DIFFERENTIAL APPROXIMATION FOR SATISFIABILITY AND RELATED PROBLEMS

CAHIER N° 180
avril 2001

Vangelis PASCHOS ¹
Cristina BAZGAN ¹

received: January 2001.

¹ LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France ((paschos,bazgan)@lamsade.dauphine.fr).
Table of Contents

Résumé ......................................................... ii
Abstract ....................................................... ii

1 Introduction .................................................. 1

2 Preliminaries .................................................. 2

3 Satisfiability problems ......................................... 3
   3.1 Approximation preserving reductions for optimal satisfiability ................. 3
   3.2 MIN SAT and MIN VERTEX COVER ............................................. 4
   3.3 A positive differential approximation result for MAX NAE 3SAT .................. 4

4 Optimal satisfiability and MIN INDEPENDENT DOMINATING SET ................... 5

5 MIN INDEPENDENT DOMINATING SET .......................... 5

6 Discussion ................................................... 7

References ...................................................... 8
Approximation différentielle pour le problème de la satisfaisabilité et autres problèmes connexes

Résumé

Nous étudions l'approximabilité différentielle de divers problèmes de satisfaisabilité optimale. Nous démontrons que, sauf si \( \text{co-RP} = \text{NP} \), MIN SAT n'est pas approximable à rapport différentiel \( 1/m^{1-\varepsilon} \), pour tout \( \varepsilon > 0 \), où \( m \) est le nombre des clauses de la formule. En mettant en évidence que chaque algorithme d'approximation différentielle pour le problème de MAX MINIMAL VERTEX COVER peut être transformé en un algorithme d'approximation différentielle pour le problème de MIN \( k \)SAT garantisant le même rapport de performance, nous sommes amenés à étudier l'approximabilité différentielle des problèmes MAX MINIMAL VERTEX COVER et MIN INDEPENDENT DOMINATING SET ; tous les deux sont équivalents pour l'approximation différentielle. Pour ces problèmes, nous montrons un résultat fort d'inapproximabilité, informellement, si \( P \neq \text{NP} \), alors tout algorithme d'approximation différentielle a un rapport d'approximation égal à 0.

Mots-clé : optimisation combinatoire, théorie de la complexité, satisfaisabilité, approximation.

Differential approximation for satisfiability and related problems

Abstract

We study the differential approximability of several optimal satisfiability problems. We prove that, unless \( \text{co-RP} = \text{NP} \), MIN SAT is not differential \( 1/m^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), where \( m \) is the number of clauses. Brought to the fore that any differential approximation algorithm for MAX MINIMAL VERTEX COVER can be transformed into a differential approximation algorithm for MIN \( k \)SAT achieving the same differential performance ratio, we are lead to study the differential approximability of MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET. Both of them are equivalent for the differential approximation. For these problems we prove a strong inapproximability result, informally, unless \( P = \text{NP} \), any approximation algorithm has worst-case approximation ratio equal to 0.

Keywords: combinatorial optimization, complexity theory, satisfiability, approximation.
1 Introduction

In this paper we deal with the approximation of classical optimal satisfiability problems as MAX and MIN SAT, MAX and MIN DNF, as well as of restrictive versions of these problems as the ones where the size of any clause is bounded, or/and the number of the occurrences of any literal is bounded. We also deal with some graph-problems as MAX and MIN INDEPENDENT DOMINATING SET and MAX and MIN MINIMAL VERTEX COVER. We study the approximability of all these problems using the so-called differential approximation ratio which, informally, for an instance $I$ measures the relative position of the value of an approximated solution in the interval [worst-value feasible solution of $I$, optimal-value solution of $I$].

All these problems have no polynomial time approximation schemata for the standard approximation (where one measures the ratio between the value of the approximate solution of an instance and the value of an optimal one). The SAT problems admit algorithms achieving constant standard approximation ratio, while algorithms for the DNF ones do not guarantee such ratios (more details about the standard approximability of all these problems can be found in [CK]). The MIN VERTEX COVER (called MIN MINIMAL VERTEX COVER in this paper) is standard 2-approximable, while the MAX INDEPENDENT SET (called MAX INDEPENDENT DOMINATING SET in the paper) cannot be approximated within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $\text{co-RP} = \text{NP}$ ([Has96]). On the other hand, MIN INDEPENDENT DOMINATING SET is standard approximable within $B/2$ where, $B$ is the maximum graph-degree ([Kann92]), while the MAX MINIMAL VERTEX COVER has, to our knowledge, not been studied yet in the standard approximation.

The initial objective of the paper was to study the differential approximability of the optimal satisfiability problems defined above. This study has brought to the fore an interesting relationship between MIN $k$SAT and MIN MINIMAL VERTEX COVER which can be informally described as follows: any differential approximation algorithm for MIN MINIMAL VERTEX COVER can be transformed into a differential approximation algorithm for MIN $k$SAT achieving the same differential performance ratio. On the other hand, as we will see just below, MAX MINIMAL VERTEX COVER is equivalent, for the differential approximation, to the well-known MIN INDEPENDENT DOMINATING SET. We are so led to study differential approximation results for MAX MINIMAL VERTEX COVER and MIN INDEPENDENT DOMINATING SET.

All the problems we deal with in this paper have the characteristic that computation of both their optimal and worst solutions is NP-hard (for example, considering an instance $\varphi$ of MAX $k$SAT, its worst solution is an assignment satisfying the minimum number of the clauses of $\varphi$, i.e., an optimal solution for MIN $k$SAT on $\varphi$). Remark also that, given a graph $G = (V, E)$, the complement, with respect to $V$ of a minimal vertex cover (resp., maximal independent set) is a maximal independent set (resp., minimal vertex cover) of $G$. In other words, the objective values of MIN (MAX) MINIMAL VERTEX COVER and of MIN (MAX) INDEPENDENT DOMINATING SET are linked by affine transformations. On the other hand, the differential approximation ratio is stable for the affine transformation, in the sense that pairs of problems, the objective values of which are linked by affine transformations, are differential equivalent. Hence the following fact holds: MIN (MAX) MINIMAL VERTEX COVER and MAX (MIN) INDEPENDENT DOMINATING SET are differential equivalent.

In what follows, we first study differential approximation preserving reductions for several optimal satisfiability problems. Combining them with a general result linking approximability of maximization problems in differential and standard approximations, we obtain interesting differential inapproximability results for optimal satisfiability. We also prove that MIN $k$SAT$(B, \overline{B})$ and MAX $k$SAT$(B, \overline{B})$ reduce to MIN MINIMAL VERTEX COVER-$B'$ and MIN INDEPENDENT DOMINATING SET-$B'$, respectively. These reductions lead us to study the differential approx-
imation of MIN INDEPENDENT DOMINATING SET. For this problem we prove a strong inap-
proximability result, informally, unless $P = NP$, any approximation algorithm has worst-case
approximation ratio equal to 0. To our knowledge, no such result was previously known for the
differential approximation.

2 Preliminaries

We first recall a few definitions about differential and standard approximabilities. Given an
instance $x$ of an optimization problem and a feasible solution $y$ of $x$, we denote by $m(x, y)$ the
value of the solution $y$, by $\text{opt}(x)$ the value of an optimal solution of $x$, and by $\omega(x)$ the value of
a worst solution of $x$. The standard performance, or approximation, ratio of $y$ is defined as

$$r(x, y) = \max \left\{ \frac{m(x, y)}{\text{opt}(x)}, \frac{\text{opt}(x)}{m(x, y)} \right\}$$

while the differential performance, or approximation, ratio of $y$ is defined as

$$\rho(x, y) = \frac{m(x, y) - \omega(x)}{\text{opt}(x) - \omega(x)}.$$ 

It is easy to see that the differential approximation ratio is stable for the affine transformation
of the objective function of a problem, while this does not hold for the standard approximation
ratio.

For a function $f$, $f(n) > 1$, an algorithm is a standard $f(n)$-approximation algorithm for a
problem $\Pi$ if, for any instance $x$ of $\Pi$, it returns a solution $y$ such that $r(x, y) \leq f(|x|)$, where $|x|
$ is the size of $x$. We say that an optimization problem is standard constantly approximable if,
for some constant $c > 1$, there exists a polynomial time standard $c$-approximation algorithm
for it. An optimization problem has a standard polynomial time approximation schema if it
has a polynomial time standard $(1 + \varepsilon)$-approximation, for every constant $\varepsilon > 0$. Similarly,
for a function $f$, $f(n) < 1$, an algorithm is a differential $f(n)$-approximation algorithm for a
problem $\Pi$ if, for any instance $x$ of $\Pi$, it returns a solution $y$ such that $\rho(x, y) \geq f(|x|)$. We say
that an optimization problem is differential constantly approximable if, for some constant $\delta < 1$,
there exists a polynomial time differential $\delta$-approximation algorithm for it. An optimization
problem has a differential polynomial time approximation scheme if it has a polynomial time
differential $(1 + \varepsilon)$-approximation, for every constant $\varepsilon > 0$. We say that two optimisation
problems are differential equivalent if a differential $\delta$-approximation algorithm for one of them
implies a differential $\delta$-approximation algorithm for the other one.

In this paper, we study the differential approximability of the following NP-hard optimal
satisfiability problems.

**MAX (MIN) SAT**

**Input:** a set of clauses $C_1, \ldots, C_m$ on $n$ variables $x_1, \ldots, x_n$.

**Output:** a truth assignment to the variables that maximizes (minimizes) the number of clauses satisfied.

**MAX (MIN) DNF**

**Input:** a set of conjunctions $C_1, \ldots, C_m$ on $n$ variables $x_1, \ldots, x_n$.

**Output:** a truth assignment to the variables that maximizes (minimizes) the number of conjunctions satisfied.

For a constant $k \geq 2$, MAX $k$SAT, MAX $k$DNF, MIN $k$SAT, MIN $k$DNF are the versions of MAX
SAT, MAX DNF, MIN SAT, MIN DNF where each clause or conjunction has size at most $k$. For a constant $B \geq 1$, MAX $k$SAT($B, \bar{B}$), MAX $k$DNF($B, \bar{B}$), MIN $k$SAT($B, \bar{B}$), MIN $k$DNF($B, \bar{B}$),
\textbf{MAX SAT}(B, \overline{B}), \textbf{MAX DNF}(B, \overline{B}), \textbf{MIN SAT}(B, \overline{B}), \textbf{MIN DNF}(B, \overline{B}) \text{ are the versions of these problems where each literal appears at most } B \text{ times.}

\textbf{MAX NAE 3SAT} \\
\textbf{Input:} a set of conjunctions \( C_1, \ldots, C_m \) of three literals on \( n \) variables \( x_1, \ldots, x_n \). \\
\textbf{Output:} a truth assignment to the variables that maximizes the number of conjunctions satisfied in such a way that any one of them has at least one true literal and at least one false literal.

\textbf{MIN (MAX) MINIMAL VERTEX COVER} \\
\textbf{Input:} a graph \( G = (V, E) \). \\
\textbf{Output:} a minimal vertex cover (a set \( S \subseteq V \) such that, \( \forall (u, v) \in E, u \in S \) or \( v \in S \) of minimum (maximum) size.

\textbf{MIN (MAX) INDEPENDENT DOMINATING SET} \\
\textbf{Input:} a graph \( G = (V, E) \). \\
\textbf{Output:} a maximal independent set (a set \( S \subseteq V \) such that, \( \forall u, v \in S, (u, v) \notin E \) and \( \forall u \notin S, \exists v \in S, (u, v) \in E \) of minimum (maximum) size.

In what follows we denote by \( \textbf{MIN (MAX) INDEPENDENT DOMINATING SET} \) and \( \textbf{MIN (MAX) MINIMAL VERTEX COVER} \).

\textbf{MINIMAL VERTEX COVER-B} the versions of the above problems on graphs with maximum degree bounded by \( B \).

3 Satisfiability problems

3.1 Approximation preserving reductions for optimal satisfiability

We first prove the differential equivalence for \textbf{MAX SAT} and \textbf{MIN DNF} and for \textbf{MIN SAT} and \textbf{MAX DNF}.

\textbf{Theorem 1.} \textbf{MAX SAT} and \textbf{MIN DNF}, as well as \textbf{MIN SAT} and \textbf{MAX DNF} are differential equivalent.

\textbf{Proof.} We construct a reduction from \textbf{MAX SAT} to \textbf{MIN DNF} that preserves the differential approximation ratio. Let \( I \) be an instance of \textbf{MAX SAT} on \( n \) variables and \( m \) clauses. The instance \( I' \) of \textbf{MIN DNF} contains \( m \) clauses and the same set of \( n \) variables. With each clause \( \ell_1 \lor \ldots \lor \ell_k \) of \( I \) we associate in \( I' \) the conjunction \( \bar{\ell}_1 \land \ldots \land \bar{\ell}_k \), where \( \bar{\ell}_i = \bar{x}_j \) if \( \ell_i = x_j \) and \( \bar{\ell}_i = x_j \) if \( \ell_i = \bar{x}_j \). It is easy to see that \( \text{opt}(I') = m - \text{opt}(I) \) and \( \omega(I') = m - \omega(I) \). Also, if \( m(I', y) \) is the value of the solution \( y \) in \( I' \), then the same solution \( y \) has in \( I \) the value \( m(I, y) = m - m(I', y) \). Thus, \( m(I', y) = m(I, y) \). The reduction from \textbf{MIN DNF} to \textbf{MAX SAT} is the same.

By an exactly similar reduction, one can prove that \textbf{MIN SAT} and \textbf{MAX DNF} are also approximate equivalent. \( \blacksquare \)

By the proof of theorem 1 one easily can deduce that for each constant \( k \geq 2 \), \textbf{MAX kSAT} and \textbf{MIN kDNF} as well as \textbf{MIN kSAT} and \textbf{MAX kDNF} are differential equivalent.

Consider an instance \( I \) of a maximization problem \( \Pi \), an approximation algorithm \( A \) for \( \Pi \) and denote by \( S \) a feasible solution of \( \Pi \) computed by \( A \) in \( I \). Then,

\[
\frac{m_A(I, S) - \omega(I)}{\text{opt}(I) - \omega(I)} \geq \delta \Rightarrow \frac{m_A(I, S)}{\text{opt}(I)} \geq \delta \Rightarrow \frac{m_A(I, S)}{\omega(I)} \geq \frac{m(I, S)}{\omega(I)} \geq \delta
\]

and the following proposition immediately holds.

\textbf{Proposition 1.} Approximation of a maximization problem \( \Pi \) within differential approximation ratio \( \delta \), implies approximation of \( \Pi \) within standard approximation ratio \( 1/\delta \).

Combining the results of theorem 1 and proposition 1 with the fact that for \( k \geq 2 \) and \( B \geq 3 \), \textbf{MAX kSAT}(B, \overline{B}) and \textbf{MAX kDNF}(B, \overline{B}) have no standard polynomial time approximation schemata ([PY91]), one deduces the following.
Corollary 1. For \( k \geq 2 \) and \( B \geq 3 \), \( \text{MAX } k\text{SAT}(B, \bar{B}) \), \( \text{MAX } k\text{DNF}(B, \bar{B}) \), \( \text{MIN } k\text{SAT}(B, \bar{B}) \), and \( \text{MIN } k\text{DNF}(B, \bar{B}) \) have no differential polynomial time approximation schema, unless \( P = NP \).

3.2 MIN SAT and MIN VERTEX COVER

MIN VERTEX COVER is as the MIN MINIMAL VERTEX COVER defined in section 2 modulo the fact that the feasible solutions for the former are not necessarily minimal. We show that the reduction used in [CST96] from MIN VERTEX COVER to MIN SAT is also differential approximation preserving. This will allow us to establish an inapproximability result for MIN SAT.

Theorem 2. Unless \( co-RP = NP \), MIN SAT is not differential \( 1/m^{1-\epsilon} \)-approximable for any \( \epsilon > 0 \), where \( m \) is the number of clauses of the instance.

Proof. The reduction of [CST96] from MIN VERTEX COVER to MIN SAT works as follows. Let \( G = (V, E) \) be a graph on \( n \) vertices and denote by \( V = \{1, \ldots, n\} \) its vertex set. In order to construct an instance \( I \) of MIN SAT, at each edge \( (i, j) \in E; i < j \) we associate a variable \( x_{ij} \). For each vertex \( i \) we define a clause \( C_i \), where

\[
C_i = \bigvee_{j : (i, j) \in E \land i < j} x_{ij} \lor \bigvee_{j : (i, j) \in E \land i > j} \bar{x}_{ij}.
\]

From a vertex cover \( C \) of \( G \) we define an assignment as follows. For each \( i \notin C \) and each \( (i, j) \in E \), \( x_{ij} = 1 \) if \( i > j \) and \( x_{ij} = 0 \) if \( i < j \). Since \( C \) is a vertex cover, this definition is not contradictory. If \( i \notin C \), then \( C_i \) is not satisfied and so \( \text{opt}(I) \leq \text{opt}(C) \).

Given an assignment \( v \) of \( I \), let \( C = \{ i : C_i \text{ is satisfied} \} \). Note that set \( C \) is a vertex cover since for \( (i, j) \in E \), at least one of \( C_i \) and \( C_j \) is satisfied and so at least one of the vertices \( i, j \) appears in \( C \). So, at each assignment \( v \) of \( I \), we associate in \( G \) a vertex cover \( C \) with \( m(G, C) = m(I, v) \). This also proves that \( \text{opt}(I) = \text{opt}(C) \).

Finally, using \( \omega(I) \leq \omega(G) \), it is easy to show that \( \rho(G) \geq \rho(I) \).

We have seen that MIN VERTEX COVER is differential equivalent to MAX IMPERFECTION.
Theorem 6. If $P \neq NP$, then, for any $\delta(n) \in (0,1)$, ($\delta$ decreasing in $n$), MIN INDEPENDENT DOMINATING SET on graphs of order $n$ is not differential $\delta(n)$-approximable.

Proof. We show that, for any $\delta(n) \in (0,1)$, a polynomial time differential $\delta(n)$-approximation algorithm $A$ for MIN INDEPENDENT DOMINATING SET, could distinguish in polynomial time if an instance of SAT on $n$ variables is satisfiable or not.

Given an instance $\varphi$ of SAT with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $C_1, \ldots, C_m$ we construct a graph $G$, instance of MIN INDEPENDENT DOMINATING SET as follows. With each positive literal $x_i$ we associate a vertex $u_i$ and for each negative literal $\bar{x}_i$ we associate a vertex $v_i$. For $i = 1, \ldots, n$ we draw edges $u_iv_i$. For any clause $C_j$ we add in $G$ a vertex $w_j$ and an edge between $w_j$ and each vertex corresponding to a literal contained in $C_j$. Finally, we add edges in $G$ in order to obtain a complete graph on $w_1, \ldots, w_m$.

Remark that an independent set of $G$ contains at most $n+1$ vertices since it contains at most one vertex among $w_1, \ldots, w_m$ and at most one vertex among $u_i$ and $v_i$ for $i = 1, \ldots, n$. An independent dominating set containing the vertices corresponding to true literals of a non satisfiable assignment and one vertex corresponding to a clause not satisfied by this assignment is a worst solution of $G$ of size $n+1$.

If $\varphi$ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying $\varphi$ is an independent dominating set (each vertex $w_j$ is dominated by a vertex corresponding to a true literal of $C_j$) of minimum size. On the other hand, if $\varphi$ is not
each positive literal $x_i$ we associate a vertex $u_i$, and with each negative literal $\overline{x}_i$ we associate a vertex $v_i$. For $i = 1, \ldots, n$ we draw in $G$ the edges $u_iv_i$. Also with each clause $C_j$ we associate $c = [(B - 1)/4]$ vertices $w_{j1}, \ldots, w_{jc}$. For each clause $C_j$ we add in $G$ an edge between each $w_{jk}$, $k = 1, \ldots, c$ and any vertex corresponding to a literal contained in $C_j$.

Suppose that each literal appears at least once. Remark that an independent set of $G$ contains at most $m \cdot c$ vertices. An independent dominating set containing the vertices corresponding to the $m$ clauses of $\varphi$ is a worst solution of size $m \cdot c$.

If $\varphi$ is satisfiable then $\text{opt}(G) = n$ since the set of vertices corresponding to the true literals of an assignment satisfying $\varphi$ is an independent dominating set (each vertex $w_{jk}$ is dominated by a vertex corresponding to a true literal of $C_j$) of minimum size. On the other hand, if the optimal value of $\varphi$ is $m' \leq (1 - \varepsilon')m$ then $\text{opt}(G) = n + (m - m') \cdot c \geq n + \varepsilon' \cdot m \cdot c$.

We show that a differential $f(B)$-approximation algorithm $A$ for MIN INDEPENDENT DOMINATING SET-B with $f(B) = 1 - (2\varepsilon'(B - 5)/(2B - 5))$ gives in the case where $\varphi$ is satisfiable a solution of value less that the value of the optimum solution in the case where $\varphi$ is not satisfiable.

Denote by $\text{val}$ the value of the solution computed by $A$. Then, $(m \cdot c - \text{val})/(m \cdot c - n) \geq f(B)$. Since $c \leq (B - 1)/4$ and $m \leq 8n/3$, $\text{val} \leq n + (m \cdot \varepsilon'(B - 5)/4) < n + m \cdot \varepsilon' \cdot c$, q.e.d.

6 Discussion

We have given in this paper differential inapproximability results for optimal satisfiability problems, as well as for MIN INDEPENDENT DOMINATING SET. For this problem we have shown that any polynomial time approximation algorithm has worst-case differential approximation ratio 0. This result brings MIN INDEPENDENT DOMINATING SET to the status of one of the hardest problems for the differential approximation.

Differential approximation for optimal satisfiability misses until now in positive results. Despite our efforts, the only one we have been able to produce is the one of section 3.3 on a class of instances of MAX NAE 3SAT, the satisfiable ones. It is interesting to produce non-trivial such results and this is a major open problem posed by our work. However, it seems to us that, in the opposite of the standard approximation, obtaining constant differential approximation ratios for optimal satisfiability is a rather hard task.

As we have already mentioned, results as the one of theorem 6 have not been produced until now. However such strongly negative results are very interesting since they draw the hardest of the NP-hard problems classes in the differential approximability hierarchy. Establishing such results for other problems is an equally interesting open problem.