Parameterized approximation schemes for Steiner Trees

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Steiner tree

Steiner Tree

- Input: A graph $G = (V, E)$, a set of terminals $R \subseteq V$, and a weight function $w\colon E \to \mathbb{R}^+$.
- Task: Find a tree $F \subseteq E$ of smallest weight such that (R, F) is connected.

A vertex in $V \setminus R$ is called a Steiner vertex.

Steiner tree

Directed Steiner tree

- Input: A directed graph $G = (V, A)$, a set of terminals $R \subseteq V$, a root $r \in R$, and a weight function $w\colon A \to \mathbb{R}^+$,
- Task: Find an arborescence $F \subseteq A$ of smallest weight such that every terminal is reachable from r by edges in F.

Steiner tree – parameters

We are interested in following two parameters:

- the number of terminals – $|R|$ (denoted by k),
- the number of Steiner vertices in optimal solution (denoted by p).

Steiner tree – known results

Parameterized complexity:

- FPT w.r.t. number of terminals ($3^k n^3$ algorithm in the 70s (Dreyfus & Wagner), the best known now is $2^k n^2$ (Björklund et al.))
- W[1]-hard w.r.t. p

Approximation:

- 2-approximation is trivial
- $(\ln 4 + \varepsilon)$-approximation algorithm (Byrka et al.)
- (96/95)-approximation is NP-hard (which means no approximation scheme exists unless P = NP)

Since Steiner Tree is both hard to approximate and W[1]-hard w.r.t. p, it is an excellent target for parameterized approximation.

Overview of algorithm

- In each step, our algorithm decides to greedily "buy" a tree and contracts it.
- If any of the vertices in contraction was a terminal, the resulting vertex will be a terminal.
- Once the number of terminals reaches 1, we have a solution.

Overview of algorithm

- In each step, our algorithm decides to greedily "buy" a tree and contracts it.
- If any of the vertices in contraction was a terminal, the resulting vertex will be a terminal.
- Once the number of terminals reaches 1, we have a solution.

For contractions we consider only stars such that

- all leaves are terminals,
- contain at least two terminals.

Note that we consider an edge connecting two terminals as a star.
A ratio of a star C is $w(C)/(|Q| - 1)$, where Q is the set of terminals incident to Q.
At each step we pick the star of the best ratio.

Overview of algorithm

Phase 1: We continue contracting the star with the best ratio until the number of terminals is small ($\mathcal{O}(p^2/\varepsilon^4)$)
Phase 2: Run an FPT algorithm for parameterization by $|R|$

Analysis

Correctness: Clearly, our algorithm returns a tree connecting all the terminals.

Running time: We show how to find the best-ratio star in polynomial time.

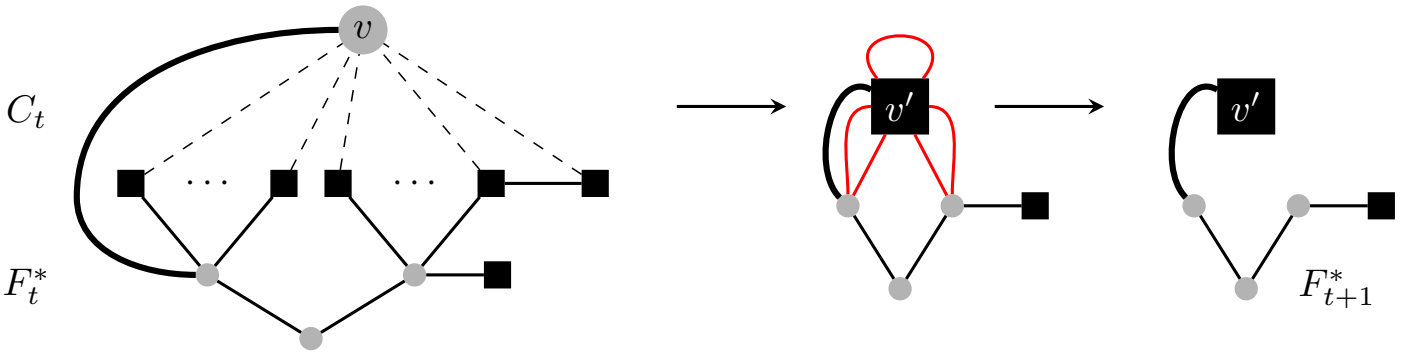
- Fix the center of the star v.
- We sort the edges going from v to terminals such that $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_x)$.
- Star of best ratio having center at v mus be of form $ve_1, \ldots, ve_q$ for some q.
- The best star can be found in time $\mathcal{O}(n^3)$.

Analysis

- For analysis we start with an optimal solution F^*.
- We maintain a tree originating from F^* in the contracted instances.
- Denote the tree after t steps of the algorithm by F_t^*.

We compare the weight of each contraction with a subset of an optimal solution F^*.

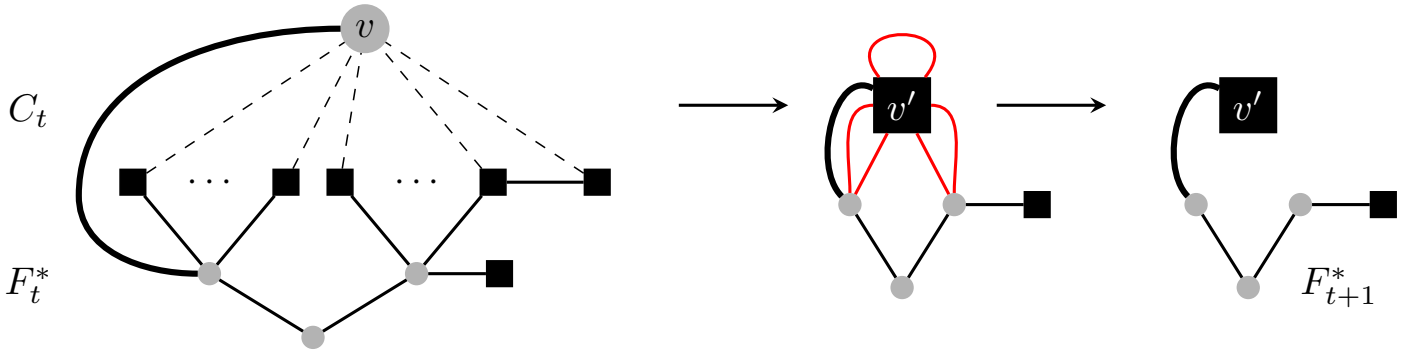
- In each step we compare weight of $w(C_t)$ with $w(D_t)$.
- D_t is a feedback edge set of F_t^* / C_t



Analysis

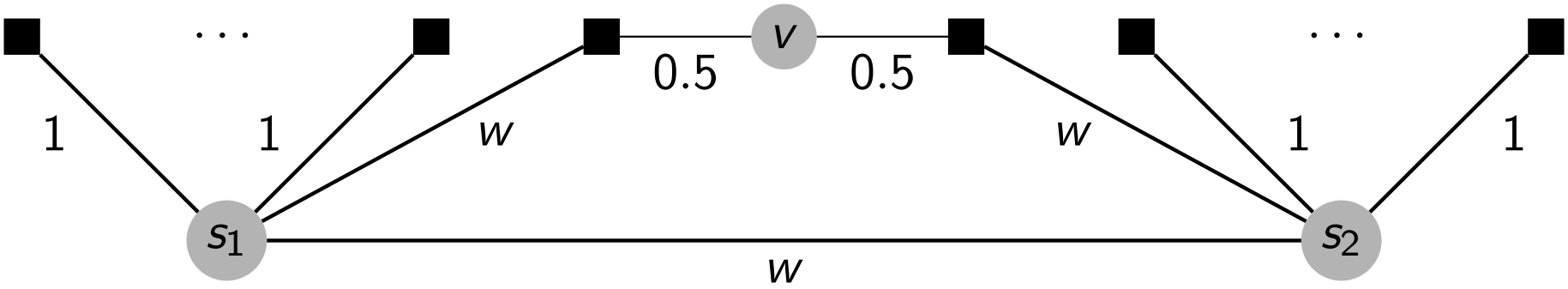
There are many feedback edge set to pick. Which one should we pick?
A few observations:

- All inclusion-wise minimal feedback edge sets have the same number of edges.
- We can pick a FES such that all its edges are incident with the contracted vertex



Analysis

- If $w(C_t) \leq (1+\varepsilon)w(D_t)$ for every t our algorithm is $(1+\varepsilon)$-approximating.
- Unfortunately, this is not the case.

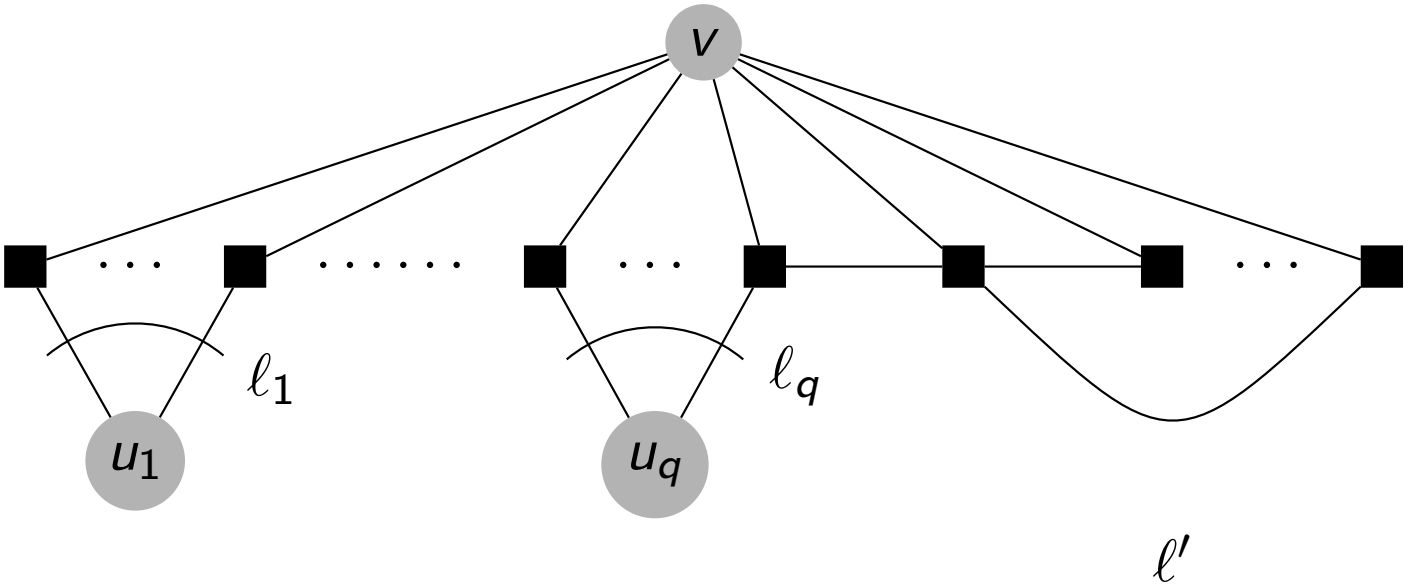


Bounding bad contractions

- Idea: show that all bad contractions together have weight at most $\varepsilon \cdot OPT$.
- The good contractions + phase 2 cost at most $(1+\varepsilon)OPT$.
- Together with bad contractions we have cost at most $(1+2\varepsilon)OPT$.

Bounding bad contractions

If a star C contains at least $(1+\varepsilon)p/\varepsilon$ terminals then contraction of C is good.



Bounding bad contractions

- the weight of C is $r(|Q| - 1)$.
- We can split D into at most p stars $D_1, \ldots, D_q$ (denote $\ell_i = |D_i|$) and ℓ' edges that lead between terminals.
- The star D_i has weight at least $r(\ell_i - 1)$.
- Each edge between terminals has weight at least r.
- The total weight of D is at least

$$r\ell' + \sum_{i=1}^{q} r(\ell_i - 1) \geq r(|Q| - p - 1).$$

Kernelization

The phase 1 of our algorithm resembles preprocessing.
Can we rephrase our result as a kernelization?

Kernelization

For decision problem, the kernelization is a polynomial time algorithm $\mathcal{R}$ (called a reduction algorithm) that given an instance (I, p) returns an instance (I', p') such that

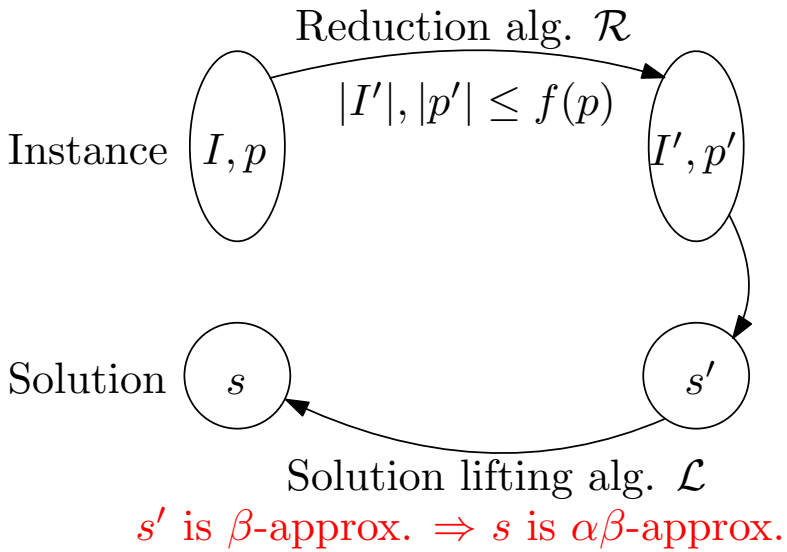
- $|I'|, p' \leq f(p)$ for some function f, and
- (I, p) is a yes instance iff (I', p') is a yes instance.

The function g is the size of the kernel.
A problem admits kernelization iff it has an FPT algorithm.
We are usually interested in polynomial kernels.

Kernelization

For optimization problem, the α-lossy kernelization is a pair of polynomial time algorithm $\mathcal{R}$ and $\mathcal{L}$ (a reduction algorithm and a solution lifting algorithm) such that

- given an instance (I, p) the algorithm $\mathcal{R}$ returns an instance (I', p') with $|I'|, p' \leq f(p)$ for some function g, and
- given a solution s' of an instance (I', p') returns a solution s of an instance (I, p),
- if s' is a β-approximation then s is an $(\alpha \cdot \beta)$-approximation.



Kernelization

For optimization problem, the α-lossy kernelization is a pair of polynomial time algorithm $\mathcal{R}$ and $\mathcal{L}$ (reduction algorithm and solution lifting algorithm) such that

- given an instance (I, k) the algorithm $\mathcal{R}$ returns an instance (I', k') with $|I'|, k' \leq g(k)$ for some function g, and
- given a solution s' of an instance (I', k') returns a solution s of an instance (I, k),
- if s' is a β-approximation then s is an $(\alpha \cdot \beta)$-approximation.

If we can get $(1 + \varepsilon)$-lossy kernelization for every $\varepsilon > 0$ such that the kernel is polynomial w.r.t. parameter (but the degree of polynomial may depend on ε), we say we have a Polynomial size approximate kernelization scheme (PSAKS).

Kernelization

- The phase 1 leaves us with an instance with $\mathcal{O}(p^2/\varepsilon^4)$ terminals, but the instance size may still be large (unbounded in terms of p and ε).
- Fortunately, there exists a PSAKS for parameterization by $|R|$, returning kernel of size $|R|^{2^{\mathcal{O}(1/\varepsilon)}}$ (Lokshtanov et al.).
- Combining these two we get a kernel of size $(p/\varepsilon)^{2^{\mathcal{O}(1/\varepsilon)}}$.

Generalizations of Steiner Tree

Steiner Forest

- Input: A graph $G = (V, E)$, a list of terminal pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$, and a weight function $w\colon E \to \mathbb{R}^+$.
- Task: Find a forest $F \subseteq E$ of smallest weight such that s_i and t_i is in the same component of F for every i.

Generalizations of Steiner Tree

There is no FPT algorithm w.r.t. to number of Steiner vertices in optimal solution.

- We can simulate Steiner vertices using terminals.
- However, in such an instances the number of components of the optimal solution is high.
- We can use the same idea if we add parameterization by the number of components of the optimal solution.

Preprint is on arXiv:

Parameterized Approximation Schemes for Steiner Trees with Small Number of Steiner Vertices

`https://arxiv.org/abs/1710.00668`

Thank you!