Coloring k-partite sparse digraphs

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Abstract

In this paper, we study colorings of k-partite sparse digraphs. The *chromatic number* of a graph G is the smallest integer k such that the vertices of G can be colored with k colors with the property that each color class is an independent set. The *dichromatic number* of a digraph D is the minimum k such that the vertices of D can be colored with k colors with each color class inducing an acyclic subdigraph. This coloring invariant shares many similarities with the graph chromatic number and can be thought of as its analogous digraph generalisation.

Our main result in this short note shows that there exist sparse k-partite digraphs which have dichromatic number k. This, in particular, not only implies that there exist graphs with equal chromatic and dichromatic number, but that they can be taken to be somewhat sparse.

Keywords: Graph coloring, Dichromatic number, k-partite digraphs

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1 Introduction

In this paper, we consider digraphs that are oriented graphs, i.e., they contain neither loops, parallel arcs nor directed cycles of length two. A subset S of vertices of a digraph D is called *acyclic* if the induced subdigraph on S contains no directed cycle. The *dichromatic number* $\vec{\chi}(D)$ of D is the smallest integer k such that $V(D)$ can be partitioned into k sets $V_1, ..., V_k$ where each V_i is acyclic. Note that, equivalently, the dichromatic number is the smallest integer k, such that the vertices of D can be colored with k colors so that there is no monochromatic directed cycle. It is easy to see that, for any undirected graph G and its bidirected digraph D obtained from G by replacing each edge by two oppositely oriented arcs, we have $\chi(G) = \vec{\chi}(D)$. The dichromatic number was first introduced by Neumann-Lara [10]. In recent years, there has been considerable attention devoted to this topic, and many results have demonstrated that this digraph invariant generalizes many results on the graph chromatic number (see, for example [1, 2, 3, 4, 6, 7, 8]). Some evidence of the surprising relationship between chromatic number of graphs and the dichromatic number of digraphs include the generalisation of Gallai's classical theorem on list coloring to digraphs in [3], the extension of the important result of Erdős that sparse graphs can have large chromatic number to digraphs in [4], the derivation of an analog of a classical result due to Bollobas in [7], etc.

For an undirected graph G, the *dichromatic number* of G, denoted by $\vec{\chi}(G)$, is the maximum dichromatic number over all its orientations.

In 1970s, Erdős and Neumann-Lara conjectured the following:

Conjecture 1.1. *[5] For every integer k, there exists an integer f_k such that if G is any graph satisfying $\chi(G) \geq f_k$, then $\vec{\chi}(G) \geq k$.*

It is clear that f_1 and f_2 exist ($f_2 = 3$). It is unknown if f_3 exists.

A graph G is *k-partite* if $V(G)$ can be partitioned into k independent sets, i.e., $\chi(G) \leq k$. For two positive integer n and k, let K_{n*k} denote the complete k-partite graph with n vertices in each vertex class. The main result of this paper is to show that the conjecture of Erdős and Neumann-Lara holds for very sparse subgraphs of K_{n*k} in the following strong form.

Theorem 1.2. *Let $k \geq 2$. There is an integer n_0 such that for any $n \geq n_0$ the following holds: let G_{n*k} be any k-partite graph with vertex classes $V_1, ..., V_k$ with $|V_i| = n$ for all $i \in [k]$ satisfying the property that for all distinct pairs i, j and all sets $V_i' \subset V_i$ and $V_j' \subset V_j$ with $|V_i'| \geq n/k^2$ and*

$|V_j'| \geq n/k^2$, *the number of edges between* V_i' *and* V_j' *is at least* $100n \log k/k^2$. *Then* $\vec{\chi}(G_{n*k}) = k$.

Remark 1.3. *We note that a random subgraph of* K_{n*k} *where the probability of an edge between any two vertices in different partite sets is at least* $f(n)/n$, *where* $f(n)$ *is a function increasing arbitrarily slowly, satisfies the condition of Theorem 1.2, and thus, its implication.*

2 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, let us recall the following result from Manber and Tompa [9].

Let S be the set of all graphs with n vertices and m edges and let $A(m, n)$ denote the maximum number of acyclic orientations that a graph in S can have.

Theorem 2.1. *[9]* $A(m,n) \leq \prod_{v \in V}(1 + \deg(v)) \leq \left(\frac{2m}{n} + 1\right)^n$.

Lemma 2.2. *Let* i *and* j *be distinct, and let* $G_{ij} = G_{n*k}[V_i \cup V_j]$ *be the bipartite graph induced on vertex-set* V_i *and* V_j. *There is an orientation of the edges of* G_{ij} *such that the resulting digraph* D *satisfies the property that for every* $V_i' \subset V_i$ *and* $V_j' \subset V_j$ *verifying* $|V_i'|, |V_j'| \geq n/k^2$, *the digraph* $D' = D[V_i' \cup V_j']$ *has a directed cycle.*

Proof. We employ the probabilistic method. Let D be the digraph obtained from G_{ij} by taking a uniformly random orientation of its edges.

Let $V_i' \subset V_i$ and $V_j' \subset V_j$ be two fixed sets with $|V_i'| = |V_j'| = \frac{n}{k^2}\cdot$ Let $D' = (V_i' \cup V_j', E)$ be the digraph induced on V_i' and V_j'. Let m be the number of edges of D' and recall that $m \geq 100n \log k/k^2$.

Now, using Theorem 2.1, we obtain that

$$\begin{aligned}
\mathbb{P}[\exists V_i', V_j' : D[V_i' \cup V_j'] \text{ is acyclic}] &\leq \binom{n}{n/k^2}^2 \frac{\left(\frac{2m}{2n/k^2} + 1\right)^{2n/k^2}}{2^m} \\
&\leq (ek^2)^{2n/k^2} \frac{\left(\frac{100n \log k/k^2}{n/k^2} + 1\right)^{2n/k^2}}{2^{100n \log k/k^2}} \\
&\leq (ek^2)^{2n/k^2} \frac{(100 \log k + 1)^{2n/k^2}}{k^{100n/k^2}} \\
&\leq \left(\frac{ek^2(100 \log k + 1)}{k^{50}}\right)^{2n/k^2} < 1.
\end{aligned}$$

This completes the proof.

$\square$

Now we will complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let D be the digraph obtained by taking an orientation of G_{n*k} such that for every pair i, j the conclusion of Lemma 2.2 holds for G_{ij}. We will show that $[\vec{\chi}(D) = k]$.

Assume D is colored with any number of colors. We can say that if there exist two partitions V_i and V_j and subsets of vertices $V_i' \subset V_i$ and $V_j' \subset V_j$ of the same color, with $|V_i'|, |V_j'| \geq n/k^2$, then there is a monochromatic directed cycle. Indeed, this follows from the aforementioned lemma.

Note that $\vec{\chi}(D) \leq \chi(G_{n*k}) \leq k$.

To prove that $\vec{\chi}(D) = k$, we assume, for a contradiction, that it is possible to color D with $k-1$ colors. As D is validly colored with $k-1$ colors it means that there cannot be two sets of the same color in different partitions verifying $|V_i'|, |V_j'| \geq \frac{n}{k^2}\cdot$

If there are $k-1$ colors, then there is at least one color (hereupon called *the dominant color*) so that there are at least $\frac{k \cdot n}{k-1}$ vertices colored with that color. Let $(c_1, c_2, ..., c_k)$ be the number of vertices that are assigned the dominant color in $(V_1, V_2, ..., V_k)$, respectively.

As mentioned above there cannot be c_i and c_j such that both are at least $\frac{n}{k^2}\cdot$ In the best case, only one of them can be more than $\frac{n}{k^2}\cdot$ Without loss of generality assume that $c_k \geq \frac{n}{k^2}\cdot$ The following inequality holds.

$$\frac{kn}{k-1} \leq \sum_{i=1}^{k} c_i\cdot \tag{1}$$

Moreover,

$$\sum_{i=1}^{k} c_i = \sum_{i=1}^{k-1} c_i + c_k \leq (k-1) \cdot \frac{n}{k^2} + c_k$$

By Equation 1, we have the following.

$$(k-1) \cdot \frac{n}{k^2} + c_k \geq \frac{kn}{k-1}$$

$$(k-1) \cdot \frac{n}{k^2} \geq \frac{kn}{k-1} - c_k$$

As $c_k \leq n$, we have

$$(k-1)\cdot\frac{n}{k^2}\geq\frac{k\cdot n}{k-1}-n=\frac{n}{k-1}$$

But the above inequality is invalid for any n and k. Thus, there necessarily exist at least two partite sets with more than $\frac{n}{k^2}$ vertices of the dominant color, which means with probability $1-o(1)$ there will exist a monochromatic cycle.

Thus, $\vec{\chi}(D)=k$. □

Corollary 2.3. *For every k, there is a simple graph G such that $\chi(G)=\vec{\chi}(G)=k$.*

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