Cutting matroids like cakes

Laurent Gourvès, Jérôme Monnot, Lydia Tlilane

LAMSADE, CNRS - PSL, Université Paris Dauphine

January 19, 2015
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up
Allocation of a divisible resource

...also known as cake cutting

- \( n \) agents \( \{1, \ldots, n\} \)
- a divisible resource (a cake)
- A solution is a partition of the resource \( P_1 \cup \ldots \cup P_n \) such that agent \( i \) receives \( P_i \)
- Agent \( i \)'s utility for \( P_i \) is denoted by \( u_i(P_i) \)

Normalization assumption: every agent has a utility of 1 for the entire resource
Allocation of a divisible resource

...also known as cake cutting

- $n$ agents $\{1, \ldots, n\}$
- a divisible resource (a cake)
- A solution is a partition of the resource $P_1 \cup \ldots \cup P_n$ such that agent $i$ receives $P_i$
- Agent $i$’s utility for $P_i$ is denoted by $u_i(P_i)$

Normalization assumption: every agent has a utility of 1 for the entire resource
Two desirable properties

Proportionality: \( u_i(P_i) \geq 1/n \) for all \( i \in [n] \)

Envy-freeness: \( u_i(P_i) \geq u_i(P_j) \) for all \( i, j \in [n] \)

It is possible to find an envy-free (and thus proportional) solution

2 agents: Divide-and-Choose

One agent divides the resource into what she believes are equal halves, and the other agent chooses the “half” she prefers.
Two desirable properties

Proportionality: \( u_i(P_i) \geq 1/n \) for all \( i \in [n] \)

Envy-freeness: \( u_i(P_i) \geq u_i(P_j) \) for all \( i, j \in [n] \)

It is possible to find an envy-free (and thus proportional) solution

2 agents: Divide-and-Choose
One agent divides the resource into what she believes are equal halves, and the other agent chooses the “half” she prefers

Bibliographic notes

- 1940’s: proportional protocol for any number of agents
- late 1950’s: envy-free protocol for 3 agents
- 1995: envy-free protocol for any number of agents
Two desirable properties

Proportionality: \( u_i(P_i) \geq 1/n \) for all \( i \in [n] \)

Envy-freeness: \( u_i(P_i) \geq u_i(P_j) \) for all \( i, j \in [n] \)

It is possible to find an envy-free (and thus proportional) solution

2 agents: Divide-and-Choose

One agent divides the resource into what she believes are equal halves, and the other agent chooses the “half” she prefers

Bibliographic notes

- 1940’s: proportional protocol for any number of agents
- late 1950’s: envy-free protocol for 3 agents
- 1995: envy-free protocol for any number of agents
Allocation of indivisible goods

- $n$ agents $\{1, \ldots, n\}$
- $m$ indivisible items $\{1, \ldots, m\}$
- $u_i(j)$, nonnegative utility of agent $i$ for item $j$
- A solution/allocation is a partition of the set of items $A_1 \cup \ldots \cup A_n$ so that agent $i$ receives $A_i$
- Agent $i$’s utility for $A_i$ is denoted by $u_i(A_i)$ and defined as $\sum_{j \in A_i} u_i(j)$

Normalization assumption: $\sum_{j=1}^{m} u_i(j) = 1$ for all $i \in [n]$
Allocation of indivisible goods

- $n$ agents $\{1, \ldots, n\}$
- $m$ indivisible items $\{1, \ldots, m\}$
- $u_i(j)$, nonnegative utility of agent $i$ for item $j$
- A solution/allocation is a partition of the set of items $A_1 \cup \ldots \cup A_n$ so that agent $i$ receives $A_i$
- Agent $i$’s utility for $A_i$ is denoted by $u_i(A_i)$ and defined as $\sum_{j \in A_i} u_i(j)$

Normalization assumption: $\sum_{j=1}^{m} u_i(j) = 1$ for all $i \in [n]$. 

Proportionality

Proportionality: \( u_i(A_i) \geq 1/n \) for all \( i \in [n] \)

A proportional allocation is not guaranteed to exist

<table>
<thead>
<tr>
<th></th>
<th>item 1</th>
<th>item 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>agent 2</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

The agent who does not receive item 2 has total utility \( < \frac{1}{2} \)
Hill’s substitute

Ted Hill [The Annals of Probability, 1987] tried to characterize the value $t_n$ such that, in any case, there exists an allocation $A$ satisfying

$$u_i(A_i) \geq t_n \text{ for all } i \in [n]$$

The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$
Hill’s substitute

Ted Hill [The Annals of Probability, 1987] tried to characterize the value $t_n$ such that, in any case, there exists an allocation $A$ satisfying

$$u_i(A_i) \geq t_n \text{ for all } i \in [n]$$

The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$

<table>
<thead>
<tr>
<th></th>
<th>item 1</th>
<th>item 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>agent 2</td>
<td>0.1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

$\alpha = 0.9$

There exists an allocation $A$ such that $u_i(A_i) \geq 0.2$ for $i = 1, 2$
Hill’s substitute

Ted Hill [The Annals of Probability, 1987] tried to characterize the value $t_n$ such that, in any case, there exists an allocation $A$ satisfying

$$u_i(A_i) \geq t_n \text{ for all } i \in [n]$$

The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$

<table>
<thead>
<tr>
<th></th>
<th>item 1</th>
<th>item 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>agent 2</td>
<td>0.1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

$\alpha = 0.9$

There exists an allocation $A$ such that $u_i(A_i) \geq 0.2$ for $i = 1, 2$
Hill’s substitute

Ted Hill [The Annals of Probability, 1987] tried to characterize the value $t_n$ such that, in any case, there exists an allocation $A$ satisfying

$$u_i(A_i) \geq t_n \text{ for all } i \in [n]$$

The result relies on a parameter

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j)$$

<table>
<thead>
<tr>
<th></th>
<th>item 1</th>
<th>item 2</th>
<th>item 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>agent 1</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>agent 2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

$\alpha = 0.5$

There exists an allocation $A$ such that $u_i(A_i) \geq 0.5$ for $i = 1, 2$
Hill’s substitute

Hill’s characterization is a non increasing function
\( V_n : [0, 1] \rightarrow [0, n^{-1}] \)

\[
V_n(x) = \begin{cases} 
1 - p(n - 1)x, & x \in I(n, p) \\
1 - \frac{(p+1)(n-1)}{(p+1)n-1}, & x \in NI(n, p)
\end{cases}
\]

where for any integer \( p \geq 1 \),
\[
I(n, p) = \left[ \frac{p+1}{p((p+1)n-1)}, \frac{1}{pn-1} \right] \quad \text{and} \quad NI(n, p) = \left[ \frac{1}{(p+1)n-1}, \frac{p+1}{p((p+1)n-1)} \right]
\]
If there are $n$ agents and $\alpha := \max_{i \in [n], j \in [m]} u_i(j)$ then there must exist an allocation $A$ such that $u_i(A_i) \geq V_n(\alpha)$ for all $i \in [n]$
An algorithmic approach

Markakis & Psomas [WINE 2011] proposed a centralized algorithm called ALLOCATE, polynomial in $(n, m)$, that builds an allocation $A$ such that $u_i(A_i) \geq V_n(\alpha)$

Moreover, the allocation is such that $u_i(A_i) \geq V_n(\alpha_i) \geq V_n(\alpha)$

where

$$\alpha_i = \max_{j \in [m]} u_i(j)$$

and

$$\alpha := \max_{i \in [n], j \in [m]} u_i(j) = \max_{i \in [n]} \alpha_i$$

Note that the utility of the least happy agent remains the same
An algorithmic approach

Markakis & Psomas [WINE 2011] proposed a centralized algorithm called ALLOCATE, polynomial in \((n, m)\), that builds an allocation \(A\) such that \(u_i(A_i) \geq V_n(\alpha)\)

Moreover, the allocation is such that \(u_i(A_i) \geq V_n(\alpha_i) \geq V_n(\alpha)\)

where

\[
\alpha_i = \max_{j \in [m]} u_i(j)
\]

and

\[
\alpha := \max_{i \in [n], j \in [m]} u_i(j) = \max_{i \in [n]} \alpha_i
\]

Note that the utility of the least happy agent remains the same
Centralized versus decentralized algorithms

Centralized

- A third party gathers the utilities of the agents for the different portions and computes an allocation
- An agent is often reluctant to disclose his utilities (is the third party trustworthy?)
- Issue with the communication of the utilities for every possible portion

Decentralized (Mechanism design)

- There is no third party to compute the allocation; each agent takes part of the construction of the final solution
- The communication problem is avoided
Centralized versus decentralized algorithms

Centralized

- A third party gathers the utilities of the agents for the different portions and computes an allocation
- An agent is often reluctant to disclose his utilities (is the third party trustworthy?)
- Issue with the communication of the utilities for every possible portion

Decentralized (Mechanism design)

- There is no third party to compute the allocation; each agent takes part of the construction of the final solution
- The communication problem is avoided
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up
Matroids

A matroid $\mathcal{M} = (X, \mathcal{F})$ consists of a finite set of $n$ elements $X$ and a collection $\mathcal{F}$ of subsets of $X$ such that:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $F_2 \subseteq F_1$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$,

(iii) for every couple $F_1, F_2 \in \mathcal{F}$ such that $|F_1| < |F_2|$, 
$\exists \ e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$.

Every $F \subseteq X$ s.t. $F \in \mathcal{F}$ is said independent

Every $C \subseteq X$ s.t. $C \notin \mathcal{F}$ is said dependent

A base $B$ is an independent of maximal size
A matroid $\mathcal{M} = (X, \mathcal{F})$ consists of a finite set of $n$ elements $X$ and a collection $\mathcal{F}$ of subsets of $X$ such that:

(i) $\emptyset \in \mathcal{F}$,
(ii) if $F_2 \subseteq F_1$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$,
(iii) for every couple $F_1, F_2 \in \mathcal{F}$ such that $|F_1| < |F_2|$, there exists $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$.

Every $F \subseteq X$ s.t. $F \in \mathcal{F}$ is said independent.

Every $C \subseteq X$ s.t. $C \notin \mathcal{F}$ is said dependent.

A base $B$ is an independent of maximal size.
Example 1 : Forests

A forest in a graph $G$ is a subset of its edges \textit{without} any cycle.

We can define a matroid $\mathcal{M} = (X, \mathcal{F})$ over a graph $G$:

- $X$ is the edges of $G$,
- $\mathcal{F}$ is the set of all forests of $G$.

A base is a spanning tree.
Example 2: Allocation of indivisible goods

$n$ agents, $m$ indivisible items

For every item $j$, build the set $E_j = \{j^1, j^2, \ldots, j^n\}$

We can define a matroid $\mathcal{M} = (X, \mathcal{F})$ as follows

- $X = \bigcup_{j=1}^{m} E_j$
- $\mathcal{F} = \{F \subseteq X : |F \cap E_j| \leq 1, j \in [m]\}$
Example 2: Allocation of indivisible goods

3 agents, 4 goods...

An allocation is a spanning tree in this graph.

Taking edge $j^i$ means allocating item $j$ to agent $i$. 
Example 3: Seminar

- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker $j$ is available for a subset of $\{d_1, \ldots, d_\ell\}$

At most one speaker per day

Find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

One can define a matroid $(X, \mathcal{F})$

- $X = \{1, \ldots, m\}$
- $F \in \mathcal{F}$ if there exists $f : \{1, \ldots, m\} \to \{d_1, \ldots, d_\ell\}$, every $j \in F$ is available on day $f(j)$, and $f(j) \neq f(j')$ for all $(j, j') \in F^2$.
Example 3: Seminar

- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker $j$ is available for a subset of $\{d_1, \ldots, d_\ell\}$.

At most one speaker per day.

Find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$.

One can define a matroid $(X, F)$

- $X = \{1, \ldots, m\}$
- $F \in \mathcal{F}$ if there exists $f : \{1, \ldots, m\} \to \{d_1, \ldots, d_\ell\}$, every $j \in F$ is available on day $f(j)$, and $f(j) \neq f(j')$ for all $(j, j') \in F^2$
Matroids’ background

Introduced by H. Whitney (1935), a matroid is a structure that generalizes the notion of linear independence in vector spaces.

Many applications in combinatorial optimization:

- spanning trees
- assignment problem

Polynomial algorithms

- greedy: independent of maximum weight in a matroid
- maximum weight independent in the intersection of 2 matroids defined on the same set of elements
Matroids’ background

Introduced by H. Whitney (1935), a matroid is a structure that generalizes the notion of linear independence in vector spaces.

Many applications in combinatorial optimization:

- spanning trees
- assignment problem

Polynomial algorithms

- greedy: independent of maximum weight in a matroid
- maximum weight independent in the intersection of 2 matroids defined on the same set of elements

See J. Oxley, Matroid Theory, Oxford University Press, 1992. for more details
Matroids’ background

Introduced by H. Whitney (1935), a matroid is a structure that generalizes the notion of linear independence in vector spaces.

Many applications in combinatorial optimization:
- spanning trees
- assignment problem

Polynomial algorithms
- greedy: independent of maximum weight in a matroid
- maximum weight independent in the intersection of 2 matroids defined on the same set of elements

Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose: 2 agents

Divide-Ask-and-Choose: 3 agents

Wrap-up
The setting

Input

- a matroid $\mathcal{M} = (X, \mathcal{F})$
- $n$ agents
- a utility function $u_i : X \to \mathbb{R}^+$ for every agent $i$

Additive utility: Agent $i$ has utility $u_i(F) := \sum_{x \in F} u_i(x)$ for every set $F \in \mathcal{F}$
The setting

Input

- a matroid $\mathcal{M} = (X, \mathcal{F})$
- $n$ agents
- a utility function $u_i : X \to \mathbb{R}^+$ for every agent $i$

Additive utility: Agent $i$ has utility $u_i(F) := \sum_{x \in F} u_i(x)$ for every set $F \in \mathcal{F}$

Goal: Build a “collective solution” $F \in \mathcal{F}$ and consider $(u_i(F))_{i \in [n]}$, the profile of the agents’ utilities
The setting

Input

- a matroid $\mathcal{M} = (X, \mathcal{F})$
- $n$ agents
- a utility function $u_i : X \to \mathbb{R}^+$ for every agent $i$

Additive utility: Agent $i$ has utility $u_i(F) := \sum_{x \in F} u_i(x)$ for every set $F \in \mathcal{F}$

Goal: Build a “collective solution” $F \in \mathcal{F}$ and consider $(u_i(F))_{i \in [n]}$, the profile of the agents’ utilities
The setting

Remark
Since the utility for every element is nonnegative, one can restrict ourselves to the bases of $M$.

Normalization to 1
For every agent $i$ and $F \in \mathcal{F}$ we have $0 \leq u_i(F) \leq 1$

$u_i(B_i^*) = 1$ where $B_i^* \in \mathcal{F}$ is the base that agent $i$ likes the most

No loss of generality (rescaling)
The setting

**Remark**
Since the utility for every element is nonnegative, one can restrict ourselves to the bases of $\mathcal{M}$

**Normalization to 1**
For every agent $i$ and $F \in \mathcal{F}$ we have $0 \leq u_i(F) \leq 1$

$u_i(B_i^*) = 1$ where $B_i^* \in \mathcal{F}$ is the base that agent $i$ likes the most

No loss of generality (rescaling)
Example: Seminar

- $n$ agents $\{a_1, \ldots, a_n\}$
- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker is available for a subset of $\{d_1, \ldots, d_\ell\}$, at most one speaker per day, find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>0.4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.3</td>
<td>0.4</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$n = 2$
Example: Seminar

- $n$ agents $\{a_1, \ldots, a_n\}$
- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker is available for a subset of $\{d_1, \ldots, d_\ell\}$, at most one speaker per day, find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$n = 2$
Example: Seminar

- $n$ agents $\{a_1, \ldots, a_n\}$
- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_{\ell}\}$

A speaker is available for a subset of $\{d_1, \ldots, d_{\ell}\}$, at most one speaker per day, find a subset of speakers that we can assign to $\{d_1, \ldots, d_{\ell}\}$

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$n = 2$
**Example : Seminar**

- $n$ agents $\{a_1, \ldots, a_n\}$
- $m$ speakers $\{1, \ldots, m\}$
- $\ell$ days $\{d_1, \ldots, d_\ell\}$

A speaker is available for a subset of $\{d_1, \ldots, d_\ell\}$, at most one speaker per day, find a subset of speakers that we can assign to $\{d_1, \ldots, d_\ell\}$

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$n = 2$
Example: Seminar

<table>
<thead>
<tr>
<th>maximal feasible sets</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$\min{u_1, u_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, 2, 3}</td>
<td>0.8</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>{1, 2, 5}</td>
<td>1</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>{1, 3, 4}</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>{1, 3, 5}</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>{1, 4, 5}</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>{2, 3, 5}</td>
<td>0.8</td>
<td>0.9</td>
<td>0.8</td>
</tr>
<tr>
<td>{3, 4, 5}</td>
<td>0.7</td>
<td>1</td>
<td>0.7</td>
</tr>
</tbody>
</table>
Centralized algorithm

One can extend \textsc{allocate}, the centralized algorithm by Markakis & Psomas, to matroids.

For every number of agents $n$, and every matroid $\mathcal{M}$, there exists a centralized algorithm which outputs a base $B$ of $\mathcal{M}$ such that $u_i(B) \geq V_n(\alpha_i)$ for all agent $i$.

\begin{align*}
V_n(\alpha) &= 1 - \frac{p(n-1)\alpha}{(p+1)n-1} \\
V_n(\alpha) &= 1 - \frac{(p+1)(n-1)}{(p+1)n-1}
\end{align*}

G, Monnot, Tlilane IJCAI 2013
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up
Divide-and-Choose on a matroid (2 agents)

The input is a matroid \((X, \mathcal{F})\)

1. Agent 1 computes a base and cuts it in two parts \(S_1 \cup T_1\)
2. Agent 2 finds \(S_2\) and \(T_2\) s.t. \(S_1 \cup S_2\) and \(T_1 \cup T_2\) are bases
3. Agent 2 chooses the base that she prefers between \(S_1 \cup S_2\) and \(T_1 \cup T_2\)
The viewpoint of Agent 1

Lemma
Given a base $B$ of a matroid partitioned in $S \cup T$ and another base $B^*$ with maximal utility, one can always partition $B^*$ in $S^* \cup T^*$ such that $\min\{u(S^*), u(T^*)\} \geq \min\{u(S), u(T)\}$

Hence, in the DaC mechanism, the first agent should compute a base which maximizes her utility (greedy algorithm) and partition it such that the lightest part is maximized is an NP-hard problem (SUBSET-SUM); but one can use ALLOCATE which is polynomial
The viewpoint of Agent 1

Lemma
Given a base $B$ of a matroid partitioned in $S \cup T$ and another base $B^*$ with maximal utility, one can always partition $B^*$ in $S^* \cup T^*$ such that $\min\{u(S^*), u(T^*)\} \geq \min\{u(S), u(T)\}$

Hence, in the DaC mechanism, the first agent should compute a base which maximizes her utility (greedy algorithm) and partition it such that the lightest part is maximized is an $NP$-hard problem (SUBSET-SUM); but one can use ALLOCATE which is polynomial.

By doing so the first agent is guaranteed to have a utility of $V_2(\alpha_1)$
The viewpoint of Agent 1

Lemma

Given a base $B$ of a matroid partitioned in $S \cup T$ and another base $B^*$ with maximal utility, one can always partition $B^*$ in $S^* \cup T^*$ such that $\min\{u(S^*), u(T^*)\} \geq \min\{u(S), u(T)\}$

Hence, in the DaC mechanism, the first agent should compute a base which maximizes her utility (greedy algorithm) and partition it

Partitioning such that the lightest part is maximized is an \textbf{NP}-hard problem (\textsc{subset-sum}); but one can use \textsc{allocate} which is polynomial

By doing so the first agent is guaranteed to have a utility of $V_2(\alpha_1)$
Divide-and-Choose on a matroid (2 agents)

The input is a matroid \((X, \mathcal{F})\)

1. Agent 1 computes a base and cut its in two parts \(S_1 \cup T_1\)
2. Agent 2 finds \(S_2\) and \(T_2\) s.t. \(S_1 \cup S_2\) and \(T_1 \cup T_2\) are bases
3. Agent 2 chooses the base that she prefers between \(S_1 \cup S_2\) and \(T_1 \cup T_2\)
The viewpoint of Agent 2

Theorem [Brylawski ’73, Greene ’73, Woodall ’74]

Given two bases $B_1$ and $B_2$ of a matroid $\mathcal{M}$, and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are two bases of $\mathcal{M}$

Hence, in the DaC mechanism, the second agent can compute a base $B_2^*$ which maximizes her utility and find the partition $S_2 \cup T_2$ of it such that $S_1 \cup S_2$ and $T_1 \cup T_2$ are both bases

- the first step is polynomial (greedy algorithm)
- the second step is also polynomial (matroid intersection)
The viewpoint of Agent 2

Theorem [Brylawski ’73, Greene ’73, Woodall ’74]
Given two bases $B_1$ and $B_2$ of a matroid $M$, and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are two bases of $M$

Hence, in the DaC mechanism, the second agent can compute a base $B_2^*$ which maximizes her utility and find the partition $S_2 \cup T_2$ of it such that $S_1 \cup S_2$ and $T_1 \cup T_2$ are both bases

- the first step is polynomial (greedy algorithm)
- the second step is also polynomial (matroid intersection)

By doing so the second agent is guaranteed to have a utility of $\max\{u_2(S_1 \cup S_2), u_2(T_1 \cup T_2)\} \geq \max\{u_2(S_2), u_2(T_2)\} \geq u(B_2^*)/2 = 1/2 > V_2(\alpha_2)$
The viewpoint of Agent 2

Theorem [Brylawski ’73, Greene ’73, Woodall ’74]

Given two bases $B_1$ and $B_2$ of a matroid $M$, and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that $X_1 \cup Y_2$ and $X_2 \cup Y_1$ are two bases of $M$.

Hence, in the DaC mechanism, the second agent can compute a base $B_2^*$ which maximizes her utility and find the partition $S_2 \cup T_2$ of it such that $S_1 \cup S_2$ and $T_1 \cup T_2$ are both bases.

- the first step is polynomial (greedy algorithm)
- the second step is also polynomial (matroid intersection)

By doing so the second agent is guaranteed to have a utility of

$$\max\{u_2(S_1 \cup S_2), u_2(T_1 \cup T_2)\} \geq \max\{u_2(S_2), u_2(T_2)\} \geq u(B_2^*)/2 = 1/2 > V_2(\alpha_2)$$
Summary

With the matroid extension of Divide-and-Choose, Agent 1 and 2 can guarantee to themselves $V_2(\alpha_1)$ and $0.5 > V_2(\alpha_2)$, respectively.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th></th>
<th>d₁</th>
<th>d₂</th>
<th>d₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.1</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.3</td>
<td></td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.2</td>
<td></td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>0.4</td>
<td></td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.3</td>
<td>0.4</td>
<td></td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Summary

With the matroid extension of Divide-and-Choose, Agent 1 and 2 can guarantee to themselves $V_2(\alpha_1)$ and $0.5 > V_2(\alpha_2)$, respectively.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Agent 1 proposes $\{\{1, 5\}, \{2\}\}$. Agent 2 can partition his base $\{3, 4, 5\}$ into $\{\{4\}, \{3, 5\}\}$ such that $\{1, 4, 5\}$ and $\{2, 3, 5\}$ are two bases. Agent 2 likes them equally; suppose $\{1, 4, 5\}$ is returned. The two agents have utility 0.9.
Summary

With the matroid extension of Divide-and-Choose, Agent 1 and 2 can guarantee to themselves $V_2(\alpha_1)$ and $0.5 > V_2(\alpha_2)$, respectively.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>d₁</th>
<th>d₂</th>
<th>d₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Agent 1 proposes $\{\{1, 5\}, \{2\}\}$. Agent 2 can partition his base $\{3, 4, 5\}$ into $\{\{4\}, \{3, 5\}\}$ such that $\{1, 4, 5\}$ and $\{2, 3, 5\}$ are two bases. Agent 2 likes them equally; suppose $\{1, 4, 5\}$ is returned. The two agents have utility 0.9.
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose : 2 agents

Divide-Ask-and-Choose : 3 agents

Wrap-up
A protocol for 3 agents on a matroid

1. Agent 1 computes a base and cuts it in 3 parts $A_1 \cup A_2 \cup A_3$
2. Agent 2 chooses one part, say $A_i$, and asks Agent 3 if he agrees to “give” this part to Agent 1
3. If Agent 3 agrees then $A_i$ is in the final solution and Agent 2 and 3 apply Divide-and-Choose on the contraction of $\mathcal{M}$ by $A_i$
4. Otherwise, Agents 1 and 2 apply Divide-and-Choose on the contraction of $\mathcal{M}$ by $A_i$; $R$ denotes the resulting independent set
5. Agent 3 completes $R$ into a base of $\mathcal{M}$
Theorem
Using Divide-Ask-and-Choose, Agents 1, 2 and 3 can guarantee to themselves $V_3(\alpha_1)$, $\frac{2}{3} V_2(\alpha_2) \geq V_3(\alpha_2)$ and $1/3 \geq V_3(\alpha_3)$, respectively.

This result partially relies on the following lemma

Multiple exchange property (Greene & Magnanti 1975)
Let $A$ and $B$ be bases of a matroid $\mathcal{M}$ such that $A$ is partitioned into $\{A_1, \ldots, A_n\}$. Then there exists a partition of $B$ into $\{B_1, \ldots, B_n\}$ such that $(A - A_i) \cup B_i$ is a base for all $i \in [n]$. 
Theorem
Using Divide-Ask-and-Choose, Agents 1, 2 and 3 can guarantee to themselves $V_3(\alpha_1)$, $\frac{2}{3} V_2(\alpha_2) \geq V_3(\alpha_2)$ and $1/3 \geq V_3(\alpha_3)$, respectively.

This result partially relies on the following lemma

Multiple exchange property (Greene & Magnanti 1975)
Let $A$ and $B$ be bases of a matroid $\mathcal{M}$ such that $A$ is partitioned into $\{A_1, \ldots, A_n\}$. Then there exists a partition of $B$ into $\{B_1, \ldots, B_n\}$ such that $(A - A_i) \cup B_i$ is a base for all $i \in [n]$. 
Outline

Allocation of goods

Matroids

Our problem(s)

Divide-and-Choose: 2 agents

Divide-Ask-and-Choose: 3 agents

Wrap-up
The fair allocation of (in)divisible goods is a challenging problem with a lot of applications.

We work with matroids, a structure that extends the allocation of indivisible goods.
The fair allocation of (in)divisible goods is a challenging problem with a lot of applications.

We work with matroids, a structure that extends the allocation of indivisible goods.

Known results for the worst case utility of the agents in the problem of allocating indivisible goods apply to matroids:

- centralized (any number of agents)
- decentralized (limited number of agents, \( n \leq 8 \))
The fair allocation of (in)divisible goods is a challenging problem with a lot of applications.

We work with matroids, a structure that extends the allocation of indivisible goods.

Known results for the worst case utility of the agents in the problem of allocating indivisible goods apply to matroids:

- centralized (any number of agents)
- decentralized (limited number of agents, $n \leq 8$)
THANK YOU FOR YOUR ATTENTION