RESOLVING INCONSISTENCIES AMONG CONSTRAINTS
ON THE PARAMETERS OF AN MCDA MODEL

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Resolving inconsistencies among constraints
on the parameters of an MCDA model

ABSTRACT: We consider a framework where Decision Makers (DMs) interactively define a multicriteria evaluation model by providing imprecise information (i.e., a linear system of constraints to the model's parameters) and by analyzing the consequences of the information provided. DMs may introduce new constraints explicitly or implicitly (results that the model should yield). If a new constraint is incompatible with the previous ones, then the system becomes inconsistent and the DMs must choose between removing the new constraint and removing some of the older ones. We address the problem of identifying subsets of constraints which, when removed, lead to a consistent system. The identification of such subsets is extremely useful because they state the reason for the inconsistent information given by DMs. There may exist several possibilities for the DMs to choose from, in order to resolve the inconsistency. We present some approaches based on mathematical programming to identify such possibilities and an application to an aggregation/disaggregation procedure for the ELECTRE TRI method.

Keywords: Inconsistent Linear Systems, Multiple Criteria Analysis, ELECTRE TRI, Aggregation/Disaggregation Approach, Imprecise Information.

Résolution des incohérences dans un système de contraintes sur les paramètres d'un modèle multicritère

RESUME: Nous considérons un contexte dans lequel le décideur définit de façon interactive un modèle d’évaluation multicritère en spécifiant une information imprécise (i.e., un système de contraintes linéaires sur paramètres du modèle) et en analysant les conséquences de l’information fournie. Le décideur peut introduire de nouvelles contraintes explicitement ou implicitement (sous la forme de résultats que le modèle doit reproduire). Si une nouvelle contrainte est incompatible avec les précédentes, le système devient incohérent et le décideur doit soit retirer la nouvelle contrainte, soit certaines parmi les anciennes. Nous nous intéressons au problème de l’identification de sous-ensembles de contraintes dont la suppression rend le système cohérent. L’identification de tels sous-ensembles est extrêmement utile car elle permet d’identifier la source de l’incohérence dans l’information fournie par le décideur. Il peut exister plusieurs sous-ensembles de ce type ; le décideur doit choisir parmi eux une manière de résoudre l’incohérence. Nous présentons des approches basées sur la programmation mathématique pour identifier ces sous-ensembles ainsi qu’une application dans le cadre d’une procédure d’agrégation-désagrégation pour la méthode ELECTRE TRI.

1. Introduction

Multicriteria decision aiding models usually have many preferential parameters that the decision makers (DMs) must set. These parameters influence, namely, the manner in which differences in performances are valued, and the role of each criterion in the aggregation of the performances. Providing precise figures for all parameter values is often difficult, to the extent that there may exist some imprecision, contradiction, arbitrariness, and/or lack of consensus concerning the value of the parameters (Roy & Bouyssou, 1989).

We will consider an imprecise information context (see, e.g., Weber, 1987; Dias & Climaco, 2000), where the DMs may indicate some constraints on the acceptable combinations of parameter values. Such information may be provided in an explicit manner (e.g., parameter $t_1$ belongs to $[0.2, 0.3]$, or parameter $t_1$ is larger than parameter $t_2$), or in an implicit manner (indicating a result that the model should restore, e.g., alternative $a_1$ should be better ranked than $a_2$). Methods that accept the latter type of constraints to infer parameter values are often called aggregation/disaggregation procedures (Jacquet-Lagrèze & Siskos, 1982; Mousseau & Slowinski, 1998; Nadeau et al., 1991).

In the course of an interactive process, DMs may progressively add constraints on the parameter values. Let $T_k$ denote the set of parameter values that are acceptable to the DMs (according to the constraints they provided) at the $k$-th iteration. Given this set, it is possible to provide some output to support the DMs in revising $T_k$:

- **robust conclusions** – the results that are valid for all the combinations $t \in T_k$ (Roy, 1998; Vincke, 1999); for instance, “$a_i$ is never contained in the choice set” in a selection problem; “$a_1$ is always better ranked than $a_2$” in a ranking problem; or “$a_1$ can only be assigned to category “good” or “very good” in a sorting problem;

- **variability information** – the results that vary more, according to the combination chosen; for instance, “the position of $a_i$ in the ranking is very unstable: for some input values it may be the best, whereas for other combinations it is one of the worst” in a ranking problem (Kämpke, 1996);

- **inferred parameter values procedures** (Jacquet-Lagrèze & Siskos, 1982; Mousseau & Slowinski, 1998) – a “central” combination $t \in T_k$ that satisfies all the constraints, hence able to restore the results that were demanded.

We consider interactive processes in which DMs start the first iteration with very little information. Each iteration will provide an opportunity to add, delete or modify a specific supplementary constraint. Adding a single piece of information at each iteration facilitates the control of the information supplied by the DMs. This interactive process stops when DMs are satisfied and the set $T_k$ as well as the results of the model match their view of the decision problem.

We will consider that all the constraints are linear ($T_k$ is a polyhedron) and that the polyhedron $T_{k+1}$ that corresponds to the next iteration is obtained by adding a single constraint, i.e., by intersecting $T_k$ with a half-space or a hyperplane. A difficulty occurs when $T_{k+1}$ becomes empty, meaning that the new constraint contradicts some of the previous ones. To resolve the inconsistency that appeared in the linear system of constraints, one must either drop the new constraint, or some of the older ones. The choice should belong to the DMs, after they learn which are the sets of constraints that lead to a non-empty $T_{k+1}$ if removed. Notice that when we refer to the removal of one or more constraints, the DMs may choose to relax these constraints instead (e.g., increasing the right-hand side of an $Ax \leq b$ system).

Many authors have previously addressed the subject of infeasibility analysis in linear programming (see, J.W. Chinneck, (1997) for a complete summary of the state of the art in infeasibility analysis algorithms) according to different perspectives, namely:
1. Some authors (Loon, 1981; Chinneck, 1994; Tamiz et al., 1996) are interested in determining an Irreducibly Inconsistent System (IIS). An IIS is a subset of constraints that corresponds to an inconsistent system, which is minimal, in the sense that any proper subset of an IIS is a consistent system. Let us remark that the inconsistency in an IIS can be removed by deleting any constraint, but if there are other IISs, then the initial system of constraints can remain inconsistent.

2. A different problem is to determine the minimum number of constraints that has to be removed to restore the consistency in the initial system, which is equivalent to solve the minimum-cardinality IIS set-covering problem (Chinneck, 1996; Murty et al., 2000).

3. Finally, we can mention the problem of determining the minimum weight (or cost) alternative to restore the consistency in a system, which is equivalent to determine a minimum-weight IIS set-cover (Chinneck, 1996; Murty et al., 2000).

The perspective we are interested in is close to problem 2, with the following specificities:

- we are also interested in sets of constraints that restore the consistency if removed that are not of minimum cardinality, since the DMs may rather drop two constraints they consider unimportant than drop a single important one;
- we know that one of the constraints caused the inconsistency, hence removing that constraint is a trivial manner to resolve the inconsistency; the question here is what other alternatives exist.

Hence, we may formulate the problem we are addressing as: to determine the p "smallest" sets of constraints (in terms of cardinality) that, if removed, restore the consistency to the initial system.

In the context of the interactive processes we are considering, solving such problems will allow us to propose alternative ways to resolve an inconsistency that appeared at a given iteration. This helps the DMs to understand how their inputs are conflicting and to question previously expressed judgments. Analyzing and confronting the alternative solutions of such problems provide opportunities for the DMs to learn about their preferences as the interactive process evolves.

In the next section we define our problem formally and propose two techniques to solve it. One of the techniques consists in solving a succession of mixed-integer linear programs, while the second one uses only linear programming. Section 3 presents an application to the aggregation/disaggregation approach for ELECTRE TRI, reviewing this approach and including a numerical example. Section 4 indicates some extensions and concludes the paper.

2. Two different methods to cope with inconsistent systems

Consider a problem in which the DM has interactively specified constraints on the parameters by defining a polyhedron of acceptable values denoted by $T_{k-1}$ (at iteration $k-1$). This polyhedron is defined by the following (general) consistent system of $m-1$ linear constraints on $n$ variables $x_1, ..., x_n$:
\[
\begin{align*}
\sum_{j=1}^{n} \alpha_{1j} x_j & \geq \beta_1 \\
\vdots \\
\sum_{j=1}^{n} \alpha_{(m-1)j} x_j & \geq \beta_{(m-1)} \\
\end{align*}
\]

(\alpha_{1, \ldots, m}, \alpha_{(m-1)n} \in \mathbb{R}, \beta_1, \ldots, \beta_{(m-1)} \in \mathbb{R}) \quad (1)

Let \( \sum_{j=1}^{n} \alpha_{mj} x_j \geq \beta_m \) be a new constraint that, when added to the system (1), originates a system (2) of \( m \) linear constraints, which is now inconsistent:

\[
\begin{align*}
\sum_{j=1}^{n} \alpha_{1j} x_j & \geq \beta_1 \\
\vdots \\
\sum_{j=1}^{n} \alpha_{mj} x_j & \geq \beta_m \\
\sum_{j=1}^{n} \alpha_{(m-1)j} x_j & \geq \beta_{(m-1)} \\
\end{align*}
\]

(\alpha_{1, \ldots, m} \in \mathbb{R}, \beta_1, \ldots, \beta_m \in \mathbb{R}) \quad (2)

Let \( I = \{1, \ldots, m\} \) be the set of indices of the constraints defining \( T_k \) (at iteration \( k \), i.e., with the new constraint that makes \( T_k \) empty). Hence,

\[
T_k = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \alpha_{ij} x_j \geq \beta_i, \forall i \in I \right\} = \emptyset
\]

Let \( S \subseteq I \) denote a subset of indices of constraints. We will say that \( S \) resolves (2) if and only if the system \( \sum_{j=1}^{n} \alpha_{ij} x_j \geq \beta_i, \forall i \in I \setminus S \), is consistent. Let \( |S| \) denote the cardinality of the set \( S \). Formally, the problem we are addressing is to determine \( p \) distinct sets \( S_1, \ldots, S_p \) (if they exist) such that:

i. \( S_i \) resolves (2), \( i \in \{1, \ldots, p\} \);

ii. \( S_i \not\subset S_j, i, j \in \{1, \ldots, p\}, i \neq j \);

iii. \( |S_i| \leq |S_j|, i, j \in \{1, \ldots, p\}, i < j \);

iv. If there exists a set \( S \) such that \( S \) resolves (2) and \( S \not\subset S_i (i \in \{1, \ldots, p\}) \), then \( |S| \geq |S_p| \).

Since we already know that the system (1) is consistent and the system (2) is inconsistent,
A similar approach can be found in Kim & Ahn (1999): this is done through $p$-1 successive optimizations (PM$_2$, PM$_3$, ..., PM$_n$).

The program PM$_2$ minimizes the number of constraints to be removed in order to make $T_k$ feasible. The subset $S_i$={$m$} is obviously the smallest subset verifying (i) to (iv). The first problem PM$_2$ to be solved has the following form:

$$\begin{align*}
\text{PM}_2: & \\
\text{min} & \sum_{i=1}^{m-1} y_i \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j + M y_i \geq \beta_i, \text{ for } i \in I \setminus \{m\} \\
& \sum_{j=1}^{n} a_{ij} x_j \geq \beta_i, \text{ for } i = m \\
& x_j \geq 0 \quad j = 1,...,m, \quad y_i \in \{0,1\} \quad i = 1,...,m-1
\end{align*}$$

where $M$ is a positive large number. The variables $y_i, i=1, ..., m-1,$ are binary variables intended to work as a "switch". The indices of constraints for which $y_i=1$ are dropped from the system.
Figure 1: Feasible set.

We then build the following PM model:

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{7} y_i \\
\text{s.t.:} & \quad -0.05x_1 - x_2 \geq -40 - My_1 \quad [1] \\
& \quad -0.5x_1 - x_2 \geq -50 - My_2 \quad [2] \\
& \quad -1.2x_1 - x_2 \geq -70 - My_3 \quad [3] \\
& \quad -4.5x_1 - x_2 \geq -179 - My_4 \quad [4] \\
& \quad -x_1 \geq -35 - My_5 \quad [5] \\
& \quad -1.3x_1 + x_2 \geq -30 - My_6 \quad [6] \\
& \quad -0.75x_1 + x_2 \geq -14 - My_7 \quad [7] \\
& \quad 0.6x_1 + x_2 \geq 55 \quad [8]
\end{align*}
\]

where \( y_i \in \{0,1\} \) for \( i = 1, \ldots, 7 \).

Let us compute, step by step, all the feasible solutions for the above example as follows,

0. \( S_1=\{8\} \) (trivial solution).

1. In the first stage, we obtain \( S_2=\{1,2\} \), i.e., \( y_1^* = y_2^* = 1 \). The constraint \( y_1 + y_2 \leq I \) is then added to the constraint set.

2. In the second stage, after optimizing PM the solution is \( S_3=\{2,3\} \), i.e., \( y_2^* = y_3^* = 1 \). We add the constraint \( y_2 + y_3 \leq I \) to the model.

3. In the third stage, we obtain \( S_4=\{3,4,5,6\} \), i.e., \( y_3^* = y_4^* = y_5^* = y_6^* = 1 \). We add the constraint \( y_3 + y_4 + y_5 + y_6 \leq 3 \).
4. The problem becomes infeasible, meaning that there are no more alternatives to solve our problem.

2.2. An algorithm to propose solutions for inconsistency using LP

In this section we propose a second algorithm to solve the problem we are addressing, i.e., to find the sets $S_2, ..., S_p$ (since we are considering $S_1=\{m\}$). The algorithm is based on the results presented below:

**Proposition 1.** Let $S \subseteq \{1, ..., m-1\}$ be a set of indices of constraints. Let $LP(S)$ denote the linear program to maximize $\sum_{j=1}^{n} a_{mj} x_j$, subject to constraints of system (1), excluding the constraints in $S$:

$$LP(S): \quad \max \left\{ \sum_{j=1}^{n} a_{mj} x_j : \sum_{j=1}^{n} a_{ij} x_j \geq \beta_i, \forall i \in I \setminus (S \cup \{m\}) \right\}$$

Then,

a) $LP(S)$ is always feasible, and

b) $S$ resolves (2) $\iff$ $LP(S)$ is unbounded or its optimal value is not less than $\beta_m$.

**Proof:**

a) Since the system (1) is consistent, it remains consistent after removing some of its constraints; hence, the linear program is feasible.

b) When $LP(S)$ is unbounded or when the optimal value of $LP(S)$ is greater than, or

<table>
<thead>
<tr>
<th>Constraints except the ones in $S$ are satisfied, including</th>
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\[ \max \left\{ \sum_{j=1}^{n} \alpha_{mj} x_j : \sum_{j=1}^{n} \alpha_{yj} x_j \geq \beta_i, \forall i \in B(S) \right\} \]

otherwise it would not be optimal to \( \text{LP}(S) \).
During the while loop, at a given moment, the candidates of cardinality \( k \) \((1 \leq k < |R|)\) will start to appear at the head of the Candidates FIFO. If one of these elements \( S \) is such that \( S \subseteq \mathcal{R} \), then solving \( \text{LP}(S) \) yields an optimal value which is less than \( \beta_m \). From Proposition 2, \( B(S) \cap \mathcal{R} \neq \emptyset \), i.e., there exists an element \( s(j) \) that belongs to \( \mathcal{R} \) and does not belong to \( S \). This element is appended to \( S \) to constitute a set \( \{s(1), \ldots, s(j)\} \subseteq \mathcal{R} \) that enters the FIFO Candidates.

Some iterations later, the candidates of cardinality \(|\mathcal{R}|\) will start to appear at the head of the Candidates FIFO. One of these candidates is \( \mathcal{R} \), which will be declared a solution since \( \text{LP}(\mathcal{R}) \) will either be unbounded, or have an optimal value that is not less than \( \beta_m \).

**On the computer implementation of the algorithm**

One way to implement this algorithm is to solve the linear program before the while loop, and then another linear program for each iteration. An alternative way is to solve the initial \( \text{LP}(\emptyset) \) and to save the simplex tableau corresponding to the removal of each constraint put in Candidates. In the while loop, solving \( \text{LP}(S) \) will amount to perform a single simplex iteration.
The algorithms provide the information (and the corresponding constraints) to remove in order to retrieve consistency. In this case, the DM should choose to delete one of the three following constraints: [6], [7] or [10].

4. Conclusion and further research

In this paper, we have proposed two alternative algorithms to deal with inconsistencies among constraints on the parameters of a MCDA model. The inconsistencies considered here correspond to situations in which the DM specifies a list of linear constraints on preferential parameters value that originate an empty polyhedron. More specifically, these algorithms allow us to compute subsets of constraints that, when removed, yield a non-empty polyhedron of acceptable values for preferential parameters.

The algorithms presented in section 2 are particularly useful within the context of preference elicitation through an aggregation/disaggregation process. Section 3 described and illustrated how these algorithms can be used when inferring the weights in the Electre Tri method from assignment examples.

The results presented in this paper suggest further research. First, it is obvious that the proposed algorithms can be used to solve inconsistencies on preferential parameters in various aggregation models.

Second, if some ordinal confidence index is attached to each constraint provided by the DM, it might be interesting to find the "smallest" subsets of constraints in which the DM has the least confidence. This problem leads to complex ordinal optimization programs.

Third, the proposed algorithms are specifically designed for the case in which the last constraint added causes infeasibility. Further research should be performed so as to figure out whether a variation of the algorithms could be used in the general case in which a set of constraints which is infeasible is given to begin with.

Lastly, we consider the reduction of inconsistencies through the deletion of subsets of constraints. It might be very interesting to try to relax some constraints (rather than to delete them) in order to restore consistency.

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References


