Clique-width of Hereditary Graph Classes

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Motivation

Many natural problems in algorithmic graph theory are NP-complete.

Want to find restricted classes of graphs where we can solve some problems in polynomial time.

Best if we can find classes where lots of problems can be solved in polynomial time.
Why Clique-width?


Any problem expressible in “monadic second-order logic with quantification over vertices” (and certain other classes of problems) can be solved in polynomial time on any graph class of bounded clique-width.

This includes:

- Vertex Colouring
- Maximum Independent Set
- Minimum Dominating Set
- Hamilton Path/Cycle
- Partitioning into Perfect Graphs
- ...
Clique-width

The clique-width is the minimum number of labels needed to construct $G$ by using the following four operations:

(i) creating a new graph consisting of a single vertex $v$ with label $i$ (represented by $i(v)$)
(ii) taking the disjoint union of two labelled graphs $G_1$ and $G_2$ (represented by $G_1 \oplus G_2$)
(iii) joining each vertex with label $i$ to each vertex with label $j$ ($i \neq j$) (represented by $\eta_{i,j}$)
(iv) renaming label $i$ to $j$ (represented by $\rho_{i \rightarrow j}$)

For example, $P_4$ has clique-width 3.

An expression for a graph can be represented by a rooted tree.
Clique-width

\[ \eta_{3,2}(3(d) \oplus \rho_{3\to2}(\rho_{2\to1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))) \]

\[ \eta_{3,2} \oplus \rho_{3\to2} \rho_{2\to1} \eta_{3,2} \oplus \eta_{2,1} \oplus 1(a) \]

\[ 3(d) \quad 3(c) \quad 2(b) \]
Clique-width

1

(a)

1(a)
Clique-width
Clique-width

\[ 2(b) \oplus 1(a) \]
Clique-width

\[ \eta_{2,1}(2(b) \oplus 1(a)) \]

\[ \eta_{2,1} \quad \oplus \quad 1(a) \]

\[ 2(b) \]
Clique-width

\[ 3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)) \]

\[ \eta_{2,1} \oplus 1(a) \]

\[ 3(c) \quad 2(b) \]
Clique-width

$$3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))$$

\[ \oplus \quad \eta_{2,1} \quad \oplus \quad 1(a) \]
\[ \quad \quad \quad \quad \quad | \quad \quad \quad \quad | \]
\[ 3(c) \quad \quad 2(b) \]
Clique-width

\[ \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))) \]

\[ \eta_{3,2} \quad \oplus \quad \eta_{2,1} \quad \oplus \quad 1(a) \]

\[ 3(c) \quad \quad \quad \quad \quad \quad \quad \quad 2(b) \]
Clique-width

\[ \rho_{2 \rightarrow 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))) \]

\[ \rho_{2 \rightarrow 1} \quad \eta_{3,2} \quad \oplus \quad \eta_{2,1} \quad \oplus \quad 1(a) \]

\[ 3(c) \quad 2(b) \]
Clique-width

\[ \rho_{3 \rightarrow 2}(\rho_{2 \rightarrow 1}(\eta_{3,2}(3(c)) \oplus \eta_{2,1}(2(b) \oplus 1(a)))) \]

\[ \rho_{3 \rightarrow 2} \rightarrow \rho_{2 \rightarrow 1} \rightarrow \eta_{3,2} \rightarrow \eta_{2,1} \rightarrow 1(a) \]

\[ 3(c) \quad 2(b) \]
Clique-width

\[ 3(d) \rho_3 \rightarrow_2 (\rho_2 \rightarrow_1 (\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))) \]

\[ \rho_3 \rightarrow_2 \quad \rho_2 \rightarrow_1 \quad \eta_{3,2} \quad \oplus \quad \eta_{2,1} \quad \oplus \quad 1(a) \]

\[ 3(d) \quad 3(c) \quad 2(b) \]
Clique-width

\[ 3(d) \oplus \rho_{3\to2}(\rho_{2\to1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))) \]

\[ \oplus \xrightarrow{\rho_{3\to2}} \xrightarrow{\rho_{2\to1}} \eta_{3,2} \oplus \eta_{2,1} \oplus 1(a) \]

\[ 3(d) \quad 3(c) \quad 2(b) \]
Clique-width

\[
\eta_{3,2}(3(d) \oplus \rho_{3 \rightarrow 2}(\rho_{2 \rightarrow 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))
\]
Calculating clique-width

Theorem (Fellows, Rosamond, Rotics, Szeider 2009)

*Calculating clique-width is NP-hard.*

Theorem (Corneil, Habib, Lanlignel, Reed, Rotics 2012)

*Can detect graphs of clique-width at most 3 in polynomial time.*

It’s not known if this is also the case for graphs of clique-width 4.

Theorem (Oum 2008)

*Can find a c-expression for a graph G where \( c \leq 8^{cw(G)} - 1 \) in cubic time.*

The clique-width of all graphs up to 10 vertices has been calculated (Heule & Szeider 2013).
Why clique-width?

- “Equivalent” to rank-width and NLC-width
- Generalises tree-width
- “Equivalent” to tree-width on graphs of bounded degree

The following operations don’t change the clique-width by “too much”

- Complementation
- Bipartite complementation
- Vertex deletion
- Edge subdivision (for graphs of bounded-degree)

Need only look at graphs that are

- prime
- 2-connected
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Aim

Underlying Research Question

What kinds of graph properties ensure bounded clique-width?

By knowing what the bounded cases are, we may be able to reduce other classes down to known cases and get polynomial algorithms.
Hereditary Classes

A graph $H$ is an induced subgraph of $G$ if $H$ can be obtained by deleting vertices of $G$, written $H \subseteq_i G$.

\[ P_4 \quad 3P_1 \quad P_1 + P_2 \]

So $P_1 + P_2 \subseteq_i P_4$, but $3P_1 \not\subseteq_i P_4$.

A class of graphs is hereditary if it is closed under taking induced subgraphs.

Let $\mathcal{H}$ be a set of graphs. The class of $\mathcal{H}$-free graphs is the set of graphs that do not contain any graph in $\mathcal{H}$ as an induced subgraph.

For example: bipartite graphs are the $(C_3, C_5, C_7, \ldots)$-free graphs.

We will consider classes defined by finite set of forbidden induced subgraphs.
Graphs of large clique-width

Walls are bipartite and have unbounded clique-width, even if we subdivide each edge $k$ times.

If $H$ contains a $C_k$ or $I_k$, then the $k$-subdivided walls are $H$-free.
Which classes have bounded clique-width?

If the class of $H$-free graphs has bounded clique-width then $H$ must contain no cycles and no $I_k$.

Every component of $H$ must be a subdivided claw, path or isolated vertex. The set of such graphs is called $S$. 

![Graphs](image)
A graph has clique-width at most 2 if and only if it is a $P_4$-free graph (Courcelle and Olariu. 2004).

Theorem (Dabrowski, P. 2014)

The class of $H$-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$. 

\[
\text{\begin{circle}[fill=black]1\end{circle} = \begin{circle}[fill=black]1\end{circle} = \begin{circle}[fill=black]1\end{circle} = \begin{circle}[fill=black]1\end{circle}}
\]
Colouring $H$-free graphs

Theorem (Král’, Kratochvíl, Tuza & Woeginger, 2001)

The Vertex Colouring problem is polynomial-time solvable for $H$-free graphs if and only if $H \subseteq P_1 + P_3$ or $P_4$, otherwise it is NP-complete.

\[ P_1 + P_3 \]

\[ P_4 \]
Colouring \((H_1, H_2)\)-free graphs

The Vertex Colouring problem is polynomial-time solvable for \((H_1, H_2)\)-free graphs if

1. \(H_1\) or \(H_2\) is an induced subgraph of \(P_1 + P_3\) or of \(P_4\)
2. \(H_1 \subseteq_i K_{1,3}\), and \(H_2 \subseteq_i C_3^{++}\), \(H_2 \subseteq_i C_3^*\) or \(H_2 \subseteq_i P_5\)
3. \(H_1 \neq K_{1,5}\) is a forest on at most six vertices or \(H_1 = K_{1,3} + 3P_1\), and \(H_2 \subseteq_i P_1 + P_3\)
4. \(H_1 \subseteq_i sP_2\) or \(H_1 \subseteq_i sP_1 + P_5\) for \(s \geq 1\), and \(H_2 = K_t\) for \(t \geq 4\)
5. \(H_1 \subseteq_i sP_2\) or \(H_1 \subseteq_i sP_1 + P_5\) for \(s \geq 1\), and \(H_2 \subseteq_i P_1 + P_3\)
6. \(H_1 \subseteq_i P_1 + P_4\) or \(H_1 \subseteq_i P_5\), and \(H_2 \subseteq_i P_1 + P_4\)
7. \(H_1 \subseteq_i P_1 + P_4\) or \(H_1 \subseteq_i P_5\), and \(H_2 \subseteq_i P_5\)
8. \(H_1 \subseteq_i 2P_1 + P_2\), and \(H_2 \subseteq_i 2P_1 + P_3\) or \(H_2 \subseteq_i P_2 + P_3\)
9. \(H_1 \subseteq_i 2P_1 + P_2\), and \(H_2 \subseteq_i 2P_1 + P_3\) or \(H_2 \subseteq_i P_2 + P_3\)
10. \(H_1 \subseteq_i sP_1 + P_2\) for \(s \geq 0\) or \(H_1 = P_5\), and \(H_2 \subseteq_i tP_1 + P_2\) for \(t \geq 0\)
11. \(H_1 \subseteq_i 4P_1\) and \(H_2 \subseteq_i 2P_1 + P_3\)
12. \(H_1 \subseteq_i P_5\), and \(H_2 \subseteq_i C_4\) or \(H_2 \subseteq_i 2P_1 + P_3\).
Colouring \((H_1, H_2)\)-free graphs

The Vertex Colouring problem is polynomial-time solvable for \((H_1, H_2)\)-free graphs if

1. \(H_1\) or \(H_2\) is an induced subgraph of \(P_1 + P_3\) or of \(P_4\)

2. \(H_1 \subseteq P_1 + S_{1,1,2}, P_1 + S_{1,1,3}, P_2, P_3, P_6\) or \(H_2 \subseteq P_1 + P_3\)

3. \(H_1 \neq K_{1,5}\) is a forest on at most six vertices

\((H_1 \subseteq K_{1,3} + P_2, P_1 + S_{1,1,2}, P_6\) or \(S_{1,1,3}\)) or \(H_1 = K_{1,3} + 3P_1, H_2 \subseteq P_1 + P_3\)

4. \(H_1 \subseteq sP_2\) or \(H_1 \subseteq sP_1 + P_5\) for \(s \geq 1\), and \(H_2 = K_t\) for \(t \geq 4\)

5. \(H_1 \subseteq sP_2\) or \(H_1 \subseteq sP_1 + P_5\) for \(s \geq 1\), and \(H_2 \subseteq P_1 + P_3\)

6. \(H_1 \subseteq P_1 + P_4\) or \(H_1 \subseteq P_5\), and \(H_2 \subseteq P_1 + P_4\)

7. \(H_1 \subseteq P_1 + P_4\) or \(H_1 \subseteq P_5\), and \(H_2 \subseteq P_5\)

8. \(H_1 \subseteq 2P_1 + P_3\), and \(H_2 \subseteq 2P_1 + P_3\) or \(H_2 \subseteq P_2 + P_3\)

9. \(H_1 \subseteq 2P_1 + P_3\), and \(H_2 \subseteq 2P_1 + P_3\) or \(H_2 \subseteq P_2 + P_3\)

10. \(H_1 \subseteq sP_1 + P_2\) for \(s \geq 0\) or \(H_1 = P_5\), and \(H_2 \subseteq tP_1 + P_2\) for \(t \geq 0\)

11. \(H_1 \subseteq 4P_1\) and \(H_2 \subseteq 2P_1 + P_3\)

12. \(H_1 \subseteq P_5\), and \(H_2 \subseteq C_4\) or \(H_2 \subseteq 2P_1 + P_3\).
The class of \((H_1, H_2)\)-free graphs has bounded clique-width if:

1. \(H_1 \text{ or } H_2 \subseteq \overline{P_4}\);
2. \(H_1 = sP_1 \text{ and } H_2 = K_t \text{ for some } s, t\);
3. \(H_1 \subseteq \overline{P_1 + P_3} \text{ and } \overline{H_2} \subseteq K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,2}, P_6 \text{ or } S_{1,1,3}\);
4. \(H_1 \subseteq 2P_1 + P_2 \text{ and } \overline{H_2} \subseteq 2P_1 + P_3, 3P_1 + P_2 \text{ or } P_2 + P_3\);
5. \(H_1 \subseteq P_1 + P_4 \text{ and } \overline{H_2} \subseteq P_1 + P_4 \text{ or } P_5\);
6. \(H_1 \subseteq 4P_1 \text{ and } \overline{H_2} \subseteq 2P_1 + P_3\);
7. \(H_1, H_2 \subseteq K_{1,3}\).

and it has unbounded clique-width if:

1. \(H_1 \not\in S \text{ and } H_2 \not\in S\);
2. \(\overline{H_1} \not\in S \text{ and } \overline{H_2} \not\in S\);
3. \(H_1 \supseteq K_{1,3} \text{ or } 2P_2 \text{ and } \overline{H_2} \supseteq 4P_1 \text{ or } 2P_2\);
4. \(H_1 \supseteq P_1 + P_4 \text{ and } \overline{H_2} \supseteq P_2 + P_4\);
5. \(H_1 \supseteq 2P_1 + P_2 \text{ and } \overline{H_2} \supseteq K_{1,3}, 5P_1, P_2 + P_4 \text{ or } P_6\);
6. \(H_1 \supseteq 3P_1 \text{ and } \overline{H_2} \supseteq 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2 \text{ or } 2P_3\);
7. \(H_1 \supseteq 4P_1 \text{ and } \overline{H_2} \supseteq P_1 + P_4 \text{ or } 3P_1 + P_2\).
Theorem (Dabrowski, P. 2015)

This leaves 13 cases where it is unknown if the clique-width of $(H_1, H_2)$-free graphs is bounded or not (up to some equivalence relation).

1. $H_1 = 3P_1$, $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2$, $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$;
3. $H_1 = P_1 + P_4$, $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ or
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

There are 15 classes of $(H_1, H_2)$-free graphs for which both boundedness of clique-width and computational complexity of vertex colouring are open.
The Orange Row

Theorem (Dabrowski, Huang, P., 2015)

The class of \((H_1, H_2)\)-free graphs has bounded clique-width when
\[ H_1 = 2P_1 + P_2 \text{ and } H_2 \subseteq 2P_1 + P_3, \ 3P_1 + P_2 \text{ or } P_2 + P_3. \]
Our Technique for the Orange Row

A graph $G$ is perfect if for every induced subgraph $H$ of $G$, the chromatic number of $H$ is equal to the size of a maximum clique of $H$.

The clique covering number of a graph $G$ is the smallest number of (mutually vertex-disjoint) cliques such that every vertex of $G$ belongs to exactly one clique.

An alternative definition:

A graph $G$ is perfect if for every induced subgraph $H$ of $G$, the clique covering number of $H$ is equal to the size of a maximum independent set of $H$. 
Our Technique for the Orange Row

Theorem (Strong Perfect Graph Theorem
Chudnovsky, Robertson, Seymour, Thomas 2006)

Perfect graphs are precisely the \((C_5, \overline{C_5}, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots)\)-free graphs.

Our technique can be summarised as follows:

1. Reduce the input graph to a graph that is in some subclass of perfect graphs.
2. While doing so, bound the clique covering number.
The Clique Covering Lemma

Lemma
Let $G$ be a $(2P_1 + P_2, 2P_2 + P_4)$-free graph. If the vertex set of $G$ can be partitioned into at most $k$ cliques, then the clique-width of $G$ is bounded by a function depending only on $k$.

Sketch of proof:
- Divide $G$ into $k$ cliques ($k$ minimal) $X_1, \ldots, X_k$.
- May assume every clique is big (at least $k + 7$ vertices).
- If $x \in X_i$ has two neighbours in $X_j$ then it is complete to $X_j$.
- If there are two such vertices then $X_i$ is complete to $X_j$, contradicting minimality of $k$.
- Delete all vertices in every clique that are complete to any other clique.
The Clique Covering Lemma

\[ 2P_1 + P_2 \quad \text{and} \quad 2P_2 + P_4 \]

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The Clique Covering Lemma

Every vertex in a clique $X_i$ has at most one neighbour in a different clique $X_j$ and every clique has at least 8 vertices.

- If $k \leq 3$, complement each clique to get a graph of maximum degree $\leq 2$, clique-width $\leq 4$.
- If $k \geq 4$, if $G$ is a disjoint union of cliques, clique-width is 2.
- Otherwise, can find a $P_4$ using vertices from two cliques. Can find two disjoint $P_2$s in two other cliques, leading to a contradiction.
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The Clique Covering Lemma

Every vertex in a clique $X_i$ has at most one neighbour in a different clique $X_j$ and every clique has at least 8 vertices.

If $k \leq 3$, complement each clique to get a graph of maximum degree $\leq 2$, clique-width $\leq 4$.

If $k \geq 4$, if $G$ is a disjoint union of cliques, clique-width is 2.

Otherwise, can find a $P_4$ using vertices from two cliques. Can find two disjoint $P_2$s in two other cliques, leading to a contradiction.
Example

\[ 2P_1 + P_2 \quad \quad 3P_1 + P_2 \]

Lemma (Dabrowski, Lozin, Raman, Ries, 2012)
The class of \((K_3, K_{1,3} + P_2)\)-free graphs has bounded clique-width.

Lemma (Dabrowski, Golovach, P. 2014)
Let \(s \geq 0\) and \(t \geq 0\). Then every \((sP_1 + P_2, tP_1 + P_2)\)-free graph is \((K_{s+1}, tP_1 + P_2)\)-free or \((sP_1 + P_2, (s^2(t - 1) + 2)P_1)\)-free.

Consequences: If \(G\) is a \((2P_1 + P_2, 3P_1 + P_2)\)-free graph then it is \(K_3\)-free (in which case we use the first lemma) or \(10P_1\)-free.
Example

\[
G \text{ is a } (2P_1 + P_2, 3P_1 + P_2)-\text{free graph that is } 10P_1-\text{free}.
\]

▶ If \(G\) contains an induced \(C_5\) or \(C_7\), partition other vertices according to their neighbourhood in the cycle.

▶ Each such set is a clique and there are a bounded number of them. Apply Clique Covering Lemma.

▶ Since \(G\) is \(3P_1 + P_2\)-free, it contains no odd cycles of length 9 or more.

▶ Since \(G\) is \(2P_1 + P_2\)-free it contains no complements of cycles of length 7 or more.

▶ So \(G\) is perfect and \(10P_1\)-free, so it can be partitioned into at most 9 cliques. Apply Clique Covering Lemma.
Example

\[
\begin{align*}
2P_1 + P_2 & \quad 3P_1 + P_2
\end{align*}
\]

\(G\) is a \((2P_1 + P_2, 3P_1 + P_2)\)-free graph that is 10\(P_1\)-free.

- If \(G\) contains an induced \(C_5\) or \(C_7\), partition other vertices according to their neighbourhood in the cycle.
  - Each such set is a clique and there are a bounded number of them. Apply Clique Covering Lemma.
  - Since \(G\) is 3\(P_1 + P_2\)-free, it contains no odd cycles of length 9 or more.
  - Since \(G\) is 2\(P_1 + P_2\)-free it contains no complements of cycles of length 7 or more.
  - So \(G\) is perfect and 10\(P_1\)-free, so it can be partitioned into at most 9 cliques. Apply Clique Covering Lemma.
Example

$G$ is a $(2P_1 + P_2, 3P_1 + P_2)$-free graph that is $10P_1$-free.

- If $G$ contains an induced $C_5$ or $C_7$, partition other vertices according to their neighbourhood in the cycle.
- Each such set is a clique and there are a bounded number of them. Apply Clique Covering Lemma.
- Since $G$ is $3P_1 + P_2$-free, it contains no odd cycles of length 9 or more.
- Since $G$ is $2P_1 + P_2$-free it contains no complements of cycles of length 7 or more.
- So $G$ is perfect and $10P_1$-free, so it can be partitioned into at most 9 cliques. Apply Clique Covering Lemma.
Example

\[2P_1 + P_2\] \hspace{1cm} \[3P_1 + P_2\]

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Example

\[ 2P_1 + P_2 \quad 3P_1 + P_2 \]

\( G \) is a \((2P_1 + P_2, 3P_1 + P_2)\)-free graph that is \(10P_1\)-free.

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Example

\[ G \text{ is a } (2P_1 + P_2, 3P_1 + P_2)\text{-free graph that is } 10P_1\text{-free.} \]

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**H-Free Bipartite Graphs**

**Theorem (Dabrowski, P. 2014)**

The class of *H-free bipartite* graphs has bounded clique-width if and only if *H* is an induced subgraph one of:

- $K_{1,3} + 3P_1$
- $P_1 + S_{1,1,3}$
- $K_{1,3} + P_2$
- $S_{1,2,3}$

$sP_1$ for some $s$
($s = 5$ shown)
A graph $G$ is \textit{weakly chordal} if both $G$ and $\overline{G}$ are $(C_5, C_6, \ldots)$-free.

**Theorem (Brandstädt, Dabrowski, Huang, P. 2015+)**

Let $H$ be a graph. Then the class of $H$-free weakly chordal graphs has bounded clique-width if and only if $H \subseteq_i P_4$.

\[ \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet \\
 \end{array} \]

A graph $G$ is \textit{chordal} if $G$ is $(C_4, C_5, \ldots)$-free.
**H-Free Chordal Graphs**

![Graphs F1 and F2](image)

**Theorem (Brandstädt, Dabrowski, Huang, P. 2015)**

Let $H$ be a graph with $H \notin \{F_1, F_2\}$. The class of $H$-free chordal graphs has bounded clique-width if and only if $H$ is an induced subgraph of one of:

- $S_{1,1,2}$
- $K_{1,3} + 2P_1$
- $P_1 + P_1 + P_3$
- $P_1 + 2P_1 + P_2$
- Bull
- $K_r$ for some $r \geq 1$
- $P_1 + P_4$
- $P_1 + P_4$
Theorem (Brandstädt, Dabrowski, Huang, P. 2015)

Let $H$ be a graph such that neither $H$ nor $\overline{H}$ is in $\{F_4, F_5\}$. The class of $H$-free split graphs has bounded clique-width if and only if $H$ or $\overline{H}$ is

- isomorphic to $rP_1$ for some $r \geq 1$ or
- an induced subgraph of one of:

\[ K_{1,3} + 2P_1 \quad F_1 \quad F_2 \quad F_3 \quad \text{bull} + P_1 \quad Q \]
For which graphs $H$ does the class of $H$-free chordal graphs have bounded clique-width? (2 open cases)

For which graphs $H$ does the class of $H$-free split graphs have bounded clique-width? (2 open cases)

For which pairs of graphs $(H_1, H_2)$ does the class of $(H_1, H_2)$-free graphs have bounded clique-width. (13 open cases)
References