A proof of a conjecture of Barát and Thomassen

Julien Bensmail (Sophia-Antipolis), Ararat Harutyunyan, Tien-Nam Le (ENS Lyon), Martin Merker (Tech. U. Denmark) and Stéphan Thomassé (ENS Lyon)

Institut de Mathématiques, University of Toulouse III (Paul Sabatier)

Bordeaux Graph Workshop
November 8, 2016
Decomposition of graphs

\(T\)-decomposition: **edge-partition** into copies of \(T\).
\( T \)-decomposition: \textbf{edge-partition} into \textbf{copies of} \( T \).

\( S_4 \)-decomposition
Decomposition of graphs

\(T\)-decomposition: edge-partition into copies of \(T\).

\(S_4\)-decomposition

\(P_3\)-decomposition
Wilson’s Theorem

**Theorem (Wilson 1976)**

*For any tree $T$, $K_n$ admits a $T$-decomposition, for $n$ sufficiently large (provided divisibility condition).*
Minimum degree condition

Theorem (Barber, Kuhn, Lo, Osthus 2016)

For every $T$, $\exists \epsilon_T > 0$ s.t. if $G$ has minimum degree $(1 - \epsilon_T)|V(G)|$, then $G$ has $T$-decomposition (provided divisibility condition).
Barát-Thomassen conjecture

Conjecture [Barát, Thomassen – 2006]

For every fixed tree $T$, there exists a positive constant $c_T$ such that every $c_T$-edge-connected graph with size divisible by $|E(T)|$ admits a $T$-decomposition.
Barát-Thomassen conjecture

Conjecture [Barát, Thomassen – 2006]

For every fixed tree $T$, there exists a positive constant $c_T$ such that every $c_T$-edge-connected graph with size divisible by $|E(T)|$ admits a $T$-decomposition.

Verified for $T$ being

- stars [Thomassen – 2012]
- $(k, k + 1)$-bistars [Thomassen – 2013]
- of deg. sequence $(1, 1, 1, 2, 3)$ [Barát, Garbner – 2014]
Conjecture [Barát, Thomassen – 2006]

For every fixed tree $T$, there exists a positive constant $c_T$ such that every $c_T$-edge-connected graph with size divisible by $|E(T)|$ admits a $T$-decomposition.

Verified for $T$ being

- stars [Thomassen – 2012]
- $(k, k + 1)$-bistars [Thomassen – 2013]
- of deg. sequence $(1, 1, 1, 2, 3)$ [Barát, Garbner – 2014]

... and actually whenever $\text{diam}(T) \leq 4$ [Merker – 2015+].
When $T$ is a path: $T = P_\ell$

- $\ell \in \{3, 4\}$ [Thomassen – 2008],
- $\ell = 2^k$ for any $k$ [Thomassen – 2013],
- $\ell = 5$ [Botler, Mota, Oshiro, Wakabayashi – 2015+],
- $\ell$ is any value [Botler, Mota, Oshiro, Wakabayashi – 2015+].
Is edge-connectivity necessary?

The following would be best optimal:

- 3-edge-connectivity,
Is edge-connectivity necessary?

The following would be best optimal:

- 3-edge-connectivity,

Note: 2-edge-connectivity does not suffice; e.g. for

... and make $\delta$ increase with preserving non $P_9$-decomposability.
Relations to Tutte’s nowhere zero 3-flow conjecture

Theorem (Tutte’s Conjecture)

*Every 4-edge-connected graph admits a nowhere zero 3-flow.*

- $K_{1,3}$-decompositions relate to flows: Tutte’s conjecture implies every 10-e.c. graph has $K_{1,3}$-decomposition.
- Conversely, if every 8-e.c. $G$ admits a $K_{1,3}$-decomposition, then Tutte holds with e.c = 8.
The Barát-Thomassen Conjecture

Theorem (Bensmail, Le, Merker, Thomassé, H. – 2015+)

The Barát-Thomassen conjecture is true.
Theorem (Barát-Gerbner (2014), also Thomassen (2013))

*It is sufficient to prove the conjecture for $G$ bipartite.*
Proof technique

1. Prove that from $G$ can extract a 'rich/stable' structure $S$.

2. Use probabilistic tools to get a 'nearly good' decomposition on $S$.

3. Use the structure $S$ to repair 'blemishes'.

Absorbing' technique

STABILITY RESULT + NOISE
Proof technique

- 'Absorbing' technique

STABILITY RESULT + NOISE
Proof technique

- 'Absorbing' technique  STABILITY RESULT + NOISE

1. Prove that from $G$ can extract a 'rich/stable' structure $S$
2. Use probabilistic tools to get a 'nearly good' decomposition on $S$. 
3. Use the structure $S$ to repair 'blemishes'. 
**Preliminaries: $T$-equitable coloring**

**Definition**

$G = (A, B)$ bipartite, $T = (T_A, T_B)$ a tree. An edge-colouring $\phi : E(G) \to E(T)$ is called **$T$-equitable**, if for any pair of vertices $v \in V(G)$, $t \in V(T)$ in the same part, we have $d_j(v) = d_k(v)$ for all pair of colors $j, k$ incident to $t$.
Preliminaries: \( T \)-equitable coloring

**Definition**

\( G = (A, B) \) bipartite, \( T = (T_A, T_B) \) a tree. An edge-colouring \( \phi : E(G) \to E(T) \) is called \( T \)-equitable, if for any pair of vertices \( v \in V(G), t \in V(T) \) in the same part, we have \( d_j(v) = d_k(v) \) for all pair of colors \( j, k \) incident to \( t \).
Preliminaries: \( T \)-equitable coloring

**Definition**

\( G = (A, B) \) bipartite, \( T = (T_A, T_B) \) a tree. An edge-colouring \( \phi : E(G) \to E(T) \) is called **\( T \)-equitable**, if for any pair of vertices \( v \in V(G), t \in V(T) \) in the same part, we have \( d_j(v) = d_k(v) \) for all pair of colors \( j, k \) incident to \( t \).

**Theorem (Merker 2015+)**

A highly edge connected bipartite \( G \) (+ other divisibility assumptions) has a \( T \)-equitable coloring where the min. degree in each color is large.
For each \( v \in V(G) \) and \( t \in V(T) \) in the same part, let \( v \) “play the role” of \( t \) by matching randomly all the colored edges around \( t \) on \( v \).
For each \( v \in V(G) \) and \( t \in V(T) \) in the same part, let \( v \) “play the role” of \( t \) by matching randomly all the colored edges around \( t \) on \( v \).
For each $v \in V(G)$ and $t \in V(T)$ in the same part, let $v$ “play the role” of $t$ by matching randomly all the colored edges around $t$ on $v$.

This yields a natural decomposition, but... some trees in $G$ may intersect themselves.
For each $v \in V(G)$ and $t \in V(T)$ in the same part, let $v$ “play the role” of $t$ by matching randomly all the colored edges around $t$ on $v$.

This yields a natural decomposition, but... some trees in $G$ may intersect themselves.

Overwhelming majority of copies are isomorphic to $T$. 
Number of non-isomorphic trees

$X_v(t_0, t_j) := \text{number of non-isomorphic trees where } v \text{ is the root where images of } t_i \text{ and } t_j \text{ are the same.}$

$E[X_v(t_0, t_j)] \leq 1^{14/18}$
Number of non-isomorphic trees

$X_v := \text{number of non-isomorphic trees where } v \text{ is the root.}$

$X_v(t_i, t_j) := \text{number of non-isomorphic trees where } v \text{ is the root where images of } t_i \text{ and } t_j \text{ are the same.}$
Number of non-isomorphic trees

$X_v :=$ number of non-isomorphic trees where $v$ is the root.

$X_v(t_i, t_j) :=$ number of non-isomorphic trees where $v$ is the root where images of $t_i$ and $t_j$ are the same.

$\mathbb{E}[X_v(t_0, t_j)] \leq 1!$
McDiarmid’s Inequality (Simplified version)

Let $X$ be a non-negative random variable, determined by $m$ independent random permutations $\Pi_1, \ldots, \Pi_m$ satisfying the following conditions for some $d, r > 0$

- interchanging two elements in any one permutation can affect $X$ by at most $d$;
- for any $s$, if $X \geq s$ then there is a set of at most $rs$ choices whose outcomes certify that $X \geq s$,

then for any $0 \leq t \leq \mathbb{E}[X]$,

$$
Pr[|X - \mathbb{E}[X]| > t + 60d\sqrt{r\mathbb{E}[X]}] \leq 4e^{-\frac{t^2}{8d^2r\mathbb{E}[X]}}.
$$
Probabilistic Machinery involved

Lovász Local Lemma

Let $A_1, \ldots, A_n$ be events in some probability space $\Omega$ with $\mathbb{P}[A_i] \leq p$ for all $i \in \{1, \ldots, n\}$. Suppose that each $A_i$ is mutually independent of all but at most $d$ other events $A_j$. If $4pd < 1$, then $\mathbb{P}[\cap_{i=1}^{n} A_i] > 0$. 
Fixing bad ’trees’

- A non-isomorphic $T$ is $i$-good if the images of $t_0, \ldots, t_i$ are pairwise distinct.
Fixing bad ’trees’

- A non-isomorphic $T$ is \textit{i-good} if the images of $t_0, \ldots, t_i$ are pairwise distinct.
- Use one isomorphic copy to fix each homomorphic copy which is bad at $t_4$ by switching subtrees at the parent of $t_4$.
Fixing bad 'trees'

- A non-isomorphic $T$ is $i$-good if the images of $t_0, \ldots, t_i$ are pairwise distinct.
- Use one isomorphic copy to fix each homomorphic copy which is bad at $t_4$ by switching subtrees at the parent of $t_4$.
- This creates even more bad 'trees', but all of them 4-good!
Fixing bad 'trees'

- A non-isomorphic $T$ is $i$-good if the images of $t_0, ..., t_i$ are pairwise distinct.
- Use one isomorphic copy to fix each homomorphic copy which is bad at $t_4$ by switching subtrees at the parent of $t_4$.
- This creates even more bad 'trees', but all of them 4-good!
- Repeat for $t_5, t_6$ etc.
**Conjecture:** There is a function $f$ such that, for any fixed tree $T$ with maximum degree $\Delta_T$, every $f(\Delta_T)$-edge-connected graph with its number of edges divisible by $|E(T)|$ and minimum degree at least $f(|E(T)|)$ can be $T$-decomposed.
Conjecture: There is a function \( f \) such that, for any fixed tree \( T \) with maximum degree \( \Delta_T \), every \( f(\Delta_T) \)-edge-connected graph with its number of edges divisible by \( |E(T)| \) and minimum degree at least \( f(|E(T)|) \) can be \( T \)-decomposed.

Theorem (Bensmail, Le, Thomassé, H. 2016+)

Let \( G \) be a 24-e.c. graph with \( \ell \mid |E(G)| \) and of sufficiently large minimum degree (wrt to \( \ell \)). Then \( G \) admits a \( P_\ell \)-decomposition.
Conjecture: There is a function $f$ such that, for any fixed tree $T$ with maximum degree $\Delta_T$, every $f(\Delta_T)$-edge-connected graph with its number of edges divisible by $|E(T)|$ and minimum degree at least $f(|E(T)|)$ can be $T$-decomposed.

Theorem (Bensmail, Le, Thomassé, H. 2016+)

Let $G$ be a 24-e.c. graph with $\ell \mid |E(G)|$ and of sufficiently large minimum degree (wrt to $\ell$). Then $G$ admits a $P_\ell$-decomposition.

Thank you.