Differential Approximation for Satisfiability and Related Problems

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Approximation différentielle pour le problème de la satisfaisabilité et autres problèmes connexes

Résumé

Nous étudions l’approximabilité différentielle de divers problèmes de satisfaisabilité optimale. Nous démontrons que, sauf si co-RP = NP, Min SAT n’est pas approximable à rapport différentiel $1/m^{1-\varepsilon}$, pour tout $\varepsilon > 0$, où $m$ est le nombre des clauses de la formule. En mettant en évidence que chaque algorithme d’approximation différentielle pour le problème de Max Minimal Vertex Cover peut être transformé en un algorithme d’approximation différentielle pour le problème de Min k-SAT garantissant le même rapport de performance, nous sommes amenés à étudier l’approximabilité différentielle des problèmes Max Minimal Vertex Cover et Min Independent Dominating Set; tous les deux sont 

1 Introduction

In this paper we deal with the approximation of classical optimal satisfiability problems as Max and Min Sat, Max and Min DNF, as well as of restrictive versions of these problems as the ones where the size of any clause is bounded, or/and the number of the occurrences of any literal is bounded. We also deal with some graph-problems as Max and Min Independent Dominating Set and Max and Min Minimal Vertex Cover. We study the approximability of all these problems using the so-called differential approximation ratio which, informally, for an instance $I$ measures the relative position of the value of an approximated solution in the interval [worst-value feasible solution of $I$, optimal-value solution of $I$].

All these problems have no polynomial time approximation schemata for the standard approximation (where one measures the ratio between the value of the approximate solution of an instance and the value of an optimal one). The Sat problems admit algorithms achieving constant standard approximation ratio, while algorithms for the DNF ones do not guarantee such ratios (more details about the standard approximability of all these problems can be found in [CK]). The Min Vertex Cover (called Min Minimal Vertex Cover in this paper) is standard 2-approximable, while the Max Independent Set (called Max Independent Dominating Set in the paper) cannot be approximated within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $co-RP = NP$ ([Has96]). On the other hand, Min Independent Dominating Set is standard approximable within $B/2$ where, $B$ is the maximum graph-degree ([Kann92]), while the Max Minimal Vertex Cover has, to our knowledge, not been studied yet in the standard approximation.

The initial objective of the paper was to study the differential approximability of the optimal satisfiability problems defined above. This study has brought to the fore an interesting relationship between Min $k$Sat and Min Minimal Vertex Cover which can be informally described as follows: any differential approximation algorithm for Min Minimal Vertex Cover can be transformed into a differential approximation algorithm for Min $k$Sat achieving the same differential performance ratio. On the other hand, as we will see just below, Max Minimal Vertex Cover is equivalent, for the differential approximation, to the well-known Min Independent Dominating Set. We are so led to study differential approximation results for Max Minimal Vertex Cover and Min Independent Dominating Set.
imation of MIN INDEPENDENT DOMINATING SET. For this problem we prove a strong inapproximability result, informally, unless $P = NP$, any approximation algorithm has worst-case approximation ratio equal to 0. To our knowledge, no such result was previously known for the differential approximation.

2 Preliminaries

We first recall a few definitions about differential and standard approximabilities. Given an instance $x$ of an optimization problem and a feasible solution $y$ of $x$, we denote by $m(x, y)$ the value of the solution $y$, by $\text{opt}(x)$ the value of an optimal solution of $x$, and by $\omega(x)$ the value of a worst solution of $x$. The standard performance, or approximation, ratio of $y$ is defined as

$$r(x, y) = \max \left\{ \frac{m(x, y)}{\text{opt}(x)}, \frac{\text{opt}(x)}{m(x, y)} \right\}$$

while the differential performance, or approximation, ratio of $y$ is defined as

$$\rho(x, y) = \frac{m(x, y) - \omega(x)}{\text{opt}(x) - \omega(x)}.$$  

It is easy to see that the differential approximation ratio is stable for the affine transformation of the objective function of a problem, while this does not hold for the standard approximation ratio.

For a function $f$, $f(n) > 1$, an algorithm is a standard $f(n)$-approximation algorithm for a problem $\Pi$ if, for any instance $x$ of $\Pi$, it returns a solution $y$ such that $r(x, y) \leq f(|x|)$, where $|x|$ is the size of $x$. We say that an optimization problem is standard constantly approximable if, for some constant $c > 1$, there exists a polynomial time standard $c$-approximation algorithm for it. An optimization problem has a standard polynomial time approximation schema if it has a polynomial time standard $(1 + \varepsilon)$-approximation, for every constant $\varepsilon > 0$. Similarly, for a function $f$, $f(n) < 1$, an algorithm is a differential $f(n)$-approximation algorithm for a problem $\Pi$ if, for any instance $x$ of $\Pi$, it returns a solution $y$ such that $\rho(x, y) \geq f(|x|)$. We say problem $\Pi$ is differential constantly approximable if, for some constant $\delta < 1$, an algorithm is differential constant time approximation algorithm for it. An optimization time
\( \text{MAX SAT}(B, \bar{B}), \text{MAX DNF}(B, \bar{B}), \text{MIN SAT}(B, \bar{B}), \text{MIN DNF}(B, \bar{B}) \) are the versions of these problems where each literal appears at most \( B \) times.

\( \text{MAX NAE 3SAT} \)

**Input:** a set of conjunctions \( C_1, \ldots, C_m \) of three literals on \( n \) variables \( x_1, \ldots, x_n \).

**Output:** a truth assignment to the variables that maximizes the number of conjunctions satisfied in such a way that any one of them has at least one true literal and at least one false literal.

\( \text{MIN (MAX) MINIMAL VERTEX COVER} \)

**Input:** a graph \( G = (V, E) \).

**Output:** a minimal vertex cover (a set \( S \subseteq V \) such that, \( \forall (u, v) \in E, u \in S \) or \( v \in S \)) of minimum (maximum) size.

\( \text{MIN (MAX) INDEPENDENT DOMINATING SET} \)

**Input:** a graph \( G = (V, E) \).

**Output:** a maximal independent set (a set \( S \subseteq V \) such that, \( \forall u, v \in S \), \( (u, v) \notin E \) and \( \forall u \notin S, \exists v \in S, (u, v) \in E \) of minimum (maximum) size.

In what follows, we denote by \( \text{MIN (MAX) INDEPENDENT DOMINATING SET-B} \) and \( \text{MIN (MAX) MINIMAL VERTEX COVER-B} \) the versions of the above problems on graphs with maximum degree bounded by \( B \).

3 Satisfiability problems

3.1 Approximation preserving reductions for optimal satisfiability

We first prove the differential equivalence for \( \text{MAX SAT} \) and \( \text{MIN DNF} \) and for \( \text{MIN SAT} \) and \( \text{MAX DNF} \).

**Theorem 1.** \( \text{MAX SAT} \) and \( \text{MIN DNF} \), as well as \( \text{MIN SAT} \) and \( \text{MAX DNF} \) are differential equivalent.

**Proof.** We construct a reduction from \( \text{MAX SAT} \) to \( \text{MIN DNF} \) that preserves the differential approximation ratio. Let \( I \) be an instance of \( \text{MAX SAT} \) on \( n \) variables and \( m \) clauses. The instance \( I' \) of \( \text{MIN DNF} \) contains \( m \) clauses and the same set of \( n \) variables. With each clause \( \ell_1 \lor \ldots \lor \ell_k \) of \( I \) we associate in \( I' \) the conjunction \( \bar{\ell}_1 \land \ldots \land \bar{\ell}_k \), where \( \bar{\ell}_i = \bar{x}_j \) if \( \ell_i = x_j \) and \( \bar{\ell}_i = x_j \) if \( \ell_i = \bar{x}_j \). It is easy to see that \( \text{opt}(I') = m - \text{opt}(I) \) and \( \omega(I') = m - \omega(I) \). Also, if \( m(I', y) \) is the value of the solution \( y \) in \( I' \), then the same solution \( y \) has in \( I \) the value \( m(I, y) = m - m(I', y) \).

Thus, \( \rho(I, y) = \rho(I', y) \). The reduction from \( \text{MIN DNF} \) to \( \text{MAX SAT} \) is the same.

By an exactly similar reduction, one can prove that \( \text{MIN SAT} \) and \( \text{MAX DNF} \) are also approximate equivalent.

By the proof of theorem 1 one easily can deduce that for each constant \( k \geq 2 \), \( \text{MAX kSAT} \) and \( \text{MIN kDNF} \) as well as \( \text{MIN kSAT} \) and \( \text{MAX kDNF} \) are differential equivalent.

Consider an instance \( I \) of a maximization problem \( \Pi \), an approximation algorithm \( A \) for \( \Pi \) and denote by \( S \) a feasible solution of \( \Pi \) computed by \( A \) in \( I \). Then,

\[
\frac{m_A(I, S) - \omega(I)}{\text{opt}(I) - \omega(I)} \geq \delta \implies \frac{m_A(I, S)}{\text{opt}(I)} \geq \delta + (1 - \delta)\omega(I) \overset{\omega(I) \geq 0}{\geq} \frac{m_A(I, S)}{\beta(I)} \geq \delta
\]

and the following proposition immediately holds.

**Proposition 1.** Approximation of a maximization problem \( \Pi \) within differential approximation ratio \( \delta \), implies approximation of \( \Pi \) within standard approximation ratio \( 1/\delta \).

Combining the results of theorem 1 and proposition 1 with the fact that for \( k \geq 2 \) and \( B \geq 3 \), \( \text{MAX kSAT}(B, \bar{B}) \) and \( \text{MAX kDNF}(B, \bar{B}) \) have no standard polynomial time approximation schemata ([PY91]), one deduces the following.
Corollary 1. For \( k \geq 2 \) and \( B \geq 3 \), \( \text{MAX } k\text{SAT}(B, \overline{B}) \), \( \text{MAX } k\text{DNF}(B, \overline{B}) \), \( \text{MIN } k\text{SAT}(B, \overline{B}) \) and \( \text{MIN } k\text{DNF}(B, \overline{B}) \) have no differential polynomial time approximation schemes, unless \( P = \text{NP} \).

3.2 MIN SAT and MIN VERTEX COVER

MIN VERTEX COVER is as the MIN MINIMAL VERTEX COVER defined in section 2 modulo the fact that the feasible solutions for the former are not necessarily minimal. We show that the reduction used in [CST96] from MIN VERTEX COVER to MIN SAT is also differential approximation preserving. This will allow us to establish an inapproximability result for MIN SAT.

Theorem 2. Unless \( \text{co-RP} = \text{NP} \), MIN SAT is not differential \( 1/m^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), where \( m \) is the number of clauses of the instance.

Proof. The reduction of [CST96] from MIN VERTEX COVER to MIN SAT works as follows. Let \( G = (V, E) \) be a graph on \( n \) vertices and denote by \( V = \{1, \ldots, n\} \) its vertex set. In order to construct an instance \( I \) of MIN SAT, at each edge \( (i, j) \in E; i < j \) we associate a variable \( x_{ij} \). For each vertex \( i \) we define a clause \( C_i \), where

\[
C_i = \bigvee_{j: (i, j) \in E \land i < j} x_{ij} \lor \bigvee_{j: (i, j) \in E \land i > j} \overline{x_{ij}}.
\]

From a vertex cover \( C \) of \( G \) we define an assignment as follows. For each \( i \notin C \) and each \( (i, j) \in E \), \( x_{ij} = 1 \) if \( i > j \) and \( x_{ij} = 0 \) if \( i < j \). Since \( C \) is a vertex cover, this definition is not contradictory. If \( i \notin C \), then \( C_i \) is not satisfied and so \( \text{opt}(I) \leq \text{opt}(G) \).

Given an assignment \( v \) of \( I \), let \( C = \{i : C_i \text{ is satisfied}\} \). Note that set \( C \) is a vertex cover since for \( (i, j) \in E \), at least one of \( C_i \) and \( C_j \) is satisfied and so at least one of the vertices \( i, j \) appears in \( C \). So, at each assignment \( v \) of \( I \), we associate in \( G \) a vertex cover \( C \) with \( m(G, C) = m(I, v) \). This also proves that \( \text{opt}(I) = \text{opt}(G) \).

Finally, using \( \omega(I) \leq \omega(G) \), it is easy to show that \( \rho(G) \geq \rho(I) \).

We have seen that MIN VERTEX COVER is differential equivalent to MAX INDEPENDENT SET (which is as MAX INDEPENDENT DOMINATING SET modulo the fact that the independent set to compute has not to be minimal). On the other hand since the worst solution for MAX INDEPENDENT SET is the empty set (in other words, \( \omega(I) = 0, \forall I \)), standard and differential approximation ratios coincide. Furthermore, MAX INDEPENDENT SET is not differential \( 1/n^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), unless \( \text{co-RP} = \text{NP} \) ([Has96]). Consequently, MIN VERTEX COVER is not differential \( 1/n^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), unless \( \text{co-RP} = \text{NP} \) and the result claimed follows.

Corollary 2. MIN SAT\((B, \overline{B})\) for \( B \geq 1 \) is not differential \( 1/m^{1-\varepsilon} \)-approximable for any \( \varepsilon > 0 \), unless \( \text{co-RP} = \text{NP} \).

3.3 A positive differential approximation result for MAX NAE 3Sat

We show in this section that a restrictive version of MAX NAE 3Sat, the one on satisfiable instances is differential constantly approximable by the standard 1.096-approximation algorithm of [Zwick98].

Theorem 3. MAX NAE 3Sat on satisfiable instances is differential \( 0.649 \)-approximable.
Proof. Consider a satisfiable instance $\varphi$ of MAX NAE 3SAT defined on $m$ clauses; obviously, \[\text{opt}(\varphi) = m.\] Run the standard 1.096-approximation algorithm of [Zwick98] on $\varphi$ to obtain a
Theorem 6. If $P \neq NP$, then, for any $\delta(n) \in (0,1)$, $(\delta$ decreasing in $n)$, MIN INDEPENDENT DOMINATING SET on graphs of order $n$ is not differential $\delta(n)$-approximable.

Proof. We show that, for any $\delta(n) \in (0,1)$, a polynomial time differential $\delta(n)$-approximation algorithm $A$ for MIN INDEPENDENT DOMINATING SET, could distinguish in polynomial time if an instance of SAT on $n$ variables is satisfiable or not.
each positive literal \( x_i \) we associate a vertex \( u_i \), and with each negative literal \( \overline{x}_i \) we associate a vertex \( v_i \). For \( i = 1, \ldots, n \) we draw in \( G \) the edges \( u_i v_i \). Also with each clause \( C_j \) we associate \( c = [(B - 1)/4] \) vertices \( w_{j1}, \ldots, w_{jc} \). For each clause \( C_j \) we add in \( G \) an edge between each \( w_{jk}, k = 1, \ldots, c \) and any vertex corresponding to a literal contained in \( C_j \).

Suppose that each literal appears at least once. Remark that an independent set of \( G \) contains at most \( m \cdot c \) vertices. An independent dominating set containing the vertices corresponding to the \( m \) clauses of \( \varphi \) is a worst solution of size \( m \cdot c \).

If \( \varphi \) is satisfiable then \( \text{opt}(G) = n \) since the set of vertices corresponding to the true literals of an assignment satisfying \( \varphi \) is an independent dominating set (each vertex \( w_{jk} \) is dominated by a vertex corresponding to a true literal of \( C_j \)) of minimum size. On the other hand, if the optimal value of \( \varphi \) is \( m' \leq (1 - \epsilon')m \) then \( \text{opt}(G) = n + (m - m') \cdot c \geq n + \epsilon' \cdot m \cdot c \).

We show that a differential \( f(B) \)-approximation algorithm \( A \) for \text{MIN INDEPENDENT DOMINATING SET-B} with \( f(B) = 1 - (2\epsilon'(B - 5)/(2B - 5)) \) gives in the case where \( \varphi \) is satisfiable a solution of value less that the value of the optimum solution in the case where \( \varphi \) is not satisfiable.

Denote by \( \text{val} \) the value of the solution computed by \( A \). Then, \( (m \cdot c - \text{val})/(m \cdot c - n) \geq f(B) \). Since \( c \leq (B - 1)/4 \) and \( m \leq 8n/3 \), \( \text{val} \leq n + (m \cdot \epsilon'(B - 5)/4) < n + m \cdot \epsilon' \cdot c \), q.e.d.

6 Discussion

We have given in this paper differential inapproximability results for optimal satisfiability problems, as well as for \text{MIN INDEPENDENT DOMINATING SET}. For this problem we have shown that any polynomial time approximation algorithm has worst-case differential approximation ratio 0. This result brings \text{MIN INDEPENDENT DOMINATING SET} to the status of one of the hardest problems for the differential approximation.

Differential approximation for optimal satisfiability misses until now in positive results. Despite our efforts, the only one we have been able to produce is the one of section 3.3 on a class of instances of \text{MAX NAE 3SAT}, the satisfiable ones. It is interesting to produce non-trivial such results and this is a major open problem posed by our work. However, it seems to us that, in the opposite of the standard approximation, obtaining constant differential approximation ratios for optimal satisfiability is a rather hard task.

As we have already mentioned, results as the one of theorem 6 have not been produced until now. However such strongly negative results are very interesting since they draw the hardest of the NP-hard problems classes in the differential approximability hierarchy. Establishing such results for other problems is an equally interesting open problem.