NUMERICAL REPRESENTATION OF \textit{PQI} INTERVAL ORDERS

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Résumé

Nous considérons le problème de la représentation numérique des ordres d’intervalle PQI. Une structure de préférence sur un ensemble fini $A$ qui contient les relations $P$ (préférence stricte), $Q$ (préférence faible) et $I$ (indifférence) est un ordre d’intervalle PQI si chaque élément de l’ensemble $A$ est représentable par un intervalle de façon que la relation $P$ est vraie si un intervalle est complètement à droite de l’autre, la relation $I$ est vraie si un intervalle est inclus dans l’autre et la relation $Q$ est vraie si un intervalle est à droite de l’autre mais leur intersection n’est pas vide. Cette structure a été axiomatisée seulement récemment. Dans ce papier, nous analysons des concepts tels que la magnitude, la matrice caractéristique ou le graphe synthétique. Enfin, nous présentons deux algorithmes : le premier en $O(n^2)$ pour la détermination d’une représentation générale, le deuxième en $O(n)$ pour trouver une représentation minimale.

Mots clés : Intervalles, Ordres d’intervalle PQI, Représentation numérique, Représentation minimale.

Abstract

We consider the problem concerning the numerical representation of PQI interval orders. A preference structure on a finite set $A$ with three relations $P$, $Q$, $I$ standing for “strict preference”, “weak preference” and “indifference”, respectively, is defined as a PQI interval order if there exists a representation of each element of $A$ by an interval in such a way that $P$ holds when one interval is completely to the right of the other, $I$ holds when one interval is included to the other and $Q$ holds when one interval is to the right of the other, but they do have a non empty intersection ($Q$ modelling the hesitation between $P$ and $I$). Only recently necessary and sufficient conditions for a PQI preference structured to be identified as a PQI interval order have been established. In this paper, we are interested in the problem concerning the representation of a PQI interval order, particularly numerical representations. We will investigate some concepts aiming at characterising a PQI interval order such as: magnitude, characteristic matrix, synthetic graph (SG). Finally, we present two algorithms, the first one in $O(n^2)$ to determine a general numerical representation, and the second one, in $O(n)$, to minimise this representation.

Keywords: Intervals, PQI interval orders, Numerical representation, Minimal representation.
1 Introduction

In preference modelling and decision support we often have to compare intervals instead of discrete values. This is due to the fact that the comparison of alternatives is usually realised through their evaluations on numerical scales, subject to the unavoidable lack of precision and certainty. The conventional structure adopted in order to compare two intervals, considers that “x is preferred to y” \( P(x, y) \) iff the interval associated to x is completely to the “right” (in the sense of the line representing the reals) of the interval associated to y. In all other cases “x is indifferent to y”. Such a model (where indifference is not transitive) may conceal the fact that “x being to the right of y” (the intersection being not empty) is a situation intuitively different from the case where one interval (let’s say x) is included in the other (let’s say y). The second case can be considered a “sure indifference” as much as can be considered a “sure preference” the case \( P(x, y) \). Under such a perspective the first case is a situation of hesitation between preference and indifference which merits to be considered separately (see Tsoukiás and Vincke, 1997). We may denote such a situation as “weak preference” and represented it as \( Q(x, y) \).

The PQI interval order has been discussed since 1988 by Vincke. The problem of characterising such a structure was left open until recently. Tsoukiás and Vincke, 2000, presented a theorem providing necessary and sufficient conditions for a PQI preference structure to be identified as a PQI interval order. The operational problem of detecting if a given PQI preference structure satisfies the conditions of the theorem was solved in Ngo The et al., 2000, through an algorithm which is demonstrated to run in polynomial time.

In this paper, we are interested in the problem of the numerical representation of a PQI interval order. For this purpose, our paper is dedicated to investigate some aspects of the representation of a PQI interval order (once detected). We introduce and study some concepts aiming to characterise a PQI interval order such as: magnitude, characteristic matrix, synthetic graph (SG). These theoretical results lead to two algorithms: the first one is to determine a general representation and the second one a minimal one.

The paper is organised as follows. Section 2 provides the basic notations and definitions. In section 3 we recall some definitions and previous results concerning the numerical representation of interval orders. Section 4 introduces a general PQI interval orders. Section 5 gives the two algorithms to construct a general representation of a PQI interval order and to minimise this representation. Some conclusions are given at the end of the paper.
2 Basic notations, definitions and results

Further on, if not indicated differently, all the relations under consideration are binary relations defined on a finite set $A$ and denoted by $P, Q, I, R, S, T$. The fact that $(x, y) \in S$ is denoted either by $S(x, y)$ or $xSy$. We adopt the following notation.

\[ S^{-1} = \{ (y, x) : (x, y) \in S \} \]
\[ S^e = \{ (x, y) : \neg S(x, y) \} \]
\[ S^d = \{ (x, y) : \neg S^{-1}(y, x) \} \]
\[ S^\sim = A^2 \setminus (S \cup S^{-1}) \]
\[ S^\bowtie = \{ (x, y) : \forall z, S(x, z) \Leftrightarrow S(y, z) \text{ and } S(x, x) \Leftrightarrow S(z, y) \} \]
\[ S \subset T : \forall x, y, S(x, y) \Rightarrow T(x, y) \]
\[ S \cap T = \{ (x, y) : S(x, y) \land T(x, y) \} \]
\[ S^\Delta = S \cdot S \]
\[ S \cup T = \{ (x, y) : S(x, y) \lor T(x, y) \} \]
\[ S \cap T = \{ S(x, y) \land T(x, y) \} \]
\[ S^+(a) = \{ x \in A : S(a, x) \}. \]

If $R$ is an equivalence relation on $A$ then the equivalence class containing $a \in A$ is denoted by $[a]_R$. When there is no ambiguity, we can use simply $[a]$. A binary relation $R$ on a finite set $A = \{a_1, a_2, \ldots, a_n\}$ can be represented by an $n \times n$ 0–1 matrix $M^R$ with $M^R_{ij} = 1$ iff $(a_i, a_j) \in R$. Further on we use the following definitions (see Roubens and Vincke, 1985).

Definition 2.1 A binary relation $S$ is:
- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence relation iff it is reflexive, symmetric and transitive.

It is easy to verify that
- $S^\sim = S^e \cap S^d$
- $S^\bowtie = \{ (x, y) : \forall z, S^\sim(x, z) \Leftrightarrow S^\sim(y, z) \}$
- a weak order is also a partial order and a linear order is also a weak order.

Let’s introduce now the concept of rank function.

Definition 2.2 Let $S$ be a linear order on a finite set $A$. Its rank function is defined as:
\[ g : A \rightarrow \mathbb{N} \]
\[ g(a) = |S^+(a)| + 1 \]
We have the two following fundamental results from Fishburn 1985:

**Theorem 2.1** If $S$ is a partial order then
i) $S^\approx$ is an equivalence relation;
ii) $S^\approx, S^\approx, S^\approx = S^\approx$;
iii) $S^\approx(x, y) \iff \{z : S(x, z)\} = \{z : S(y, z)\}$ and $\{z : S(z, x)\} = \{z : S(z, y)\}$;
iv) $(A/S^\approx, S)$ is a partial order;

**Theorem 2.2** If $S$ is a partial order then the following are equivalent:
i) $S$ is a weak order;
i) $S^\approx$ is transitive;
ii) $S^\approx = S^\approx$;
iv) $S = S^\approx S^\approx = S^\approx S^\approx$;
v) $(A/S^\approx, S)$ is a linear order;
In addition, $S$ is a linear order iff $S^\approx$ is the identity relation $I_0 = \{(x, x) : x \in A\}$.

In this paper we will consider relations representing strict preference, weak preference and indifference, respectively denoted as $P, Q, I$. Such relations satisfy some “natural” properties announced in the following two definitions.

**Definition 2.3** A $(P, I)$ preference structure on a set $A$ is a couple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ is asymmetric;
- $I \cup P$ is complete;
- $P$ and $I$ are mutually exclusive ($P \cap I = \emptyset$).

By definition, a $(P, I)$ preference structure is perfectly characterised by $P$ or $S = P \cup I$. This means that we can represent it by a matrix just as the case of a binary relation.

**Definition 2.4** A $(P, Q, I)$ preference structure is a triple of binary relations, defined on $A$, such that:
- $I$ is reflexive and symmetric;
- $P$ and $Q$ are asymmetric;
- $I \cup P \cup Q$ is complete;
- $P, Q$ and $I$ are mutually exclusive ($P \cap Q = P \cap I = Q \cap I = \emptyset$).
In this case, we have three relations. As they are mutually exclusive, we also can represent it by a matrix \( M^{PQI} \) with \( M^{PQI}_{ij} = X \) where \( X = P, Q, I \) and \( X(a_i, a_j), \forall a_i, a_j \in A \).

The notion "ex æquo" is formally defined as follows:

**Definition 2.5** The equivalence relation associated to a set of relations \( B = \{P, Q, R, \ldots\} \) defined on a set \( A \) is the binary relation \( E \), defined on the set \( A \), such that, \( \forall x, y \in A : E(x, y) \iff \forall z \in A : R(x, z) \leftrightarrow R(y, z), \ R \text{ or } R^{-1} \in B. \) All \( x, y \in A \) such that \( E(x, y) \) are called "ex æquo" w.r.t. \( B. \) When there is no ambiguity, \( B \) is not mentioned.

By definition, it is obvious that \( E = P^{\equiv} \cap Q^{\equiv} \cap R^{\equiv} \ldots \). Particularly, \( E = P^{\equiv} \) in a \( \langle P, I \rangle \) preference structure and \( E = P^{\equiv} \cap Q^{\equiv} \) in a \( \langle P, Q, I \rangle \) preference structure.

A useful tool to study the minimal numerical representation of preference structures is the potential function in a valued graph. Let \( G = (A, U, v) \) be a valued graph on a finite set of nodes \( A \); a real value \( v(a, b) \) is attached to each arc \((a, b)\) of \( U \).

**Definition 2.6** A potential function of the valued graph \( G = (A, U, v) \) is a function \( g : A \to \mathbb{R} \) such that, \( \forall (a, b) \in U, g(a) \geq g(b) + v(a, b). \)

It is easy to see that if \( g \) is a potential function whose minimal value is 0, then \( g(a) \) cannot be smaller than the maximal value of the paths starting from \( a. \) A fundamental result is the following (Roy 1969).

**Theorem 2.3** A valued graph admits potential functions iff there is no circuit of strictly positive value in the graph. The smallest non-negative potential function assigns to each node the maximal value of the paths starting from the node.

### 3 Interval orders

**Definition 3.1** A \( \langle P, I \rangle \) preference structure on a finite set \( A \) is an interval order iff \( \exists r : A \to \mathbb{R}^+ \) such that, \( \forall x, y \in A: \)

i) \( r(x) \geq l(x); \)

ii) \( P(x, y) \leftrightarrow l(x) > l(y); \)

iii) \( I(x, y) \leftrightarrow l(x) \leq r(y) \) and \( l(y) \leq r(x); \)

Any couple \((l, r)\) satisfying the above conditions is a general representation of the interval order.
For a finite set $A$, definition 3.1 is equivalent to the condition $P.I.P \subset P$ which is an alternative definition of an interval order (see Fishburn 1985).

Since $A$ is finite, given a general representation $(l, r)$ of an interval order, there exists a positive constant $\varepsilon = \min_{(a, b) \in P} \{l(b) - r(a)\}$. The triple $(l, r, \varepsilon)$ is called an $\varepsilon$-representation of the interval order. With an $\varepsilon$-representation, condition ii of definition 3.1 can be rewritten as:

$$P(x, y) \Leftrightarrow l(x) \geq l(y) + \varepsilon.$$

Among all the possible $\varepsilon$-representations (with the same $\varepsilon$), the minimal $\varepsilon$-representation is of special interest. Naturally, it is defined as an $\varepsilon$-representation $(l^*, r^*, \varepsilon)$ satisfying, for any other $\varepsilon$-representation $(l, r, \varepsilon)$, $\forall a \in A$, $l^*(a) \leq l(a)$ and $r^*(a) \leq r(a)$. The construction of the minimal representation is based on the following results.

**Theorem 3.1** Let $(P, I)$ be an interval order on a finite set $A$, and let $T_l = P.I, T_r = I.P$. Then

i) $T_l, T_r$ are weak orders on $A$;

ii) $T^*_l, T^*_r$ are equivalence relations and $T_l, T_r$ are linear orders on $A/T^*_l, A/T^*_r$;

iii) $E = T^*_l \cap T^*_r$.

**Proof** See Fishburn 1985.

Let define two copies of $A$, say $A_l$ and $A_r$.

We define $T_0$ on $A_l \cup A_r$ as follows:

- $T_0(a_l, b_l) \Leftrightarrow T_l(a_l, b_l)$;
- $T_0(a_r, b_r) \Leftrightarrow T_r(a_r, b_r)$;
- $T_0(a_l, b_r) \Leftrightarrow P(a_l, b_r)$;
- $T_0(a_r, b_l) \Leftrightarrow I(a_l, b_r)$ or $P(a_l, b_r)$.

**Theorem 3.2** Let $(P, I)$ be an interval order on a finite set $A$, and let $T^*_l, T^*_r$ defined as above. Then

i) $T^*_l$ is a weak order on $(A_l \cup A_r)$;

ii) $T^*_r$ is an equivalence relation and $T_0$ is a linear order on $(A_l \cup A_r)/T^*_r$;

iii) $(A_l \cup A_r)/T^*_r = (A_l/T^*_l) \cup (A_r/T^*_r)$;

$\forall x \in A_l/T^*_l \Rightarrow \exists y \in A_l$ such that $y \leq x$, $\forall y \leq x \Rightarrow \exists y \in A_l/T^*_l$;

$y \in A_r/T^*_r \Rightarrow \exists y \in A_r$ such that $y \geq x$, $\forall y \geq x \Rightarrow \exists y \in A_r/T^*_r$;

$T_0(y, x) \Leftrightarrow y \in A_l = \exists x \in A_l/T^*_l$;

$T_0(y, x) \Leftrightarrow y \in A_r = \exists x \in A_r/T^*_r$.

**Proof** See Fishburn 1985.
$T_i$ $(T_r)$ represents the order of the left (right) end points of the intervals associated to elements of $A$. Each equivalence class in $A/T_i$, $(A/T_r)$ represents a group of elements whose left (right) end points can be identical. Two elements are ex aequo if both their two end points are identical. $T_0$ represents the order of all end points. Theorem 3.2 shows that the equivalence classes of left and right end points are alternative, i.e., after a class of left end points there is a class of right end points.

Theorem 3.3 Let $(P, I)$ be an interval order on a finite set $A$, and $T_i, T_r, T_0$ defined as above, then

i) $A/T_i$ and $A/T_r$ have the same cardinality, say $m$;

ii) If $A/T_i = \{A_m T_0 A_{m-1} T_0 ... T_0 A_1\}$
and $A/T_r = \{B_m T_0 B_{m-1} T_0 ... T_0 B_1\}$ then
$(A_i \cup A_r/T_0) = \{B_m, A_m, ..., B_1, A_1\}$,
and
$B_m T_0 A_m T_0 B_{m-1} T_0 A_{m-1} ... T_0 B_1 T_0 A_1$


The construction of the minimal $\epsilon$-representation of an interval order is direct from theorems 2.3, 3.3. The number $m$ is called magnitude of the interval order. With $\epsilon = 1$, the minimal 1-representation is a representation on the smallest possible interval of the set of integer numbers.

Example 3.1

Let's consider the following $(P, I)$ interval order.

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<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
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<tbody>
<tr>
<td>a</td>
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<td>P</td>
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<td>P</td>
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<td>P</td>
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<tr>
<td>b</td>
<td>I</td>
<td>P</td>
<td>P</td>
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</tr>
</tbody>
</table>

$A/T_i = \{A_1 = \{a, v, g, c\}, A_2 = \{f\}, A_3 = \{d, c\}, A_4 = \{b\}, A_5 = \{a_1\}\}$

$A/T_r = \{B_1 = \{s\}, B_2 = \{e, f\}, B_3 = \{d, r\}, B_4 = \{c, b\}, B_5 = \{a_1\}\}$

The 1-minimal representation of the interval order is:
4 PQI interval orders

First, we recall some definitions and fundamental results concerning PQI interval orders.

Definition 4.1 A PQI preference structure on a finite set $A$ is a PQI interval order iff $\exists: I, r: A \rightarrow \mathbb{R}^+$, such that $\forall x, y \in A$:
- $r(x) \geq l(x)$;
- $P(x, y) \leftrightarrow l(x) > r(y)$;
- $Q(x, y) \leftrightarrow r(x) > r(y) \geq l(x) > l(y)$;
- $I(x, y) \leftrightarrow r(x) \geq r(y) \geq l(x), l(y)$, or $r(y) \geq r(x) > l(x), l(y)$.

A couple $(I, r)$ satisfying these conditions is a general representation of the PQI interval order.

Theorem 4.1 A $(P, Q, I)$ preference structure on a finite set $A$ is a PQI interval order iff there exists a partial order $I_t$ such that:
- $I = I_t \cup I_r \cup I_0$ where $I_0 = \{(x, x), x \in A\}$ and $I_r = I_t^{-1}$;
- $(P \cup Q \cup I_0) P \subset P$;
- $(P \cup Q \cup I_r) P \subset P$;
- $(P \cup Q \cup I_t) Q \subset P \cup Q \cup I_t$;
- $(Q \cup Q \cup I_r) Q \subset P \cup Q \cup I_r$.


An algorithm to detect a PQI interval order, i.e., to construct $I_t$, was presented in Ngo The et al., 2000. In this paper, we assume that $I_t$ is known. From the above results, $I_t(x, y)$ iff $r(y) \geq r(x) \geq l(x) \geq l(y)$, and $I_r(x, y)$ iff $r(y) \geq r(x) \geq l(y) \geq l(x)$ with at least one strict inequality.

Since $A$ is finite, there exists

$$\epsilon = \min \{ \min_{(a,b) \in P} \{l(a) - r(b)\}, \min_{(a,b) \in Q} \{r(a) - r(b), l(a) - l(b)\} \}$$

The triple $(I, r, \epsilon)$ is called an $\epsilon$-representation of the PQI interval order. With an $\epsilon$-representation, conditions ii, iii of definition 4.1 can be rewritten as $P(x, y) \leftrightarrow l(x) \geq l(y) + \epsilon$ and $Q(x, y) \leftrightarrow r(x) \geq r(y) + \epsilon$ and $r(y) \geq l(x) \geq l(y) + \epsilon$. The minimal $\epsilon$-representation of a PQI interval order is defined similarly to that of the interval order.

The following theorem presents the interval order associated to a PQI interval through the reduction of the two relations $I, Q$ into $I_t$. 

\begin{tabular}{|c|c|c|c|c|c|c|c|} 
\hline
$a$ & $b$ & $c$ & $d$ & $e$ & $f$ & $g$ & $h$ \\
\hline
l & 5 & 4 & 3 & 1 & 2 & 1 & 1 \\
\hline
r & 5 & 4 & 3 & 2 & 2 & 2 & 3 \\
\hline
\end{tabular}
Theorem 4.2 If \((P, Q, I)\) is a PQI interval order and \(\hat{I} = I \cup Q \cup Q^{-1}\) then \((P, \hat{I})\) is an interval order.


Let's define the following relations:
\[
\hat{T}_l = P \hat{I}; \\
\hat{T}_r = \hat{I} P;
\]
We introduce two copies of \(A\), say \(A_l\) and \(A_r\) and we construct the relation \(\hat{T}_0\) on \(A_l \cup A_r\) as follows:
\[
\hat{T}_0(a_l, b_r) \iff \hat{T}_1(a_l, b_r), \\
\hat{T}_0(a_r, b_r) \iff \hat{T}_r(a_r, b_r), \\
\hat{T}_0(a_l, b_l) \iff P(a_l, b_l), \\
\hat{T}_0(a_r, b_l) \iff \lnot P(b_l, a_r).
\]
Since \((P, \hat{I})\) is an interval order, we can apply theorems 3.1, 3.2, and 3.3 for the relations \(\hat{T}_l, \hat{T}_r, \hat{T}_0\). We obtain:
\[
- m = |A_l/\hat{T}_l^-| = |A_r/\hat{T}_r^-| \text{ the magnitude of the interval order } \langle P, \hat{I} \rangle; \\
- (A_l \cup A_r)/\hat{T}_0^+ = (A_l/\hat{T}_l^+) \cup (A_r/\hat{T}_r^+); \\
- A_l/\hat{T}_l^- = \{A_m \hat{T}_0 A_{m-1} \hat{T}_0 ... A_1\}; \\
- A_r/\hat{T}_r^- = \{B_m \hat{T}_0 B_{m-1} \hat{T}_0 ... B_1\}; \\
- B_m \hat{T}_0 A_m \hat{T}_0 B_{m-1} \hat{T}_0 A_{m-1} ... \hat{T}_0 B_1 \hat{T}_0 A_1.
\]
We extend now the relations \(\hat{T}_l, \hat{T}_r, \hat{T}_0\) into \(T_l, T_r, T_0\) as follows:
\[
Q_l = Q \cup I, Q \cup Q, I \cup I, Q, I; \\
Q_r = Q \cup I, Q \cup Q, I \cup I, Q, I; \\
T_l = \hat{T}_l \cup Q_l; \\
T_r = \hat{T}_r \cup Q_r; \\
T_0(a_l, b_r) \iff \hat{T}_1(a_l, b_r), \\
T_0(a_r, b_r) \iff \hat{T}_r(a_r, b_r), \\
T_0(a_l, b_l) \iff P(a_l, b_l), \\
T_0(a_r, b_l) \iff \lnot P(b_l, a_r).
\]
It is obvious that \(T_0 \subseteq T_0\), as \(\hat{T}_l \subseteq T_l\) and \(\hat{T}_r \subseteq T_r\).

Proposition 4.1 Let \((P, Q, I)\) be a PQI interval order on a finite set \(A\), and let \(I_l, I_r, \hat{T}_l, \hat{T}_r, Q_l, Q_r, T_l, T_r\) defined as above. Then
\[
i) Q_l \subseteq Q \cup I_l \text{ and } I_r, Q \subseteq I_r \cup Q; \\
ii) P \subseteq P \cup Q \cup I_l \text{ and } I_r, P \subseteq P \cup Q \cup I_r; \\
iii) P \subseteq (P \cup Q \cup I_l) \text{ and } Q^{-1}, P \subseteq (P \cup Q \cup I_r); \\
iv) Q_l \cap \hat{T}_l = Q_r \cap \hat{T}_r = \emptyset; \\
v) P \cup Q \subseteq T_l \cup I_l \cup P \cup Q \text{ and } P \cup Q \subseteq T_r \cup I_r \cup P \cup Q; \\
w) (P^{-1} \cup Q^{-1} \cup I_l) \subseteq \hat{T}_l \subseteq (P^{-1} \cup Q^{-1} \cup I_l \cup I_r), \text{ and } (P^{-1} \cup Q^{-1} \cup I_l) \subseteq \hat{T}_r \subseteq (P^{-1} \cup Q^{-1} \cup I_l \cup I_r).
\]
vii) \( T_1 \subseteq P \subseteq P \) and \( P \subseteq T_r \)

viii) \( P \subseteq T_1 \) and \( T_r \subseteq P \)

**Proof**

We provide only the proofs for \( T_1 \) (those of \( T_r \) are similar).

i) \( aQbIc \Rightarrow [(r(a) > r(b)) \geq l(a) > l(b)] \) and \( (r(c) > r(b) \geq l(b) \geq l(c))] \Rightarrow (c) \geq l(a) > l(c) \Rightarrow (a, c) \in Q \cup I_1.

ii) \( aPbIc \Rightarrow [(l(a) > r(b)) \text{ and } (r(c) > r(b) \geq l(b) \geq l(c))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup I_1.

iii) \( aPbQ^{-1}c \Rightarrow [(l(a) > r(b)) \text{ and } (r(c) > r(b) \geq l(c) \geq l(b))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup I_1.

iv) Otherwise, \( \exists x, (x, x) \in (Q \cup I_1.Q \cup Q.Q \cup I_1.Q.I_1) \). By theorem 4.1 and i, we have \((Q \cup I_1.Q \cup Q.Q \cup I_1.Q.I_1) \subseteq (Q \cup P \cup I_1).P \subseteq P.

v) As \( P \subseteq P.I \subset \hat{T}_1 \subseteq T \) and \( Q \subseteq T \), then \( P \cup Q \subseteq T \).

vi) Direct consequence of v.

vii) \( T_1 \subseteq P \subseteq P \)

\[ T_1 = PI_1.P \cup Q,P \cup I_1.Q.P \cup Q,I_1.P \cup I_1.Q.I_1 \subseteq P \] (as \( I_1.P \subseteq P \) and \( Q.P \subseteq P \)).

viii) \( P.T_1 \subseteq T_1 \) and \( T_r.P \subseteq T_r \)

\[ P.T_1 = P.P.I_1.P.P.Q.P \cup Q,I_1.P \cup Q.I_1.P \cup I_1.Q.I_1 \subseteq P \] (by i, ii). Therefore, \( T_1 \subseteq P \cup Q \cup I_1 \).

For the construction of the minimal \( e \)-representation of a PQI interval order, we will extend theorems 3.1, 3.2, 3.3 using \( T_1, T_r, T_0 \).

**Theorem 4.3** Let \( (P, Q, I) \) be a PQI interval order on a finite set \( A \), and let \( T_1, T_r \) be defined as above. Then

i) \( T_1, T_r \) are weak orders on \( A \);

ii) \( T_1, T_r \) are equivalence relations

and \( T_1, T_r \) are linear orders on \( A/T_1, A/T_r \);

iii) \( T_1 \cap T_r \subseteq E \).

iv) \( \forall a \in A : [a]_{T_1} \subseteq [a]_{T_r} \text{ and } [a]_{T_r} \subseteq [a]_{T_r} \).

9
Proof We consider only $T_I$ ($T_r$ is similar).

i) We show that $T_I$ is asymmetric and negatively transitive.

- **Asymmetry.** We recall that if $R, S$ are two asymmetric relations and $R \cap S^{-1} = \emptyset$ then $R \cup S$ is asymmetric.
  $P, Q, I_t$ are asymmetric and mutually exclusive \(\Rightarrow (P \cup Q \cup I_t)\) is asymmetric \(\Rightarrow Q_t \subset (P \cup Q \cup I_t)\) is asymmetric too.
  As $\hat{T}_I$ and $Q_t$ are asymmetric, furthermore $Q_t \cap \hat{T}_I^{-1} = \emptyset$ (proposition 4.1.iv), $T_I$ is asymmetric.

- **Negative transitivity.** We recall that the formula:
  \[\forall x, y, z \models -T_I(a, b) \wedge -T_I(b, c) \rightarrow -T_I(a, c)\]
  can be reformulated (through simple logical equivalences) as:
  \[\forall x, y, z \models -T_I(b, c) \wedge T_I(a, c) \rightarrow T_I(a, b).\]
  We will demonstrate this second formulation.

By proposition 4.1.vi, we have $(b, c) \in -T_I \Rightarrow (c, b) \in (P \cup Q \cup I_I \cup I_r)$.

Since $T_I \subset P \cup Q \cup I_I$, we consider three cases.

1 - $(a, c) \in P$. Then, if $(c, b) \in (P \cup Q \cup I_I)$, we have $(a, b) \in P \subset T_I$. If $(a, b) \in I_I$ then $(a, b) \in P, I_I \subset \hat{T}_I \subset T_I$.

2 - $(a, c) \in Q$. Then, if $(c, b) \in (P \cup Q)$ we have $(a, b) \in (P \cup Q) \subset T_I$. If $(c, b) \in I_r \Rightarrow (a, b) \in Q, I_I \subset T_I$. If $(a, b) \in I_r \Rightarrow (a, b) \in (P \cup Q \cup I_r)$. If $(a, b) \in (P \cup Q) \Rightarrow (a, b) \in T_I$, otherwise, \((b, a) \in I_I \Rightarrow (b, c) \in I_r, Q \subset T_I\), impossible as $(b, c) \in -T_I$.

3 - $(a, c) \in (T_I \setminus (P \cup Q)) \subset I_r$. We also have $(a, c) \in \hat{T}_I, Q \cup I_I, Q \cup I_I, Q \cup I_I \setminus (P \cup Q) \subset P, Q, I_I \cup I_r, Q, I_I \cup I_r, Q, I_I$ (theorem 4.1, proposition 4.1).

Let's consider different possibilities of $(c, b)$.

- $(c, b) \in (P \cup Q)$ then $(a, b) \in (I_r, P \cup I_I, Q \subset (P \cup I_I, Q) \subset T_I$.

- $(c, b) \in I_I$.
  We have $(a, b) \in (P, Q^{-1} \cup P, I_I \cup I_I, Q \cup Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I $.$ I_I \subset (P, Q^{-1} \cup P, I_I \cup I_I, Q \cup Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I$.
  By proposition 4.1, $I_r \subset I_r \cup Q \Rightarrow Q^{-1} \cup I_r \subset Q^{-1} \cup I_r$.
  Therefore, $P, Q^{-1}, I_r \subset P, Q^{-1} \cup P, I_I \subset T_I$.
  $P, I_I \subset P, I_I \subset T_I$.
  $I_r, Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I \subset I_r, Q, I_I \cup I_I, Q, I_I \cup I_I, Q, I_I \subset T_I$.

- $(c, b) \in I_r$.
  We consider five possibilities for $(a, c)$.

  * $(a, c) \in P, Q^{-1} \Rightarrow \exists x \in A, s.t. (aP_xQ^{-1}c)$ and $bI_ncQx \Rightarrow (b, x) \in (P \cup Q \cup I_I)$.
If \((b, x) \in P \Rightarrow bPxQ^{-1}c \Rightarrow (b, c) \in T_1\), impossible as \((b, c) \in -T_1\).
If \((b, x) \in Q \Rightarrow aPxQ^{-1}b \Rightarrow (a, b) \in T_1\).
If \((b, x) \in I_1 \Rightarrow aPxI_1b \Rightarrow aPb \Rightarrow aT_1b\).

* \((a, c) \in P, I_1 \Rightarrow \exists x \in A, \text{s.t.} (aPxI_1c)\)
  If \((b, x) \in P \Rightarrow bPxQ^{-1}c \Rightarrow (b, c) \in T_1\), impossible.
  If \((b, x) \in (Q \cup P^{-1} \cup Q^{-1}) \Rightarrow aPx(P \cup (Q \cup Q^{-1} \cup I_1)b) \Rightarrow aPx(P \cup I_1)b \Rightarrow (a, b) \in (P, P \cup T_1) \subset T_1\).

* \((a, c) \in I_1, Q \Rightarrow \exists x \in A, \text{s.t.} (aIxQc)\)
  If \((b, x) \in (P \cup Q \cup I_1) \Rightarrow (b, c) \in (P \cup Q \cup I_1)Q \subset (P \cup Q \cup I_1, Q) \subset T_1\), impossible.
  If \((b, x) \in I_1 \Rightarrow xI_1bI_1c \Rightarrow xI_1c\), impossible as \((x, c) \in Q\).
  If \((x, b) \in (P \cup Q) \Rightarrow aIxQ(P \cup Q)b \Rightarrow (a, b) \in I_1, P \cup I_1, Q \subset P \cup I_1, Q \subset T_1\).

* \((a, c) \in Q, I_1 \Rightarrow \exists x \in A, \text{s.t.} (aQxI_1c)\)
  If \((b, a) \in (P \cup Q \cup I_1) \Rightarrow (b, c) \in (P \cup Q \cup I_1)Q, I_1 \subset (P \cup Q \cup I_1, Q, I_1)\). But we have \(P, Q, I_1 \subset P, I_1 \subset T_1\), and \(Q, Q, I_1 \subset (P \cup Q), I_1 \subset T_1\) and \(I_1, Q, I_1 \subset T_1\), then \((b, c) \in T_1\), impossible.
  If \((b, a) \in I_r \Rightarrow (b, x) \in I_r, Q \subset I_r \cup Q\). If \((b, x) \in I_r \Rightarrow (x, b) \in I_1 \Rightarrow aQxI_1b \Rightarrow (a, b) \in T_1\), otherwise \((b, x) \in Q \Rightarrow bQxI_1c \Rightarrow (b, c) \in T_1\), impossible.
  Therefore \((a, b) \in P \cup Q \subset T_1\).

* \((a, c) \in I_1, Q, I_1 \Rightarrow \exists x, y \in A, \text{s.t.} (aIxQyI_1c)\)
  If \(b(P^{-1} \cup Q^{-1})a \Rightarrow aT_1b\).
  If \((b, a) \in (P \cup Q \cup I_1) \Rightarrow (b, c) \in (P \cup Q \cup I_1)Q, I_1 \subset P, I_1, Q, I_1 \subset Q, I_1 \cup I_1, Q, I_1 \subset P, I_1, Q, I_1 \subset P, I_1 \cup Q, I_1 \subset T_1\). We have \(I_1, Q, I_1 \subset T_1\), and \(P, (P \cup Q \cup I_1), I_1 \subset Q, P, I_1 \cup Q, I_1 \subset T_1\). If \((b, a) \in I_r, \text{we continue to consider the five possibilities of} (b, y)\).

  * \(b(P \cup Q)yI_1 \Rightarrow (b, c) \in T_1\), impossible.
  * \(bHy\).
    If \(b(P \cup P^{-1})x\) then we have either \((bPa)\), impossible as \((bI_1a)\), or \((xPa)\), impossible as \(xI_1a\).
    If \(b(Q \cup I_1)xQyI_1c \Rightarrow (b, c) \in Q, Q, I_1 \cup I_1, Q, I_1 \subset T_1\), impossible.
    If \(bQ^{-1}x \Rightarrow aI_1xQb \Rightarrow aT_1b\).
    If \(bI_1x \Rightarrow xI_1bI_1y \Rightarrow xI_1y\), impossible as \(xQy\).
  * \(b(P^{-1} \cup Q^{-1})y \Rightarrow y(P \cup Q)b \Rightarrow aI_1xQy(P \cup Q)b \Rightarrow aI_1xQyI_1b\).
\[(a, b) \in I_i Q \cup I_i Q \subseteq I_i P \cup I_i Q \subseteq T_i. \]
\[b \in y \implies y \in T_i \implies y \in T_i x \implies y \in T_i x y \implies y \in T_i x. \]

ii) Immediate from theorems 2.1, 2.2 and i.

iii) \(T_i^{-1} \cap T_r^{-1} \subseteq E.\) If \((x, y) \in T_i^{-1} \cap T_r^{-1} \implies (x, y) \notin T_1 \cup T_1^{-1} \cup T_r \cup T_r^{-1}.\)
Suppose that \((x, y) \notin E\) then \(\exists z \in A, z R_1 x \text{ and } z R_2 y \) with \(R_1 \neq R_2.\)
Consider, for example, \(R_1 = P,\) we have:
\(z P^{-1} y \Rightarrow y P z P z \Rightarrow y T_i x,\) impossible.
\(z Q y \Rightarrow z Q^{-1} z P x \Rightarrow y T_r x,\) impossible.
\(z Q^{-1} y \Rightarrow y Q z P x \Rightarrow y T_i x,\) impossible.
The other cases are quite similar.

iv) Immediate from \(T_i \subseteq T_i \) and \(T_r \subseteq T_r.\)

\[\square\]

Remark 4.1 In general, we don't have \(E = T_i^{-1} \cap T_r^{-1}.\) For example, consider the following PQI interval order:

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<tr>
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One of its representations is:

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We have \((a, b) \notin E \setminus T_i^{-1} \cap T_r^{-1}.\)
If we assume the absence of ex aequo, then obviously, \(T_i^{-1} \cap T_r^{-1} = E = T_0.\)

Theorem 4.4 Let \((P, Q, I)\) be a PQI interval order on a finite set \(A,\) and \(\tilde{T}_i, \tilde{T}_r, \tilde{T}_0, T_i, T_r, T_0\) defined as above, then
i) \(T_0\) is a weak order on \((A_1 \cup A_r);\)
ii) \(T_0^{-1}\) is an equivalence relation and \(T_0\) is a linear order on \((A_1 \cup A_r)/T_0^{-1};\)
iii) \((A_1 \cup A_r)/T_0^{-1} = (A_1/T_1^{-1}) \cup (A_r/T_r^{-1});\)
Proof.

i) We first demonstrate that $T_0$ is asymmetric and negatively transitive.

- **Asymmetry.**
  $T_0 = (T_0 \cap A_l \times A_l) \cup (T_0 \cap A_r \times A_r) \cup (T_0 \cap (A_l \times A_r \cup A_r \times A_l)),$
  where $(T_0 \cap A_l \times A_l)$ (resp. $(T_0 \cap A_r \times A_r)$) is in fact isomorphic to $T_l$ (resp. $T_r$). As each component of $T_0$ is asymmetric and belongs to, respectively, $A_l \times A_l, A_r \times A_r, A_l \times A_r \cup A_r \times A_l$ which are mutually exclusive, $T_0$ is asymmetric.

- **Negative transitivity.**
  $\neg T_0(x, y), \neg T_0(y, z).$ We have the following case with $x, y, z.$

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<th>x</th>
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We only have to provide a proof for cases 1, 2, 3, 4.

* Case 1: $a_l \neg T_0 b_l \neg T_0 c_l \Rightarrow a_l \neg T_1 b_l \neg T_1 c_l$ (by definition).
  $\Rightarrow a_l \neg T_1 c_l,$ ($T_1$ is a weak order).
  $\Rightarrow a_l \neg T_0 c_l,$ (by definition).

* Case 2: $a_l \neg T_0 b_l \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$
  i.e. $a_l \neg T_1 b_l \neg P(b, c) \Rightarrow \neg P(a, c)$
  i.e. $P(a, c), \neg P(b, c) \Rightarrow T_1(a, b)$
  where $P = P^{-1} \cup Q \cup Q^{-1} \cup \hat{r} = P^{-1} \cup \hat{r}$
  $P(a, c), \neg P(b, c) \Rightarrow (a, b) \in P \cup \hat{r} \subset T_1$

* Case 3: $a_l \neg T_0 b_r \neg T_0 c_l \Rightarrow a_l \neg T_0 c_l$
  i.e. $\neg P(a, b), P(c, b) \Rightarrow \neg T_1(a, c)$
  i.e. $T_1(a, c), P(c, b) \Rightarrow P(a, b)$
  $T_1(a, c), P(c, b) \Rightarrow (a, b) \in T_1 \cup P \subset P,$ (by proposition 4.1.vii).

* Case 4: $a_l \neg T_0 b_r \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$
  i.e. $\neg P(a, b), \neg T_1(b, c) \Rightarrow \neg P(a, c)$
  Similar to case 2.

ii) Immediate from theorems 2.1, 2.2 and i.
iii) Consider $[x]_{T_i^-}$, $x \in A_t \cup A_r$. We will demonstrate that “if $x = a_t(x = a_r)$ for some $a \in A$ then $[x]_{T_i^-} = [a_t]_{T_i^-} = [a_r]_{T_i^-}$”. By construction of $T_0$, if $\neg T_0(x, y)$ and $\neg T_1(y, x)$ then $(x, y) \notin A_t \times A_r \cup A_r \times A_t$.

Suppose now that $x = a_t$, if $y \in [x]_{T_i^-}$ then $y = b_t$ for some $b \in A$, and $\neg T_0(a_t, b_t)$ and $\neg T_1(b_t, a_t)$$\iff \neg T_1(a_t, b_t)$ and $\neg T_1(b_t, a_t) \iff b_t \in [a_t]_{T_i^-}$.

The case $x = a_r$ is similar.

\[\square\]

**Theorem 4.5** Let $(P,Q,I)$ be a PQI interval order on a finite set $A$, and $\bar{A}, \bar{T}, \bar{I}, \bar{A}_t, \bar{T}_t, \bar{T}_r, \bar{T}_0, m = |A|/|\bar{T}_i|, l = |A|/|\bar{T}_r|, r = |A|/|\bar{T}_r|, \bar{A}/\bar{I}^- = \{A_i, i = 1..m\}, \bar{A}/\bar{T}_{r}^- = \{B_i, i = 1..m\}$ defined above, then

i) classes of $A_t/\bar{T}_i^-, A_r/\bar{T}_r^-$ can be arranged in such a way that

$A_t/\bar{T}_i^- = \{X_{1t}, X_{0t}, X_{1t-1}, T_0, \ldots, X_{1}, X_{0}, X_{1-1}, T_0, X_{1-2}, T_0, \ldots, X_{1}, \ldots\}$

$A_r/\bar{T}_r^- = \{Y_{1r}, Y_{0r}, Y_{1r-1}, T_0, \ldots, Y_{1}, Y_{0}, Y_{1-1}, T_0, Y_{1-2}, T_0, \ldots, Y_{1}, \ldots\}$

ii) with this arrangement, the linear order $T_0$ on $(A_t \cup A_r)/\bar{T}_0^-$ becomes:

$Y_{1}, T_0, Y_{1r-1}, T_0, X_{1t}, T_0, X_{1t-1}, T_0, \ldots, X_{1t}, T_0, \ldots, Y_{1}, T_0, X_{1m-1}, T_0, X_{1m-2}, T_0, \ldots, X_{1}$.

**Proof.**

i) Immediate from $\forall a \in A, [a]_{\bar{T}_i^-} \subseteq [a]_{\bar{T}_r^-}, \bar{I} \subseteq \bar{T}$.

ii) Immediate from i and theorem 3.3.

We can now arrange the elements of $(A_t \cup A_r)/\bar{T}_0^-$ using its rank function (i is the rank of $Z_i$).

$(A_t \cup A_r)/\bar{T}_0^-, T_0 = \{Z_{i+r}, T_0, Z_{i+r-1}, T_0, \ldots, Z_1\}$.

The relation between $T_0$ and an $\varepsilon$-representation is shown in the following proposition.
Proposition 4.2 Let \((l, r, \varepsilon)\) be an \(\varepsilon\)-representation of a PQI interval order on a finite set. We have:

i) \(T_0(a_1, b_1) \Rightarrow l(a) \geq l(b) + \varepsilon;\)

ii) \(T_0(a_r, b_r) \Rightarrow r(a) \geq r(b) + \varepsilon;\)

iii) \(T_0(a_1, b_r) \Rightarrow l(a) \geq r(b) + \varepsilon;\)

iv) \(T_0(a_r, b_1) \Rightarrow r(a) \geq l(b);\)

Proof.

i) \(T_0(a_1, b_1) \Rightarrow T_1(a, b) \Rightarrow (a, b) \in P \cup P.Q \cup P.Q^{-1} \cup P.I_1 \cup P.I_r \cup Q \cup I_1.Q \cup Q.I_1 \cup I_1.Q.I_1 \subset P \cup Q \cup P.I_1 \cup P.I_r \cup Q.I_1 \cup Q.I_1 \cup I_1.Q.I_1.\) If \(aPb\) then \(l(a) \geq r(b) + \varepsilon \geq l(b) + \varepsilon.\)

If \(aQb\) then \(l(a) \geq l(b) + \varepsilon.

If \(aPcQb\) then \(l(a) \geq r(c) + \varepsilon \geq l(c) + \varepsilon \geq l(b) + \varepsilon.

If \(aIcQb\) then \(l(a) \geq l(c) \geq l(b) + \varepsilon.

If \(aQcQb\) then \(l(a) \geq l(c) + \varepsilon \geq l(b) + \varepsilon.

If \(aIcQdQb\) then \(l(a) \geq l(d) + \varepsilon \geq l(b) + \varepsilon.

ii) Similar to i.

iii) \(T_0(a_1, b_r) \iff P(a, b) \Rightarrow l(a) \geq r(b) + \varepsilon.\)

iv) \(T_0(a_r, b_1) \iff \neg P(b, a) \Rightarrow r(a) \geq l(b).\)

The construction of the minimal \(\varepsilon\)-representation of a PQI interval order is a direct consequence of proposition 4.2 and theorems 4.5, 4.4.

Corollary 4.1. Given a PQI interval order on a finite set \(A\) and a positive constant \(\varepsilon\), let define

- \(l^*(a) = (i - j + 1)\varepsilon\) where \(a_i \in Z_i \subset A_j;\)

- \(r^*(a) = (i - j)\varepsilon\) where \(a_i \in Z_i \subset B_j;\)

where \(A_j, B_j, Z_i\) defined as above.

Then \((l^*, r^*, \varepsilon)\) is the minimal \(\varepsilon\)-representation of the PQI interval order and the values of \(l^*\) and \(r^*\) are integral multiples of \(\varepsilon\).

Proof.

We consider the valued graph \(G = ((A_1 \cup A_r)/T_0^-, T_0, v)\) where \(v\) is defined as follows:

\[
v(x, y) = \begin{cases} 
0 & \text{if } x = [a_r], y = [b_1] \text{ for some } a, b \in A \\
\varepsilon & \text{otherwise}
\end{cases}
\]

Since \(T_0\) is a linear order \(\Rightarrow\) there is no circuit \(\Rightarrow\) there exists a potential function (Theorem 2.3). We will prove that the maximal value of the paths
starting from a node \( a_l \) (being also the smallest potential function) is:

\[
g(a_l) = t^*(a) \\
g(a_r) = r^*(a)
\]

The nodes of \( G \) can be presented as \( Z_l T_0 T_{l+r-1} T_0 \ldots Z_1 \). Let’s remind that \( Z_i T_0 Z_j \) iff \( i \geq j \) and all the arcs of \( G \) are either 0 or \( e > 0 \). By proposition 4.2 and theorem 4.5, in two consecutive arcs, there is at least one arc with value \( e \).

For each \( Z_k \), consider the path \( \Phi = Z_k T_0 Z_{k-1} \ldots T_0 Z_1 \) and denote \( V(\Phi) \) its value. Any other path \( \Phi' \) starting from \( Z_k \) is obtained from \( \Phi \) by applying (recursively) the following operation:
- drop out the last arc \( (x, y) \), obviously \( V(\Phi) \geq V(\Phi') \) \( (v(x, y) \geq 0) \).
- replacing a portion \( (Z_i, Z_{i-1}, \ldots, Z_j) \) by \( (Z_i, Z_j) \). As \( V(Z_i, Z_j) \leq e \) and \( V(Z_i, Z_{i-1}, \ldots, Z_j) \geq e \) then \( V(\Phi) \geq V(\Phi') \).

Therefore, \( \Phi \) is the path with maximal value starting from \( Z_k \).

By theorem 4.5, along \( \Phi \), all the arcs are \( e \), except \( (a_r, b_t) \) which are transitive arcs connecting \( B_j \) to \( A_j \). If \( Z_i = a_l \in A_j \), then there are \( (j-1) \) transitive arcs \( \Rightarrow V(\Phi) = (i-j+1) \ast e \). If \( Z_i = a_r \in B_j \), then there are \( j \) transitive arcs \( \Rightarrow V(\Phi) = (i-j) \ast e \).

\[ M = l + r - m \]

is called the magnitude of the PQI interval order as in the minimal \( e \)-representation of a PQI interval order, the leftmost endpoint is \( e \) and the rightmost endpoint is \( M e \). It is easily to verify that when \( l = r = m \), then \( Q = \emptyset \), the preference structure in question is an interval order, and its magnitude is \( M = m \).

**Example 4.1** Let’s consider the following PQI interval order (L stand for \( f_l \))

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
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<tr>
<td>b</td>
<td>Q</td>
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<td>P</td>
<td>P</td>
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<td>P</td>
<td>P</td>
</tr>
<tr>
<td>c</td>
<td>Q</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>Q</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>L</td>
<td></td>
<td></td>
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<tr>
<td>e</td>
<td></td>
<td>Q</td>
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<td>L</td>
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<td></td>
</tr>
<tr>
<td>f</td>
<td></td>
<td>Q</td>
<td>L</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>g</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>h</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We have \( m = 5, l = 7, r = 7, M = l + r - m = 9 \).

\[ A(T)^*_{l+r} = \{A_1 = \{b_l, g_l, f_l\}, A_2 = \{c_l\}, A_3 = \{d_l, c_l\}, A_4 = \{b_l, A_5 = \{a_l\}\} \]

\[ A(T)^*_{l+r} = \{B_1 = \{g_r\}, B_2 = \{f_r, c_r\}, B_3 = \{d_r, h_r\}, B_4 = \{c_r, b_r\}, B_5 = \{a_r\}\} \]
$A/T_0 = \{X_1 = \{h_1, g_1\}, X_2 = \{f_1\}, X_3 = \{c_1\}, X_4 = \{d_1\}, X_5 = \{a_1\}, X_6 = \{k_1\}, X_7 = \{a_1\}\} $

$A/T_0^* = \{Y_1 = \{g_r\}, Y_2 = \{f_r\}, Y_3 = \{e_r\}, Y_4 = \{d_r, h_r\}, Y_5 = \{e_r\}, Y_6 = \{b_r\}, Y_7 = \{a_r\}\}$

After the reassignment of indices:

$Z_1 = \{h_1, g_1\}, Z_2 = \{f_1\}, Z_3 = \{g_r\}, Z_4 = \{e_1\}, Z_5 = \{f_r\}, Z_6 = \{e_r\}, Z_7 = \{d_1\}, Z_8 = \{a_1\}, Z_9 = \{d_r, h_r\}, Z_{10} = \{b_1\}, Z_{11} = \{e_r\}, Z_{12} = \{b_r\}, Z_{13} = \{a_1\}, Z_{14} = \{a_r\}\$

The 1-minimal representation of the PQI interval order is:

<table>
<thead>
<tr>
<th>$I^*$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i^*$</td>
<td>9</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r^*$</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

5 Algorithms

A direct application of the above results in order to determine a minimal $\epsilon$-representation of a PQI interval order is rather complicated as it has to pass by the determination of $T_1, T_r, T_l, T_r, T_0, (A_l \cup A_r)/T_0^*$. In this section, we will present some more results allowing us to determine a minimal $\epsilon$-representation using two algorithms. The first algorithm (in $O(n^2)$) determines a numerical representation where all endpoints are distinct. The endpoints which should be identical will be unified in the second algorithm (in $O(n)$) to obtain a minimal $\epsilon$-representation.

**Proposition 5.1** Let $(P, Q, I)$ be a PQI interval order on a finite set $A$, $\{I, r\}$ be a representation with all distinct endpoints, $B = \{I(x), r(x), x \in A\}$, the relation $T$ defined on $(A_l \cup A_r)$ as:

- $T(a_r, a_l)$;
- $T(a_l, b_l) \leftrightarrow \langle P(a, b) \rangle$ or $Q(a, b)$ or $I_l(a, b)$;
- $T(a_r, b_r) \leftrightarrow \langle P(a, b) \rangle$ or $Q(a, b)$ or $I_r(a, b)$;
- $T(a_l, b_r) \leftrightarrow \langle P(a, b) \rangle$;
- $T(a_r, b_l) \leftrightarrow \langle P(a, b) \rangle$.

Then:

i) $T_0 \subseteq T$, i.e. $T$ is an extension of $T_0$.
ii) $(A_l \cup A_r, T)$ is a linear order and an isomorphism of the linear order $(B, \rangle)$.

**Proof.**

i) $(x, y) \in T_0$.

If $x = a_l, y = b_l$ then $(a, b) \in T_1 \subseteq P \cup Q \cup I_r$ then $T(x, y)$. The same
argument for \( x = a_r, y = b_r \).

By construction of \( T \) and \( T_0 \), if \( x = a_1, y = b_r \) or \( x = a_r, y = b_1 \) then \( T(x, y) \).

ii) It is obvious that \((B, >)\) is a linear order as \( l(x), r(x) \) have all distinct values. With the mapping \( \phi : A_I \cup A_r \mapsto B \) defined as: \( \phi(a_1) = l(a), \phi(a_r) = r(a), \forall a \in A \), it is easily to verify that \( \phi \) is a bijection and \( T(x, y) \iff \phi(x) > \phi(y) \).

We can consider now the valued graph \((A_I \cup A_r, T, v)\) where \( v(x, y) = \epsilon, \forall x, y \in A \). It is obvious that \( (I(a) = \epsilon \times g(a_1), r(a) = \epsilon \times g(a_r), \epsilon) \), where \( g \) is the rank function of \( C \), is a minimal representation with all endpoints distinct.

From proposition 5.1, we have:
\[
\forall a_1 \in A_I : T^+(a_1) = \{x_l, x_r : P(a, x), x \in A\} \cup \{x_l : Q(a, x), x \in A\} \cup \{x_l : I_l(a, x), x \in A\};
\]
\[
\forall a_r \in A_r : T^+(a_r) = \{x_l, x_r : P(a, x), x \in A\} \cup \{x_l, x_r : Q(a, x), x \in A\} \cup \{x_l : I_r(a, x), x \in A\};
\]

This result leads us to the following formula of the rank function:
\[
\forall a \in A,
\]
\[
g(a_1) = |T^+(a_1)| + 1 = 2|P^+(a)| + |Q^+(a)| + |I^+_l(a)| + 1;
\]
\[
g(a_r) = |T^+(a_r)| + 1 = 2 + 2|P^+(a)| + 2|Q^+(a)| + |Q^{-1}_l(a)| + 1 + 2|I^+_r(a)|.
\]

The function \( g \) can be implemented using the following algorithm whose complexity is \( \frac{n(n-1)}{2} \), i.e. \( O(n^2) \).

```plaintext
n=|A|
f1[1..n], fr[1..n] /* g(a), g(a) */
M[1..n, 1..n]; /* matrix representing P, Q, I */
procedure numerical_representation
for all i
    f1[i]=0
    fr[i]=1
endfor
for all i, j, j > i
    switch (M[i,j])
        case P:
            f1[i]=f1[i]+2
            fr[i]=fr[i]+2
        case P^{-1}:
            f1[j]=f1[j]+2
            fr[j]=fr[j]+2
```

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case \( Q \):
  \( f_1[i] = f_1[i] + 1 \)
  \( f_1[i] = f_1[i] + 2 \)
  \( f_1[i] = f_1[i] + 1 \)

\( Q^{-1} \):
  \( f_1[j] = f_1[j] + 1 \)
  \( f_1[j] = f_1[j] + 2 \)
  \( f_1[i] = f_1[i] + 1 \)

\( I_i \):
  \( f_1[i] = f_1[i] + 1 \)
  \( f_1[i] = f_1[i] + 1 \)
  \( f_1[j] = f_1[j] + 2 \)

\( I_r \):
  \( f_1[j] = f_1[j] + 1 \)
  \( f_1[j] = f_1[j] + 1 \)
  \( f_1[i] = f_1[i] + 2 \)

endswitch
endfor

Example 5.1 We keep on working with the same example.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>( g(a_l) )</th>
<th>( g(x_r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>15</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>Q</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>12</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>Q</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>Q</td>
<td>P</td>
<td>9</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>Q</td>
<td>P</td>
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<td>P</td>
<td>L</td>
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<td>e</td>
<td>Q</td>
<td>P</td>
<td>P</td>
<td>L</td>
<td>Q</td>
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<td>f</td>
<td>Q</td>
<td>P</td>
<td>P</td>
<td>L</td>
<td>Q</td>
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<td>P</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>g</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>h</td>
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<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

By definition, \( T_0 \subset T \), i.e., \( T \) is an extension of \( T_0 \), furthermore, this extension adds only pairs of either type \( T(a_l, b_l) \) and \( T(a_r, b_r) \) to \( T_0 \). We have seen in the previous section that the minimal \( e \)-representation is based on \( T_0 \). The unification of endpoints is indeed a reduction from \( T \) to \( T_0 \): two consecutive left (right) endpoints (in \( T \)) which are not related by \( T_0 \) can be unified. Two consecutive endpoints \( a_r, Tb_l \) can always be unified because \( T_0(a_r, b_l) \) requires only \( r(a) \geq 1(b) \).

Proposition 5.2 Let \((P, Q, I)\) be a PQI interval order on a finite set \( A \), \( T_l, T_r, T \) as defined above then:

i) if \( a_l, b_l \in T \) are two consecutive endpoints and \( T_0(a_l, b_l) \) then \( Q(a, b) \);

ii) if \( a_r, T b_r \) are two consecutive endpoints and \( T_0(a_r, b_r) \) then \( Q(a, b) \).
Proof.

i) If \((a_i, b_i) \in T_0\) then \((a, b) \in T_i = P \cup \bar{Q} \cup I_i.Q \cup Q.I_i \cup I_i.Q.I_i\). With the exception of \(Q\), there is always at least an endpoint \(x\) such that \(a_iTxTb_i\), i.e., \(a_i, b_i\) are not consecutive. For example, \((a, b) \in I_i.Q\) then \(\exists c \in A, aI_cQb\), and we have \(a_iTcTb_i\). The other cases are similar.

ii) Similar to i.

As a consequence, two consecutive endpoints \(xTy\) can be unified if, \(\exists a, b \in A\) such that one of the following conditions is satisfied
- \(x = a_r, y = b_l\);
- \(x = a_l, y = b_l\) and \(I_l(a, b)\);
- \(x = a_r, y = b_r\) and \(I_r(a, b)\).

We obtain the following algorithm in \(O(n)\) to unify endpoints:

\[
\begin{align*}
\text{Rank}[1..2n]; & \quad /* \text{1..2n rank of element } x \in A \cup A_r */ \\
\text{Id}[1..2n]; & \quad /* \text{identification of element } x \in A */ \\
\text{LR}[1..2n]; & \quad /* \text{left endpoint, right endpoint} */ \\
\text{M}[1..n,1..n]; & \quad /* \text{matrix representing } P, Q, I_* */ \\
X=0; & \quad /* \text{number of unifications realised, to be subtracted from the rank to obtain the minimal representation} */
\end{align*}
\]

procedure minimal.numerical.representation

for \(i=1..2n\) do
  \text{Rank}[i]=\text{Rank}[i]-X;
  \text{if } i=2n \text{ then stop endif;}
  \text{Rank}[i]=\text{Rank}[i]-X;
  \text{if [LR[i]=left and LR[i+1]=left and M[Id[i+1],Id[i]]=I_l]} \\
  \text{or [LR[i]=right and LR[i+1]=right and M[Id[i+1],Id[i]]=I_r]} \\
  \text{or [LR[i]=left and LR[i+1]=right], then} \\
  X=X+1;
  \text{endif;}
endfor;

Example 5.2 We keep on working with the same example.
<table>
<thead>
<tr>
<th>Id</th>
<th>Rank</th>
<th>X</th>
<th>Rank – X</th>
<th>observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>hi</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>l, l, Ii(h, h)</td>
</tr>
<tr>
<td>gi</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>l, r</td>
</tr>
<tr>
<td>fi</td>
<td>3</td>
<td>-</td>
<td>2</td>
<td>l, r</td>
</tr>
<tr>
<td>gr</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>l, r</td>
</tr>
<tr>
<td>ei</td>
<td>5</td>
<td>-</td>
<td>3</td>
<td>l, r</td>
</tr>
<tr>
<td>fr</td>
<td>6</td>
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<td>l, r</td>
</tr>
<tr>
<td>er</td>
<td>7</td>
<td>-</td>
<td>4</td>
<td>l, r</td>
</tr>
<tr>
<td>d1</td>
<td>8</td>
<td>-</td>
<td>5</td>
<td>r, r, I(a, d)</td>
</tr>
<tr>
<td>c1</td>
<td>9</td>
<td>-</td>
<td>6</td>
<td>l, r</td>
</tr>
<tr>
<td>d2</td>
<td>10</td>
<td>4</td>
<td>6</td>
<td>l, r</td>
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<tr>
<td>h2</td>
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</tr>
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<td>b2</td>
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<td>7</td>
<td>7</td>
<td>l, r</td>
</tr>
<tr>
<td>c3</td>
<td>14</td>
<td>-</td>
<td>8</td>
<td>l, r</td>
</tr>
<tr>
<td>b3</td>
<td>15</td>
<td>9</td>
<td>9</td>
<td>l, r</td>
</tr>
<tr>
<td>a4</td>
<td>16</td>
<td>7</td>
<td>9</td>
<td>l, r</td>
</tr>
</tbody>
</table>

The result confirms those from the previous example 4.1.

6 More about the representation of PQI interval order

In this section we examine some other aspects of the representation of a PQI interval orders: the characteristic matrix and the synthetic graph (SG).

6.1 Characteristic matrix

Given a PQI interval order on a finite set A, by theorem 4.5, we have:
(Ai/Ti−, Ti) = {Xi Ti Xl−1 Ti...X1};
(Ai/Ti−, Ti) = {Yi Ti Yi−1 Ti...Y1}.

With such partitions, each one being a linear order, we define the characteristic matrix M of the PQI interval order as the 0 – 1 matrix defined on 
{(i,j) : i = 1..l, j = 1..r} by:
M_{ij} = \begin{cases} 1 & \text{if } \exists x \in A, x \in X_i \cap Y_j \\ 0 & \text{otherwise} \end{cases}

Thus M_{ij} = 1 if some element x ∈ A has the left endpoint in X_i and the right endpoint in Y_j.

We can observe that this matrix does not contain enough information to characterise a PQI-interval order as it does not contains information about the order of all (classes of ) endpoints. It must be completed by a partition of the rows (Ai/Ti−) as well as of the columns (Ai/Ti−) using

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\( \hat{T}_i \) and \( \hat{T}_{r} \). These two partitions enable the merge (see theorem 4.5 and proposition 4.2) of two linear orders, \((A_i/T_i, T_i)\) and \((A_r/T_r, T_r)\), into the linear order \((A_i \cup A_r)/T_0, T_0) = \{Z_1, Z_2, \ldots Z_{l+r}\}\). The characteristic matrix \(M\) completed by \((A_i \cup A_r)/T_0, T_0)\) totally characterises the associated PQI interval order.

The characteristic matrix associated with the example 4.1 is the following one. Here, next to each 1, the corresponding element in \(A\) is presented.

<table>
<thead>
<tr>
<th></th>
<th>Z₃,2</th>
<th>Z₅,3</th>
<th>Z₆,4</th>
<th>Z₉,6</th>
<th>Z₁₁,7</th>
<th>Z₁₂,8</th>
<th>Z₁₄,9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z₁,1</td>
<td>1g</td>
<td></td>
<td></td>
<td>1h</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z₂,2</td>
<td>1f</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z₄,3</td>
<td></td>
<td>1e</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Z₅,5</td>
<td></td>
<td></td>
<td>1d</td>
<td></td>
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<tr>
<td>Z₆,6</td>
<td></td>
<td></td>
<td></td>
<td>1c</td>
<td></td>
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<tr>
<td>Z₇,7</td>
<td></td>
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<td></td>
<td>1b</td>
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<tr>
<td>Z₈,8</td>
<td></td>
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<td>1a</td>
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</tbody>
</table>

It is easy to verify that if in \(M\), an element 1 corresponding to \(Z_{li}\) (row) and \(Z_{rj}\) (column) then \(l_i < r_j\). Intuitively, as \(T_0\) represents the order of all endpoints, for any interval, its left endpoint must precede its right endpoint. We can also verify that there is at least one 1 in every row and every column.

Conversely, given an \(l \times r\) \(0-1\) matrix \(M\), let \(X = \{X_1, \ldots, X_l\}\) be the set of rows and let \(Y = \{Y_1, \ldots, Y_r\}\) be the set of columns, two partitions \(X = \{A_1, \ldots, A_m\}\), \(Y = \{B_1, \ldots, B_m\}\), then we can enumerate the rows and columns, first by all the row in \(A_1\), and then all the columns in \(B_1\), then \(A_2, B_2\). The result of this rearrangement is \(Z_1, \ldots Z_{l+r}\). If (i) there is at least one 1 in every row and every column of \(M\); (ii) for all element 1 whose row and column correspond respectively to \(Z_{li}\) and \(Z_{rj}\), we have \(l_i < r_j\), then \(M\) is a characteristic matrix of a PQI interval order.

We can also verify that two PQI interval orders without "ex æquo" are isomorphic if and only if they have the same characteristic matrix with partitions on rows and columns.

### 6.2 Synthetic graph (SG)

The idea of the synthetic graph is to find the graph as simple as possible to represent a preference structure. A PQI interval order, once identified (\(I_r\) determined), can be entirely described by the relations \(P, Q, I_r\). We will proceed by eliminating all redundant elements from the above relations in order to obtain simpler ones.
Let us consider the relation $T^R$ on $A$ defined as the restriction of $T$ on $A_r$, i.e., $T^R = P \cup Q \cup T^R$, $T^R$ being a linear order (of all right endpoints). If we arrange the elements of $A$ by the linear order $T^R$ such that $A = \{a_1, a_2, a_3, \ldots, a_n\}$, its representation matrix has the upper triangle containing only $P, Q, I_r$. On each row, all the elements on the right of each $P$ are $P$ too. We will define a relation $P'$ containing only the leftmost $P$ on each row, and then preserve $P'$ instead of $P$. On the left of $P'$ (until the diagonal), we have only $Q$ and $I_r$. So, if we represent all the $I_r$ elements, the other elements are obviously $Q$. In fact, we can choose to preserve $Q$ instead of $I_r$, but here, we prefer $I_r$ because each chain of $I_r$ is a linear order, i.e., the elimination of redundant arcs and the reconstruction of $I_r$ are quite straightforward. These observations will be formalised hereafter.

We define the following relations:

- $P'(x, y)$ if $P(x, y)$ and $\forall z \neq y, P(x, z) \Rightarrow R(y, z)$, $P'$ preserves the leftmost $P$ element on each row (if exists).
- $Q' = Q \cap (T^R \setminus T^R \cup T^R)$, $Q'$ preserves only $Q$-arcs connecting consecutive elements (ordered by $T^R$). It is necessary to arrange the elements of $A$.
- $I'_r = I_r \setminus I_r \setminus I_r$.

**Proposition 6.1** Let $T^R = T^R \setminus T^R \cup T^R$. Then $T^R \subset P' \cup Q' \cup I'_r$.

**Proof.** If $(x, y) \in (P \setminus P')$ then $\exists z \neq y, P(x, z)$ and $-T^R(y, z)$ Since $T^R$ is a linear order, $-T^R(y, z) \Rightarrow T^R(x, z)$. Therefore, $(x, y) \in (P \cup Q' \cup I'_r)$, i.e., $(x, y) \in (Q' \setminus I'_r)$, $(x, y) \in I_r \setminus I_r$.

$T^R$ represent arcs connecting consecutive elements, necessary to arrange the elements of $A$ (ordered by $T^R$). This results show that we don’t have to preserve $T^R$ as it is included in $(P' \cup Q' \cup I'_r)$.

**Proposition 6.2** The relations $P, Q, I_r$ can be reconstructed from $P', Q', I'_r$ using:

i) $T^R, I_r$ being the transitive closures of, respectively, $(P' \cup Q' \cup I'_r)$ and $I'_r$.

ii) $P = P' \cup P', T^R$.

iii) $Q = (T^R \setminus (P \cup I_r))$

**Proof** i) and iii) are obvious.

ii) $(P \subset P' \cup P', T^R)$

If $P(x, y)$ and $-P'(x, y)$ then $\exists z, P(x, z)$ and $T^R(z, y)$. 

[End of Document]
\((P' \cup P'.T^R \subseteq P)\)

We have

\(P' \subseteq P\)

\(P'.T^R \subseteq P.T^R = P.(P \cup Q \cup I_r) = P.P \cup P.Q \cup P.I_r \subseteq P.\)

Example 6.1

After the rearrangement of elements of \(A\) by \(T^R\), we have the following representation matrix (\(R\) stands for \(I_r\)).

\[
\begin{array}{cccccccccccc}
  a & b & c & d & e & f & g & h & i & j & k & l & m & n \\
  c & Q & R & Q & Q & Q & P & P & P & P & P & P & P & P \\
  d & R & R & Q & R & Q & P & P & P & P & P & P & P & P \\
  g & R & Q & P & P & P & P & P & P & P & P & P & P & P \\
  k & R & R & R & R & R & R & R & R & R & R & R & R & R \\
  m & & & & & & & & & & & & & & \\
  n & & & & & & & & & & & & & & \\
\end{array}
\]

Using \(P', I'_r\), the representation matrix become a synthetic representation matrix as follows (as \(T^R\) has already been considered in the order of rows and columns, there is no need to represent \(Q'\) in the matrix).
The synthetic graph is the graph associated to the synthetic representation matrix.

## 7 Conclusion

In this paper we try to extend some well known results concerning the numerical representation of interval orders in the case of PQI interval orders. Such preference structures appear when, while comparing intervals, it might be interesting to distinguish a situation of hesitation between “sure” preference (empty intersection of the two intervals) and “sure” indifference (one interval included in the other).

The aim of this effort is to find under which foundations it is possible to construct a numerical representation of a PQI interval order as soon as it has been demonstrated that such a representation exists. Not surprisingly we are able to demonstrate that there exist two weak orders, one representing the order of the left extreme points and one representing the order of the right extreme points of the intervals and that on this basis it is possible to construct a numerical representation.

In the paper we demonstrate the theorems which enable to show what the numerical representation of a PQI interval order represents. We also show how it is possible to obtain a “minimal” representation and we provide alternative characterisations of PQI interval orders (in terms of characteristic matrix etc.). With such results we are able to define two algorithms, the first constructing a numerical representation for a given PQI interval order, the second minimising it. Both algorithms are shown to run in polynomial
time \( O(n^2) \) for the first and \( O(n) \) for the second).

References


