrow \mathcal{I}_2$ polynomially transforms instances of $\Pi_1$ into instances of $\Pi_2$;
- $g: S(\alpha(I)) \rightarrow S(I)$ polynomially transforms solutions for $\Pi_2$ into solutions for $\Pi_1$;
- $c: [0, 1] \rightarrow [0, 1]$ ($c^{-1}(0) = \{0\}$) is such that, $\forall \epsilon \in [0, 1], \forall I \in \mathcal{I}_1, \forall S \in S(\alpha(I))$,

$$R_2[\Pi_2](\alpha(I), S) \geq \epsilon \implies R_1[\Pi_1](I, g(S)) \geq c(\epsilon).$$

The following easy lemma holds.

**Lemma 3.** Consider an NPO problem $\Pi = (\mathcal{I}, S \approx_{\text{opt}})$. If $\exists \epsilon > 0$ such that $\forall I \in \mathcal{I}$, 

$$\text{c}(\epsilon)$$

is a valid solution for $\Pi[I]$. Then, the following holds.

$$R_2[\Pi_2](\alpha(I), S) \geq \epsilon \implies R_1[\Pi_1](I, g(S)) \geq \text{c}(\epsilon).$$
Theorem 3 implies \(1/\rho_{\text{TSP12}} \geq 4/5\); in other words, \(\rho_{\text{TSP12}} \leq 5/4\). This ratio is better than the one of [5] for this particular case, but with no operational impact since it is dominated by the result of [18].

Recall that \(\text{min\_TSP12}\) and \(\text{min\_TSPab}\) are equi-approximable in the differential approximation framework. Consequently, using theorem 3 with \(\delta = 3/4\), the following corollary holds.

**Corollary 3.** \(\text{min\_TSPab}\) is approximable within

\[
\rho \leq \frac{3}{4} + \frac{1}{4a}
\]

in the standard framework. This ratio tends to \(\infty\) with \(a\).

Let us now denote by \(A_{\text{max}}\) and \(A_{\text{min}}\) a maximum and a minimum spanning trees of \(K_n\), respectively, and by \(c(A_{\text{max}})\) and \(c(A_{\text{min}})\) their respective costs. Then, the following proposition holds.

**Proposition 4.** If \(c(A_{\text{max}})/c(A_{\text{min}}) \leq \nu\), \(\nu > 1\), then \(\min\_\text{TSP}\), \(\rho \leq c'(\min\_\text{TSP}, \delta)\) with \(c'(\epsilon) = 1/(\nu (1-\epsilon) + \epsilon)\).

**Proof.** Let \(T_w(K_n)\) and \(T^*(K_n)\) be a worst-value tour and an optimal tour of \(K_n\), respectively. Set \(d_w = \min_{w, v_j \in T_w(K_n)} \{d(i,j)\}\) and \(d_\beta = \max_{\beta, v_j \in T^*(K_n)} \{d(i,j)\}\). Since \(T_w(K_n) \setminus \{\text{argmin}_{u, v_j \in T_w(K_n)} \{d(i,j)\}\}\) and \(T^*(K_n) \setminus \{\text{argmax}_{\beta, v_j \in T^*(K_n)} \{d(i,j)\}\}\) are obviously spanning trees of \(K_n\), \(c(A_{\text{max}}) \geq \omega(K_n) - d_w\), \(c(A_{\text{min}}) \leq \beta(K_n) - d_\beta\). Remark also that \(d_w < \omega(K_n)/n\) and \(d_\beta > \beta(K_n)/n\). So, \(c(A_{\text{max}}) \geq \omega(K_n)(1-1/n)\) and \(c(A_{\text{min}}) \leq \beta(K_n)(1-1/n)\). Consequently,

\[
\frac{\omega(K_n)}{\beta(K_n)} \leq \frac{c(A_{\text{max}})(1-1/n)}{c(A_{\text{min}})(1-1/n)} \leq \frac{c(A_{\text{max}})}{c(A_{\text{min}})} \leq \nu.
\]

Hence, \(\omega(K_n) - \beta(K_n) \leq (\nu - 1)\beta(K_n)\), and using lemma 3 for \(t = (\nu - 1)\) we get \(c(\epsilon) = (\nu (1-\epsilon) + \epsilon)^{-1}\). □

### 5.2 An inapproximability result

We first note that one can prove very easily (with arguments similar to the ones of theorem 6.13 in [13]) that \(\min\_\text{TSP} \text{ cannot be solved by a differential PTAS unless } P=NP\). We now restrict ourselves to \(\min\_\text{TSP12}\) and revisit theorem 3. It is easy to see that it does not only establish links between the approximabilities of \(\min\_\text{TSP12}\) in standard and differential frameworks, but it also establishes limits on its approximability in the two frameworks. Plainly, since approximation of \(\min\_\text{TSP12}\) within \(\delta = 1 - \epsilon\) implies its approximation within \(\rho = 2 - (1-\epsilon) = 1 + \epsilon\), \(0 \leq \epsilon \leq 1\), if there exists an \(\epsilon_0\) such that, under a very likely complexity theory hypothesis, \(\min\_\text{TSP12}\) is inapproximable within \(\rho_0 \leq 1 + \epsilon_0\), then it is inapproximable within \(\delta_0 \geq 1 - \epsilon_0\). In other words, the hardness thresholds for standard and differential frameworks are identical.

**Theorem 4.** If under a complexity theory hypothesis \(\min\_\text{TSP12}\) is inapproximable within

\[\rho_0 \leq 1 + \epsilon_0\]
6 Differential approximation of maximum traveling salesman

We have also mentioned that in the opposite of min_TSP, max_TSP (certainly less popular than its cousin), although it is APX-hard ([20, 18]), can be solved by a PTAA achieving standard approximation ratio \(\rho = 5/7\) (this ratio is somewhat worst – 38/63 – when the input-graph is directed).

The purpose of this section is to show that, in the differential approximation framework, the two cousins are equi-approximable establishing so a kind of natural symmetry between the two problems at hand.

Theorem 5. max_TSP is equi-approximable with min_TSP; consequently it is in D-APX.

Proof. Observe first that, given a graph \(K_n\), there exists a very interesting symmetry between min_ and max_TSP with respect to worst-case and best objective values:

\[
\begin{align*}
\beta\min(K_n) &= \omega\max(K_n) \\
\beta\max(K_n) &= \omega\min(K_n)
\end{align*}
\]  
(18)

Expression (18) confirms what we said in the introduction of the paper that the worst value of a problem can be as hard to compute as the optimal one.

Given a complete graph \(K_n\), let us denote by \(\tilde{K}_n\) the complete graph on \(n\) vertices when one replaces distance \(d(i,j)\) by \(\tilde{d}(i,j) = M - d(i,j), i,j = 1,...,n\), for \(M = \max_{u,v\in E}d(i,j) + \min_{u,v\in E}d(i,j)\). It is easy to see that \(\tilde{K}_n = K_n\). Moreover, any TSP-feasible solution for \(K_n\) is TSP-feasible for \(\tilde{K}_n\).

Given a Hamiltonian cycle \(T\), we use notation \(T_{\min}\) (resp., \(T_{\max}\)) in order to indicate that we deal with a solution of min_TSP (resp., max_TSP). We then have

\[
\begin{align*}
|T_{\min}(K_n)| &= M n - |T_{\max}(\tilde{K}_n)| \\
|T_{\max}(K_n)| &= M n - |T_{\min}(\tilde{K}_n)|
\end{align*}
\]

and, more particularly,

\[
\begin{align*}
\omega\min(K_n) &= M n - \beta\min(\tilde{K}_n) = M n - \omega\max(\tilde{K}_n) \\
\beta\min(K_n) &= M n - \omega\min(\tilde{K}_n) = M n - \beta\max(\tilde{K}_n) \\
\lambda\min(K_n) &= M n - \lambda\max(\tilde{K}_n)
\end{align*}
\]  
(19)

(20)

(21)

By the discussion above, one can immediately conclude that for every PTAA \(A\) and for every \(K_n\), \(\delta_{\min}(K_n) = \delta_{\max}(\tilde{K}_n)\) (where, once again, indices min and max are used to denote min_TSP and max_TSP, respectively). Consequently, \(\delta_{\min} = \delta_{\max}, \forall A\). Since \(\delta_{\min} > 1/2\), the same holds for \(\delta_{\max}\) and this completes the proof of the theorem.

For \(d(i,j) \in \{a,b\}, \max_{u,v\in E}d(i,j) + \min_{u,v\in E}d(i,j) = d(i,j) \in \{a,b\} \forall u,v\in E\); so, the proof of theorem 5 establishes also equi-approximability between min_TSPab and max_TSPab and the following theorem summarizes differential approximation results for max_TSP.

Theorem 6.

- max_TSP is approximable within differential approximation ratio 1/2;
- max_TSP12 and max_TSPab are approximable within differential approximation ratio 3/4;
- for every \(\varepsilon > 0\), max_TSP cannot be approximated within differential ratio greater than, or equal to, \(5379/5380 + \varepsilon\), unless \(P = NP\).
An improvement of the standard ratio for the maximum traveling salesman with distances 1 and 2

Application of lemma 3 in the case of $\max_{\text{TSP}ab}$ with $t = (b - a)/a$ gets

$$\rho = c_{b/a}(\delta) = \frac{b - a}{b} \delta + \frac{a}{b}$$

and for $\delta = 3/4$ we have

$$\rho = c_{b/a} \left( \frac{3}{4} \right) = \frac{3}{4} + \frac{1}{4} \frac{a}{b}$$

The above ratio is always bounded below by $3/4$. Here we see another impact of the asymmetry between minimization and maximization versions of TSP in the standard approximation framework. Recall that, as we have seen in section 5.1, the standard approximation ratio for $\min_{\text{TSP}ab}$ tends to $\infty$ with $b$ and this obviously holds for every PTAS.
References


