DIFFERENTIAL APPROXIMATION FOR SATISFIABILITY
AND RELATED PROBLEMS

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Table of Contents

Résumé ii
Abstract ii

1 Introduction 1

2 Preliminaries 2

3 Satisfiability problems 3
   3.1 Approximation preserving reductions for optimal satisfiability 3
   3.2 MIN SAT and MIN VERTEX COVER 4
   3.3 A positive differential approximation result for MAX NAE 3SAT 4

4 Optimal satisfiability and MIN INDEPENDENT DOMINATING SET 5

5 MIN INDEPENDENT DOMINATING SET 5

6 Discussion 7

References 8
1 Introduction

In this paper we deal with the approximation of classical optimal satisfiability problems as MAX and MIN SAT, MAX and MIN DNF, as well as of restrictive versions of these problems as the ones where the size of any clause is bounded, or/and the number of the occurrences of any literal is bounded. We also deal with some graph-problems as MAX and MIN INDEPENDENT DOMINATING SET and MAX and MIN MINIMAL VERTEX COVER. We study the approximability of all these problems using the so-called differential approximation ratio which, informally, for an instance $I$ measures the relative position of the value of an approximated solution in the interval [worst-value feasible solution of $I$, optimal-value solution of $I$].

All these problems have no polynomial time approximation schemata for the standard approximation (where one measures the ratio between the value of the approximate solution of an instance and the value of an optimal one). The SAT problems admit algorithms achieving constant standard approximation ratio, while algorithms for the DNF ones do not guarantee such ratios (more details about the standard approximability of all these problems can be found in [CK]). The MIN VERTEX COVER (called MIN MINIMAL VERTEX COVER in this paper) is standard 2-approximable, while the MAX INDEPENDENT SET (called MAX INDEPENDENT DOMINATING SET in the paper) cannot be approximated within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $\text{co-RP} = \text{NP}$ ([Has96]). On the other hand, MIN INDEPENDENT DOMINATING SET is standard approximable within $B/2$ where $B$ is the maximum graph-degree ([Kann92]), while the MAX MINIMAL VERTEX COVER has, to our knowledge, not been studied yet in the standard approximation.

The initial objective of the paper was to study the differential approximability of the optimal satisfiability problems defined above. This study has brought to the fore an interesting relationship between MIN kSAT and MIN MINIMAL VERTEX COVER which can be informally described as follows: any differential approximation algorithm for MIN MINIMAL VERTEX COVER can be transformed into a differential approximation algorithm for MIN kSAT achieving the same differential performance ratio. On the other hand, as we will see just below, MAX MINIMAL VERTEX COVER is equivalent, for the differential approximation, to the well-known MIN INDEPENDENT DOMINATING SET. We are so led to study differential approximation results for MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET.

All the problems we deal with in this paper have the characteristic that computation of both their optimal and worst solutions is NP-hard (for example, considering an instance $\varphi$ of MAX kSAT, its worst solution is an assignment satisfying the minimum number of the clauses of $\varphi$, i.e., an optimal solution for MIN kSAT on $\varphi$). Remark also that, given a graph $G = (V, E)$, the complement, with respect to $V$ of a minimal vertex cover (resp., maximal independent set) is a maximal independent set (resp., minimal vertex cover) of $G$. In other words, the objective values of MIN (MAX) MINIMAL VERTEX COVER and of MIN (MAX) INDEPENDENT DOMINATING SET are linked by affine transformations. On the other hand, the differential approximation ratio is stable for the affine transformation, in the sense that pairs of problems, the objective values of which are linked by affine transformations, are differential equivalent. Hence the following fact holds: MIN (MAX) MINIMAL VERTEX COVER and MAX (MIN) INDEPENDENT DOMINATING SET are differential equivalent.

In what follows, we first study differential approximation preserving reductions for several optimal satisfiability problems. Combining them with a general result linking approximability of maximization problems in differential and standard approximations, we obtain interesting differential inapproximability results for optimal satisfiability. We also prove that MIN kSAT($B, \overline{B}$) and MAX kSAT($B, \overline{B}$) reduce to MIN MINIMAL VERTEX COVER-$B'$ and MAX MINIMAL VERTEX COVER-$B'$, respectively. These reductions lead us to study the differential approx-
imination of MIN INDEPENDENT DOMINATING SET. For this problem we prove a strong inappro-
imation result, informally, unless P = NP, any approximation algorithm has worst-case
approximation ratio equal to 0. To our knowledge, no such result was previously known for the
differential approximation.

2 Preliminaries

We first recall a few definitions about differential and standard approximabilities. Given an
instance \( x \) of an optimization problem and a feasible solution \( y \) of \( x \), we denote by \( m(x, y) \) the
value of the solution \( y \), by \( \text{opt}(x) \) the value of an optimal solution of \( x \), and by \( \omega(x) \) the value of
a worst solution of \( x \). The standard performance, or approximation, ratio of \( y \) is defined as

\[
\tau(x, y) = \max \left\{ \frac{m(x, y)}{\text{opt}(x)}, \frac{\text{opt}(x)}{m(x, y)} \right\}
\]

while the differential performance, or approximation, ratio of \( y \) is defined as

\[
\rho(x, y) = \frac{m(x, y) - \omega(x)}{\text{opt}(x) - \omega(x)}.
\]

It is easy to see that the differential approximation ratio is stable for the affine transformation
of the objective function of a problem, while this does not hold for the standard approximation
ratio.

For a function \( f, f(n) > 1 \), an algorithm is a standard \( f(n) \)-approximation algorithm for a
problem \( P \) if, for any instance \( x \) of \( P \), it returns a solution \( y \) such that \( \tau(x, y) \leq f(|x|) \), where \(|x|\)
is the size of \( x \). We say that an optimization problem is standard constantly approximable if,
for some constant \( c > 1 \), there exists a polynomial time standard \( c \)-approximation algorithm
for it. An optimization problem has a standard polynomial time approximation schema if it
for it. An optimization problem has a standard polynomial time approximation schema if it
Max Sat($B, B$), Max DNF($B, B$), Min Sat($B, B$), Min DNF($B, B$) are the versions of these problems where each literal appears at most $B$ times.

Max NAE 3SAT

**Input:** a set of conjunctions $C_1, \ldots, C_m$ of three literals on $n$ variables $x_1, \ldots, x_n$.

**Output:** a truth assignment to the variables that maximizes the number of conjunctions satisfied in such a way that any one of them has at least one true literal and at least one false literal.

Min (Max) Minimal Vertex Cover

**Input:** a graph $G = (V, E)$.

**Output:** a minimal vertex cover (a set $S \subseteq V$ such that, $\forall (u, v) \in E, u \in S$ or $v \in S$) of minimum (maximum) size.

Min (Max) Independent Dominating Set

**Input:** a graph $G = (V, E)$.

**Output:** a maximal independent set (a set $S \subseteq V$ such that, $\forall u, v \in S, (u, v) \notin E$ and $\forall u \notin S, \exists v \in S, (u, v) \in E$) of minimum (maximum) size.

In what follows, we denote by Min (Max) Independent Dominating Set-$B$ and Min (Max) Minimal Vertex Cover-$B$ the versions of the above problems on graphs with maximum degree bounded by $B$.

3 Satisfiability problems

3.1 Approximation preserving reductions for optimal satisfiability

We first prove the differential equivalence for Max Sat and Min DNF and for Min Sat and Max DNF.

**Theorem 1.** Max Sat and Min DNF, as well as Min Sat and Max DNF are differential equivalent.

**Proof.** We construct a reduction from Max Sat to Min DNF that preserves the differential approximation ratio. Let $I$ be an instance of Max Sat on $n$ variables and $m$ clauses. The instance $I'$ of Min DNF contains $m$ clauses and the same set of $n$ variables. With each clause $\ell_1 \lor \ldots \lor \ell_k$ of $I$ we associate in $I'$ the conjunction $\ell_1 \wedge \ldots \wedge \ell_k$, where $\ell_i = \bar{x}_j$ if $\ell_i = x_j$ and $\ell_i = x_j$ if $\ell_i = \bar{x}_j$. It is easy to see that $\text{opt}(I') = m - \text{opt}(I)$ and $\omega(I') = m - \omega(I)$. Also, if $m(I', y)$ is the value of the solution $y$ in $I'$, then the same solution $y$ has in $I$ the value $m(I, y) = m - m(I', y)$. Thus, $\rho(I, y) = \rho(I', y)$. The reduction from Min DNF to Max Sat is the same.

By an exactly similar reduction, one can prove that Min Sat and Max DNF are also approximate equivalent.

By the proof of theorem 1 one easily can deduce that for each constant $k \geq 2$, Max $k$SAT and Min $k$DNF as well as Min $k$SAT and Max $k$DNF are differential equivalent.

Consider an instance $I$ of a maximization problem $\Pi$, an approximation algorithm $A$ for $\Pi$ and denote by $S$ a feasible solution of $\Pi$ computed by $A$ in $I$. Then,

$$\frac{m_A(I, S) - \omega(I)}{\text{opt}(I) - \omega(I)} \geq \delta \Rightarrow \frac{m_A(I, S)}{\text{opt}(I)} \geq \delta + (1 - \delta)\omega(I) \frac{\omega(I)}{\beta(I)} \geq \frac{m_A(I, S)}{\beta(I)} \geq \delta$$

and the following proposition immediately holds.

**Proposition 1.** Approximation of a maximization problem $\Pi$ within differential approximation ratio $\delta$, implies approximation of $\Pi$ within standard approximation ratio $1/\delta$.

Combining the results of theorem 1 and proposition 1 with the fact that for $k \geq 2$ and $B \geq 3$, Max $k$SAT($B, B$) and Max $k$DNF($B, B$) have no standard polynomial time approximation schemata ([PY91]), one deduces the following.
Corollary 1. For \( k \geq 2 \) and \( B \geq 3 \), \( \text{MAX } k\text{SAT}(B, \bar{B}) \), \( \text{MAX } k\text{DNF}(B, \bar{B}) \), \( \text{MIN } k\text{SAT}(B, \bar{B}) \) and \( \text{MIN } k\text{DNF}(B, \bar{B}) \) have no differential polynomial time approximation schema, unless \( P = NP \).

3.2 \textbf{MIN SAT and MIN VERTEX COVER}

\textbf{MIN VERTEX COVER} is as the \textbf{MIN MINIMAL VERTEX COVER} defined in section 2 modulo the fact that the feasible solutions for the former are not mandatorily minimal. We show that the reduction used in [CST96] from \textbf{MIN VERTEX COVER} to \textbf{MIN SAT} is also differential approximation preserving. This will allow us to establish an inapproximability result for \textbf{MIN SAT}.

\textbf{Theorem 2.} Unless \( \text{co-RP} = \text{NP} \), \textbf{MIN SAT} is not differential \( 1/m^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), where \( m \) is the number of clauses of the instance.

\textbf{Proof.} The reduction of [CST96] from \textbf{MIN VERTEX COVER} to \textbf{MIN SAT} works as follows. Let \( G = (V, E) \) be a graph on \( n \) vertices and denote by \( V = \{1, \ldots, n\} \) its vertex set. In order to construct an instance \( I \) of \textbf{MIN SAT}, at each edge \((i, j) \in E \), \( i < j \) we associate a variable \( x_{ij} \).

For each vertex \( i \) we define a clause \( C_i \), where

\[
C_i = \bigvee_{j: (i,j) \in E \wedge i < j} x_{ij} \lor \bigvee_{j: (i,j) \in E \wedge j > i} \bar{x}_{ij}.
\]

From a vertex cover \( C \) of \( G \) we define an assignment as follows. For each \( i \notin C \) and each \((i, j) \in E \), \( x_{ij} = 1 \) if \( i > j \) and \( x_{ij} = 0 \) if \( i < j \). Since \( C \) is a vertex cover, this definition is not contradictory. If \( i \notin C \), then \( C_i \) is not satisfied and so \( \text{opt}(I) \leq \text{opt}(G) \).

Given an assignment \( v \) of \( I \), let \( C = \{i : C_i \text{ is satisfied}\} \). Note that set \( C \) is a vertex cover since for \((i, j) \in E \), at least one of \( C_i \) and \( C_j \) is satisfied and so at least one of the vertices \( i, j \) appears in \( C \). So, at each assignment \( v \) of \( I \), we associate in \( G \) a vertex cover \( C \) with \( m(G, C) = m(I, v) \). This also proves that \( \text{opt}(I) = \text{opt}(G) \).

Finally, using \( \omega(I) \leq \omega(G) \), it is easy to show that \( \rho(G) \geq \rho(I) \).

We have seen that \textbf{MIN VERTEX COVER} is differential equivalent to \textbf{MAX INDEPENDENT SET} (which is as \textbf{MAX INDEPENDENT DOMINATING SET} modulo the fact that the independent set to compute has not to be minimal). On the other hand since the worst solution for \textbf{MAX INDEPENDENT SET} is the empty set (in other words, \( \omega(I) = 0 \), \( \forall I \)), standard and differential approximation ratios coincide. Furthermore, \textbf{MAX INDEPENDENT SET} is not differential \( 1/n^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), unless \( \text{co-RP} = \text{NP} \) ([Has90]). Consequently, \textbf{MIN VERTEX COVER} is not differential \( 1/n^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \) unless \( \text{co-RP} = \text{NP} \) and the
Proof. Consider a satisfiable instance \( \varphi \) of MAX NAE 3SAT defined on \( m \) clauses; obviously, \( \text{opt}(\varphi) = m \). Run the standard 1.096-approximation algorithm of [Zwick98] on \( \varphi \) to obtain a solution \( C \) satisfying \( m(\varphi, C) \geq m/1.096 \). On the other hand any random assignment by values in \( \{0, 1\} \) of the variables of \( \varphi \), where any of the two values is assigned with probability 1/2, will feasibly satisfy 3m/4 clauses (in other words, the assignments (1, 1, 1) and (0, 0, 0) are to be excluded from the eight possible assignments for each 3-clause); consequently, \( \omega(\varphi) \leq 3m/4 \).

Using the values for \( \text{opt}(\varphi) \), \( m(\varphi, C) \) and \( \omega(\varphi) \), we get

\[
\frac{m(\varphi, C) - \omega(\varphi)}{\text{opt}(\varphi) - \omega(\varphi)} \geq \frac{\frac{m}{1.096} - \frac{3}{4}m}{m - \frac{3}{4}m} = \frac{0.712}{1.096} = 0.64963504.
\]

On the other hand, using proposition 1 and the result of [Zwick98] that MAX NAE 3SAT is not standard approximable within 1.090, unless \( P=NP \), the following is deduced.

**Proposition 2.** \textit{MAX NAE 3SAT is not differential 0.917-approximable.}

4 \textbf{Optimal satisfiability and MIN INDEPENDENT DOMINATING SET}

The reduction of [CST96] from MIN SAT to MIN VERTEX COVER is standard approximation preserving but not differential one. However, if one considers MIN MINIMAL VERTEX COVER instead of MIN VERTEX COVER, then this reduction can work also for the differential approximation.

**Theorem 4.** \textit{MIN kSat}(B, \bar{B}) is differential reducible to MIN MINIMAL VERTEX COVER-B'.

**Proof.** Let \( I \) be an instance of \( \text{MIN kSat}(B, \bar{B}) \) with \( n \) variables and \( m \) clauses. In the instance \( G \) of MIN MINIMAL VERTEX COVER, with each clause \( C_i \) of \( I \) we associate a vertex \( i \). We draw an edge between \( i \) and \( j \) if there is a variable \( x \) such that \( C_i \) contains \( x \) and \( C_j \) contains \( \bar{x} \). The vertex-degrees of the so constructed graph are bounded above by \( B' = kB \).

From an assignment \( u \) of \( I \) we define a vertex cover \( C \) as the set of vertices that correspond to clauses satisfied by \( u \). So, \( \text{opt}(G) \leq \text{opt}(I) \).

From a vertex cover \( C \) of \( G \) we define a partial assignment \( v \) as follows: if \( i \notin C \) and \( x_j \in C_i \) then \( x_j = 0 \), and if \( i \notin C \) and \( \bar{x}_j \in C_i \) then \( x_j = 1 \). Hence, if \( i \notin C \) then \( C_i \) is not satisfied by \( v \). By the way \( v \) has been defined, the number of the non satisfied clauses in \( I \) is greater than, or equal to, the number of vertices that are not in \( C \), i.e., \( m(I, v) \leq m(G, C) \). This, together with \( \text{opt}(G) \leq \text{opt}(I) \) proved just above, implies \( \text{opt}(G) = \text{opt}(I) \).

If \( C \) is a minimal vertex cover (for each \( i \in C \) there exists \( j \notin C \) such that \((i, j) \in E \)), then \( m(I, v) = m(G, C) \) since the clause \( C_i \) is satisfied by \( v \) when \( i \in C \). Consequently, in particular, \( \omega(I) = \omega(G) \) and this concludes the proof of the theorem. 

By a proof similar to the one of theorem 2, one can show that MAX kSat(B, \bar{B}) reduces to MAX MINIMAL VERTEX COVER-B'. Since the former is differential equivalent to MIN INDEPENDENT DOMINATING SET-B' the following theorem concludes the section.

**Theorem 5.** MAX kSat(B, \bar{B}) is differential reducible to MIN INDEPENDENT DOMINATING SET-B'.

5 \textbf{MIN INDEPENDENT DOMINATING SET}

The results of section 4 naturally bring us to study the differential approximation of MIN INDEPENDENT DOMINATING SET. The reduction devised in the following theorem is inspired from [Irvc91].
Theorem 6. If $P \neq NP$, then, for any $\delta(n) \in (0, 1)$, ($\delta$ decreasing in $n$), MIN INDEPENDENT DOMINATING SET on graphs of order $n$ is not differential $\delta(n)$-approximable.

Proof. We show that, for any $\delta(n) \in (0, 1)$, a polynomial time differential $\delta(n)$-approximation algorithm $A$ for MIN INDEPENDENT DOMINATING SET, could distinguish in polynomial time if an instance of SAT on $n$ variables is satisfiable or not.

Given an instance $\varphi$ of SAT with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$ we construct a graph $G$, instance of MIN INDEPENDENT DOMINATING SET as follows. With each positive literal $x_i$ we associate a vertex $u_i$ and for each negative literal $\overline{x}_i$ we associate a vertex $v_i$. For $i = 1, \ldots, n$ we draw edges $u_iu_j$. For any clause $C_j$ we add in $G$ a vertex $w_j$ and an edge between $w_j$ and each vertex corresponding to a literal contained in $C_j$. Finally, we add edges in $G$ in order to obtain a complete graph on $w_1, \ldots, w_m$.

Remark that an independent set of $G$ contains at most $n + 1$ vertices since it contains at most one vertex among $w_1, \ldots, w_m$ and at most one vertex among $u_i$ and $v_j$ for $i = 1, \ldots, n$. An independent dominating set containing the vertices corresponding to true literals of a non satisfiable assignment and one vertex corresponding to a clause not satisfied by this assignment is a worst solution of $G$ of size $n + 1$.

If $\varphi$ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying $\varphi$ is an independent dominating set (each vertex $w_j$ is dominated by a vertex corresponding to a true literal of $C_j$) of minimum size. On the other hand, if $\varphi$ is not satisfiable then $\text{opt}(G) = n + 1$.

In fact any independent dominating set of $G$ has cardinality either $n$, or $n + 1$. Hence, if $A$ computes a solution of value $n$ then $\varphi$ is satisfiable, otherwise $\varphi$ is not satisfiable.

An interesting consequence of theorem 6 above is that unless $P = NP$, any polynomial time approximation algorithm for MIN INDEPENDENT DOMINATING SET has worst-case differential approximation ratio equal to 0. This makes MIN INDEPENDENT DOMINATING SET one of the hardest problems for the differential approximation. Let us note that, to our knowledge, no problem verifying a statement as the one of theorem 6 were known until now for the differential approximation.

Consider the refinement, due to Arora et al. [Aro92], of Cook’s theorem on the NP-hardness of 3SAT.

Theorem 7. ([Aro92]) Let $L$ be a language in NP. There exists a polynomial-time algorithm and a constant $0 < \varepsilon < 1$ such that, given any input $x$, the algorithm constructs an instance $\varphi_x$ of 3SAT which satisfies the following properties:

1. if $x \in L$, then $\varphi_x$ is satisfiable;
2. if $x \notin L$, then no assignment satisfies more than a fraction $(1-\varepsilon)$ of the clauses.

Using now the $L$-reduction of [PY91] from MAX 3SAT to MAX 3SAT($4, \overline{4}$), and observing that satisfiable instances are mapped into satisfiable instances, the above result holds also if we replace 3SAT with 3SAT($4, \overline{4}$) and $\varepsilon$ with some constant $\varepsilon'$.

Theorem 8. MIN INDEPENDENT DOMINATING SET-B is not differential $f(B)$-approximable, for $f(B) = 1 - (2\varepsilon'(B - 5)/(2B - 5))$, unless $P = NP$.

Proof. We show that if MIN INDEPENDENT DOMINATING SET-B was differential $f(B)$-approximable, then we could distinguish in polynomial time if an instance of MAX 3SAT($4, \overline{4}$) is satisfiable or at most a fraction $(1-\varepsilon')$ of the clauses are satisfied.

Given an instance $\varphi$ of 3SAT($4, \overline{4}$) with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$, we construct a graph $G$, instance of MIN INDEPENDENT DOMINATING SET-B, as follows. With
each positive literal $x_i$ we associate a vertex $u_i$, and with each negative literal $\overline{x}_i$ we associate a vertex $v_i$. For $i = 1, \ldots, n$ we draw in $G$ the edge $u_i v_i$. Also with each clause $C_j$ we associate $c = \lceil (B - 1)/4 \rceil$ vertices $w_{j1}, \ldots, w_{jc}$. For each clause $C_j$ we add in $G$ an edge between each $w_{jk}, k = 1, \ldots, c$ and any vertex corresponding to a literal contained in $C_j$.

Suppose that each literal appears at least once. Remark that an independent set of $G$ contains at most $m \cdot c$ vertices. An independent dominating set containing the vertices corresponding to the $m$ clauses of $\varphi$ is a worst solution of size $m \cdot c$.

If $\varphi$ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying $\varphi$ is an independent dominating set (each vertex $w_{jk}$ is dominated by a vertex corresponding to a true literal of $C_j$) of minimum size. On the other hand, if the