A proof of a conjecture of Barát and Thomassen

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Decomposition of graphs

$T$-decomposition: edge-partition into copies of $T$. 
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$S_4$-decomposition
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$S_4$-decomposition

$P_3$-decomposition
Wilson’s Theorem

Theorem (Wilson 1976)

For any tree $T$, $K_n$ admits a $T$-decomposition, for $n$ sufficiently large (provided divisibility condition).
Minimum degree condition

**Theorem (Barber, Kuhn, Lo, Osthus 2016)**

For every $T$, $\exists \ \tau > 0$ s.t. if $G$ has minimum degree $(1 - \tau)|V(G)|$, then $G$ has $T$-decomposition (provided divisibility condition).
Conjecture [Barát, Thomassen – 2006]

For every fixed tree \( T \), there exists a positive constant \( c_T \) such that every \( c_T \)-edge-connected graph with size divisible by \( |E(T)| \) admits a \( T \)-decomposition.
Barát-Thomassen conjecture

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Verified for \( T \) being

- stars [Thomassen – 2012]
- \((k, k + 1)\)-bistars [Thomassen – 2013]
- of deg. sequence \((1, 1, 1, 2, 3)\) [Barát, Garbner – 2014]
Barát-Thomassen conjecture

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... and actually whenever $\text{diam}(T) \leq 4$ [Merker – 2015+].
When $T$ is a path: $T = P_\ell$

- $\ell \in \{3, 4\}$ [Thomassen – 2008],
- $\ell = 2^k$ for any $k$ [Thomassen – 2013],
- $\ell = 5$ [Botler, Mota, Oshiro, Wakabayashi – 2015+],
- $\ell$ is any value [Botler, Mota, Oshiro, Wakabayashi – 2015+].
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Note: 2-edge-connectivity does not suffice; e.g. for

... and make $\delta$ increase with preserving non $P_9$-decomposability.
Relations to Tutte’s nowhere zero 3-flow conjecture

**Theorem (Tutte’s Conjecture)**

*Every 4-edge-connected graph admits a nowhere zero 3-flow.*

- $K_{1,3}$-decompositions relate to flows: Tutte’s conjecture implies every 10-e.c. graph has $K_{1,3}$-decomposition.
- Conversely, if every 8-e.c. $G$ admits a $K_{1,3}$-decomposition, then Tutte holds with $e.c = 8$. 
The Barát-Thomassen Conjecture

Theorem (Bensmail, Le, Merker, Thomassé, H. – 2015+)

*The Barát-Thomassen conjecture is true.*
Theorem (Barát-Gerbner (2014), also Thomassen (2013))

It is sufficient to prove the conjecture for $G$ bipartite.
Proof technique

1. Prove that from $G$ can extract a ‘rich/stable’ structure $S$.
2. Use probabilistic tools to get a ‘nearly good’ decomposition on $S$.
3. Use the structure $S$ to repair ‘blemishes’.
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- 'Absorbing' technique
- STABILITY RESULT + NOISE
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- 'Absorbing' technique: STABILITY RESULT + NOISE

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2. Use probabilistic tools to get a 'nearly good' decomposition on $S$
3. Use the structure $S$ to repair 'blemishes'.
Definition

\( G = (A, B) \) bipartite, \( T = (T_A, T_B) \) a tree. An edge-colouring \( \varphi : E(G) \to E(T) \) is called \textit{T-equitable}, if for any pair of vertices \( v \in V(G), t \in V(T) \) in the same part, we have \( d_j(v) = d_k(v) \) for all pair of colors \( j, k \) incident to \( t \).
Preliminaries: $T$-equitable coloring

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![Diagram](image)
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**Theorem (Merker 2015+)**

A highly edge connected bipartite $G$ (+ other divisibility assumptions) has a $T$-equitable coloring where the min. degree in each color is large.
For each $v \in V(G)$ and $t \in V(T)$ in the same part, let $v$ “play the role” of $t$ by matching randomly all the colored edges around $t$ on $v$. This yields a natural decomposition, but... some trees in $G$ may intersect themselves. Overwhelming majority of copies are isomorphic to $T$. 
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\[ T \] 

\[ G \]
Number of non-isomorphic trees

$X_v(t_i, t_j) := \text{number of non-isomorphic trees where } v \text{ is the root where images of } t_i \text{ and } t_j \text{ are the same.}$

$E[X_v(t_0, t_j)] \leq 14 / 18$
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\[ \mathbb{E}[X_v(t_0, t_j)] \leq 1! \]
McDiarmid’s Inequality (Simplified version)

Let $X$ be a non-negative random variable, determined by $m$ independent random permutations $\Pi_1, \ldots, \Pi_m$ satisfying the following conditions for some $d, r > 0$

- interchanging two elements in any one permutation can affect $X$ by at most $d$;
- for any $s$, if $X \geq s$ then there is a set of at most $rs$ choices whose outcomes certify that $X \geq s$,

then for any $0 \leq t \leq \mathbb{E}[X]$,

$$
\Pr[|X - \mathbb{E}[X]| > t + 60d^p \frac{r\mathbb{E}[X]}{\mathbb{E}[X]}] \leq 4e^{-\frac{t^2}{8d^2r\mathbb{E}[X]}}.
$$
Probabilistic Machinery involved

**Lovász Local Lemma**

Let $A_1, \ldots, A_n$ be events in some probability space $\Omega$ with $\mathbb{P}[A_i] \leq p$ for all $i \in \{1, \ldots, n\}$. Suppose that each $A_i$ is mutually independent of all but at most $d$ other events $A_j$. If $4pd < 1$, then $\mathbb{P}[\bigcap_{i=1}^{n} \overline{A_i}] > 0$. 


A non-isomorphic $T$ is $i$-good if the images of $t_0, \ldots, t_i$ are pairwise distinct.
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Use one isomorphic copy to fix each homomorphic copy which is bad at $t_4$ by switching subtrees at the parent of $t_4$.

This creates even more bad 'trees', but all of them 4-good!

Repeat for $t_5, t_6$, etc.
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Conjecture: There is a function $f$ such that, for any fixed tree $T$ with maximum degree $\Delta_T$, every $f(\Delta_T)$-edge-connected graph with its number of edges divisible by $|E(T)|$ and minimum degree at least $f(|E(T)|)$ can be $T$-decomposed.
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**Theorem (Bensmail, Le, Thomassé, H. 2016+)**

Let $G$ be a 24-e.c. graph with $|E(G)|$ and of sufficiently large minimum degree (wrt to $\ell$). Then $G$ admits a $P_\ell$-decomposition.
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**Thank you.**