Coloring dense digraphs

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Abstract

The *chromatic number* of a digraph D is the minimum number of acyclic subgraphs covering the vertex set of D. A tournament H is a *hero* if every H-free tournament T has chromatic number bounded by a function of H. Inspired by the celebrated Erdős–Hajnal conjecture, Berger et al. fully characterized the class of heroes in 2013. We extend this framework to dense digraphs: A digraph H is a *superhero* if every H-free digraph D has chromatic number bounded by a function of H and $\alpha(D)$, the independence number of the underlying graph of D. We prove here that a digraph is a superhero if and only if it is a hero, and hence characterize all superheroes. This answers a question of Aboulker, Charbit and Naserasr.

1 Introduction

Every digraph in this paper is simple, loopless and finite, where a digraph D is *simple* if for every two vertices u and v of D, there is at most one arc with endpoints $\{u, v\}$. Given a digraph D, we denote by $V(D)$ the vertex set of D. The *independence number* $\alpha(D)$ of a digraph D is the independence number of the underlying graph of D. A subset X of $V(D)$ is *acyclic* if the subgraph (i.e., subdigraph) of D induced by X contains no directed cycle. A k*-coloring* of a digraph D is a partition of $V(D)$ into k acyclic sets, and the *chromatic number* $\chi(D)$ is the minimum number k for which D admits a k-coloring. This digraph invariant was introduced by Neumann-Lara [13], and naturally generalizes many results on the graph chromatic number (see, for example, [3], [9] [10], [11], [12]).

Given digraphs D and H, we say that D is H*-free* if there is no induced subgraph of D isomorphic to H. A digraph T is a *tournament* if there is an arc between every pair of distinct vertices of T. A tournament H is a *hero* if there is a number $f(H)$ such that $\chi(T) \leq f(H)$ for every H-free tournament T. The class of heroes was fully characterized in 2013 by Berger et al [2].

One may try to study the more general question for arbitrary digraphs: for which digraph H does there exist a constant $\varepsilon(H) > 0$ such that every H-free digraph D satisfies $\chi(D) \leq |V(D)|^{1-\varepsilon(H)}$? In fact, it was conjectured in [8] that this holds for every H.

Conjecture 1.1. *For every digraph H, there is $\varepsilon > 0$ such that if D is a H-free digraph, then $\chi(D) \leq |V(D)|^{1-\varepsilon}$.*

The conjecture is open even when H is the oriented triangle, C_3. In fact, it can be viewed as a strengthening of the Erdős–Hajnal conjecture. For a comprehensive survey on the Erdős–Hajnal conjecture, we refer the reader to [5]. Given a digraph D, we denote by $\beta(D)$ the maximum number of vertices of an acyclic subset of $V(D)$. The following is one formulation of the Erdös–Hajnal conjecture.

Conjecture 1.2. *For every tournament H, there exists a constant $\varepsilon(H) > 0$ such that every H-free tournament T satisfies $\beta(T) \geq |V(T)|^{\varepsilon(H)}$.*

Motivated by Conjecture 1.1, we say that a digraph H is a *superhero* if for every integer $\alpha \geq 1$, there is a number $g(H, \alpha)$ such that $\chi(D) \leq g(H, \alpha)$ for every H-free digraph D with $\alpha(D) \leq \alpha$. There are several remarks. First, if a tournament H is a superhero, then H is a hero by definition. Second, if H is not a tournament, then H is not a superhero. Indeed, one can easily construct a tournament T with arbitrarily high chromatic number (see Lemma 3.1 for example). Note that H contains two non-adjacent vertices. Hence T contains no induced isomorphic copy of H, and so H is not a superhero. From two remarks above, it follows that every superhero is a hero. Aboulker, Charbit and Naserasr [1] conjectured that the converse is also true. The main result of this paper is the affirmation of this fact.

Theorem 1.3. *Every hero is a superhero.*

Given tournaments H_1, H_2, H_3, we denote by $H_1 \Rightarrow H_2$ the vertex-disjoint union of H_1 and H_2 with complete arcs from H_1 to H_2, and $\Delta(H_1, H_2, H_3)$ the vertex-disjoint union of H_1, H_2, H_3 with complete arcs from H_1 to H_2, from H_2 to H_3, and from H_3 to H_1. For every integer $k \geq 1$, let T_k denote the *transitive* tournament on k vertices, i.e. the acyclic tournament on k vertices. The class of heroes are constructed in [2] as follows:

- The singleton T_1 is a hero.
- If H_1 and H_2 are heroes, then $H_1 \Rightarrow H_2$ is a hero.
- If H is a hero, then $\Delta(H, T_k, T_1)$ and $\Delta(H, T_1, T_k)$ are heroes for every $k \geq 1$.

The main result of [2] is that any tournament that cannot be constructed by this process is not a hero. Obviously T_1 is a superhero. Thus to prove Theorem 1.3, it suffices to prove the following two theorems.

Theorem 1.4. *If H_1 and H_2 are superheroes, then $H_1 \Rightarrow H_2$ is a superhero.*

Theorem 1.5. *If H is a superhero, then $\Delta(H, T_1, T_k)$ and $\Delta(H, T_k, T_1)$ are superheroes for any $k \geq 1$.*

One may inquire how large $g(H, \alpha)$ needs to be for particular digraphs H. For digraphs not containing an oriented triangle, we believe that the following statement may be true.

Conjecture 1.6. *There is an integer ℓ such that if D is a C_3-free digraph with $\alpha(D) \leq \alpha$, then $\chi(D) \leq \alpha^{\ell}$.*

Indeed, if Conjecture 1.6 is true, then every C_3-free digraph D either has an independent (hence, acyclic) set of size $n^{1/2\ell}$ or has chromatic number at most $(n^{1/2\ell})^{\ell} = \sqrt{n}$, and hence has an acyclic set of size $\sqrt{n}$. Consequently, the special case of Conjecture 1.2 when $H = C_3$ would hold for $\varepsilon := 1/2\ell$.

While targeting a polynomial bound for chromatic number of C_3-free digraphs, we could not even achieve an exponential bound. However, by an algorithmic approach, we are able to obtain a factorial bound.

Theorem 1.7. *If D is a C_3-free digraph with $\alpha(D) \leq \alpha$, then $\chi(D) \leq 35^{\alpha-1}\alpha!$ and such a coloring can be found in polynomial time.*

On another front, one may be interested in the chromatic number of digraphs with simple local structure. It was conjectured in [2] (Conjecture 2.6) that there is a function g such that if T is a tournament in which the set of out-neighbors of each vertex has chromatic number at most k, then $\chi(T) \leq g(k)$. The conjectured was verified for $k = 2$ in [6] and for all k in [7]. Here, we prove a generalization to digraphs with bounded independence number.

Theorem 1.8. *There is a function g such that if D is a digraph with $\alpha(D) \leq \alpha$ and that the set of out-neighbors of each vertex has chromatic number at most k, then $\chi(D) \leq g(k, \alpha)$.*

The structure of the paper is as follows. We prove Theorem 1.8 in Section 2, which is the main tool to prove Theorem 1.3. Sections 3 and 4 are devoted to proving Theorems 1.4 and 1.5, and hence complete the proof of Theorem 1.3. In Section 5, we will prove Theorem 1.7 to support Conjecture 1.6.

1.1 Notation and remarks

Given a digraph D, we say that u *sees* v and v is *seen by* u if uv is an arc in D. For every $v \in V(D)$, we denote by $N_D^+(v)$ (respectively, $N_D^-(v)$) the set of out-neighbors (respectively, in-neighbors) of v in D. Let $N(v) = N_D^+(v) \cup N_D^-(v)$. For every $X \subseteq V(D)$, let $N_D^+(X) = \bigcup_{v \in X} N_D^+(v)$ (respectively, $N_D^-(X) = \bigcup_{v \in X} N_D^-(v)$), the set of vertices seen by (respectively, seeing) at least one vertex of X, and let $M_D^+(X) = \bigcap_{v \in X} N_D^+(v)$ (respectively, $M_D^-(X) = \bigcap_{v \in X} N_D^-(v)$), the set of vertices seen by (respectively, seeing) all vertices of X. Let $N_D(X)$ denote $N_D^+(X) \cup N_D^-(X)$. We say that two vertices u, v are *non-adjacent* if there is no arc with endpoints $\{u, v\}$. For every $v \in V(D)$, we denote by $N_D^o(v)$ the set non-adjacent vertices of v in D. For a subset X of $V(D)$, we denote by $N_D^o(X)$ the set of vertices of V non-adjacent to at least one vertex of X. When it is clear from context (most of the time), we omit the subscript D in this notation. We will use throughout the paper the fact that $V(D) \setminus X = M_D^+(X) \cup N_D^-(X) \cup N_D^o(X)$ for any $X \subseteq V(D)$.

Given a digraph D and a set $X \subseteq V(D)$, we denote by $D[X]$ the subgraph of D induced by X. When the context is clear, we often use X to denote $D[X]$, and say *chromatic number of* X to refer to the chromatic number of $D[X]$. Given a digraph D we say that a set $X \subseteq V(D)$ is a *dominating set* of D if every vertex $v \in V(D) \setminus X$ is seen by at least one vertex of X. A subset Y of $V(D)$ is *independent* (or *stable*) if any two distinct vertices in Y are non-adjacent. Given a digraph D and two disjoint sets $X, Y \subseteq V(D)$, we denote $X \rightarrow_D Y$ (or just $X \rightarrow Y$) if there is no arc from Y to X in D. A key observation is that if $X \rightarrow Y$, then $\chi(X \cup Y) = \max\big(\chi(X), \chi(Y)\big)$.

A side remark is that some proofs in this paper that proceed by induction on α use the fact that if $\alpha(D) \leq \alpha$, then $\alpha(N^o(v)) \leq \alpha - 1$ for every v, and thus $\chi(N^o(v))$ is bounded. In these inductive proofs, we often cite known results on tournaments for the base case $\alpha = 1$. However, our proofs are indeed self-contained since to prove the base case $\alpha = 1$, we just repeat the same arguments and use the fact that in a tournament, $N^o(v) = \emptyset$ for any vertex v. Hence for example, the proof of Theorem 1.5 can serve as an alternative proof for Theorem 4.1 in [2] (that if H is a hero, then so are $\Delta(H, T_k, T_1)$ and $\Delta(H, T_1, T_k)$).

2 From local to global

We start with some observations regarding the size of a dominating set in an acyclic digraph.

Proposition 2.1. *An acyclic digraph D has a dominating set which is also a stable set, and hence has size at most $\alpha(D)$.*

Proof. We proceed by induction on $|D|$ to show that every acyclic digraph D has a dominating set S which is stable. The statement clearly holds for $|D| = 1$. For $|D| > 1$, since D is acyclic, there is a vertex v with no in-neighbors. Then $V(D) \setminus \{v\} = N^+(v) \cup N^o(v)$. Applying induction to $D[N^o(v)]$, we obtain a stable dominating set S' of $D[N^o(v)]$. Clearly $S := S' \cup \{v\}$ is a dominating set of D. Note that $S' \subseteq N^o(v)$, and so S is stable. □

Given $t \geq 1$, a digraph D is *t-local* if for every vertex v we have $\chi(N^+(v)) \leq t$. Let $\mathcal{C}$ be a class of digraphs closed under taking subdigraphs. We say that $\mathcal{C}$ is *tamed* if for every integer k there exists K and ℓ such that every digraph $T \in \mathcal{C}$ with $\chi(T) \geq K$ contains a set A of ℓ vertices such that $\chi(A) \geq k$. Note that a class of digraphs with bounded chromatic number is indeed tamed.

The following proposition is straightforward.

Proposition 2.2. *Let D be a t-local digraph. Then for every subset X of $V(D)$, $\chi\big(N^+(X) \cup X\big) \leq t|X|$.*

Let us restate Theorem 1.8.

Theorem 2.3. *For every pair of positive integers α and t, there is a function $f_\alpha(t)$ such that every t-local digraph D with $\alpha(D) \leq \alpha$ has chromatic number at most $f_\alpha(t)$.*

Proof. We proceed by induction on α. The case $\alpha = 1$ was proved in [7]. Suppose that Theorem 2.3 is true for $\alpha - 1$ (i.e., $f_{\alpha-1}(t)$ exists for every t).

Claim 2.4. *For every t, the class of t-local digraphs with independence number at most α is tamed.*

Proof. We fix some arbitrary t and show the property by induction on k. The claim is trivial for $k = 1$. Assuming now that (K, ℓ) exists for k, we want to find (K', ℓ') for $k+1$. For this, we set $s := K + \ell f_{\alpha-1}(t) + \ell t$ and

$$K' := 2k(\alpha s + 1)(t + f_{\alpha-1}(t) + 1),$$

and fix ℓ' later.

Let D be a t-local digraph with vertex set V such that $\alpha(D) \leq \alpha$ and $\chi(D) \geq K'$. Let B be a dominating set of D of minimum size b. By Proposition 2.2, we have

$$\chi(D) = \chi(B \cup N^+(B)) \leq \chi(B) + \chi(N^+(B)) \leq (t+1)b.$$

In particular, $b \geq K'/(t+1) \geq 2k(\alpha s+1)$. Consider a subset W of B of size $k(\alpha s+1)$. By Proposition 2.2, we have $\chi(N^+(W)) \leq kt(\alpha s+1)$. By induction hypothesis on $\alpha-1$, for every $x \in W$, we have $\chi(N^o(x)) \leq f_{\alpha-1}(t)$ since $N^o(x)$ is t-local and $\alpha(N^o(x)) \leq \alpha - 1$. Hence, (recalling that $M^-(X) := \bigcap_{x\in X} N^-(x)$ for any $X \subseteq D$)

$$\begin{aligned}
\chi(M^-(W)) &\geq \chi(D) - \chi(N^+(W)) - \chi(N^o(W)) - \chi(W) \\
&\geq K' - kt(\alpha s + 1) - \sum_{x\in W} \chi(N^o(x)) - |W| \\
&\geq K' - k(\alpha s + 1)(t + f_{\alpha-1}(t) + 1) \\
&\geq K'/2 \\
&\geq K.
\end{aligned}$$

In particular, by the tamed property applied to k, one can find a set $A \subset M^-(W)$ such that A has ℓ vertices and $\chi(A) \geq k$.

Consider now a subset S of W of size $\alpha s+1$. We claim that $\chi(N^+(S)) \geq s$. If not, we can cover $N^+(S)$ by at most $s-1$ acyclic sets. Since every acyclic set has independence number at most α, it has a dominating set of size at most α by Proposition 2.1. Hence $N^+(S)$ has a dominating set, say S' of size at most $\alpha(s-1) \leq |S| - 2$. But this yields a contradiction since the set $(B \setminus S) \cup S' \cup \{x\}$, where x is an arbitrary vertex in A, would be a dominating set of T of size less than $|B|$. Therefore, $\chi(N^+(S)) \geq s$.

By Proposition 2.2 applied to $N^+(A)$, we have $\chi(N^+(A)) \leq \ell t$. Hence

$$\begin{aligned}
\chi(N^+(S) \cap M^-(A)) &\geq \chi(N^+(S)) - \chi(N^+(A)) - \chi(N^o(A)), \\
&\geq s - \ell t - \sum_{x\in A} \chi(N^o(x)), \\
&\geq (K + \ell f_{\alpha-1}(t) + \ell t) - \ell t - |A| f_{\alpha-1}(t), \\
&= K.
\end{aligned}$$

Thus, by the tamed property applied to k, there is a subset A_S of $N^+(S) \cap M^-(A)$ such that $|A_S| = \ell$ and $\chi(A_S) \geq k$.

We now construct our subset of V with chromatic number at least $k+1$. For this we consider the set A' consisting of vertices $A \cup W$ to which we add the collection of A_S, for all subsets $S \subseteq W$ of size $\alpha s + 1$. Observe that the number of vertices of A' is at most

$$\ell' := \ell + k(\alpha s + 1) + \ell \binom{k(\alpha s + 1)}{\alpha s + 1}.$$

To conclude, it is sufficient to show that $\chi(A') \geq k+1$. Suppose not, and for contradiction, take a k-coloring of A'. Since $|W| = k(\alpha s + 1)$ there is a monochromatic set S in W of size $\alpha s + 1$ (say, colored 1). Recall that $A_S \subseteq M^-(A)$ and $A \subseteq M^-(W) \subseteq M^-(S)$, so we have all arcs from A_S to A and all arcs from A to S, and note that since $\chi(A) \geq k$ and $\chi(A_S) \geq k$, both A and A_S have a vertex of each of the k colors. Hence there are $u \in A$ and $w \in A_S$ colored 1. Since $A_S \subseteq N^+(S)$, there is $v \in S$ such that vw is an arc. We then obtain the monochromatic cycle uvw of color 1, a contradiction. Thus, $\chi(A') \geq k+1$, completing the proof of the claim. $\diamondsuit$

We now can finish the proof of the theorem. Since the class of t-local digraphs with independence number at most α is tamed, by applying tamed property for $k = t + f_{\alpha-1}(t) + 1$, we have that there exist (K, ℓ) such that every t-local digraph D with $\alpha(D) \leq \alpha$ and $\chi(D) \geq K$ contains a set A of ℓ vertices and $\chi(A) \geq t + f_{\alpha-1}(t) + 1$. We claim that A is a dominating set. If not, then there is a vertex v such that $A \subseteq N^o(v) \cup N^+(v)$. Then $\chi(A) \leq \chi(N^o(v)) + \chi(N^+(v)) \leq t + f_{\alpha-1}(t)$, a contradiction. Hence, A is a dominating set of D. Thus, $\chi(D) = \chi(\cup_{x \in A}(N^+(x) \cup \{x\})) \leq (t+1)|A| = \ell + \ell t$. Consequently, t-local digraphs have chromatic number at most $f(t) := \max(K, \ell + \ell t)$. $\square$

By reversing the directions of all arcs, we have the following theorem.

Theorem 2.5. *For every t and α, there is c such that every directed graph D with $\alpha(D) \leq \alpha$ and $\chi(N^-(v)) \leq t$ for every $v \in V(D)$ has chromatic number at most c.*

3 Proof of Theorem 1.4

Theorem 1.4 states that if H_1 and H_2 are superheroes, then so is $H_1 \Rightarrow H_2$. We will reuse the notions of r-mountains and (r,s)-cliques introduced in [2]. Let us first give the idea of the proof of Theorem 1.4 for a special case: $(C_3 \Rightarrow C_3)$-free tournaments have bounded chromatic number. Given a $(C_3 \Rightarrow C_3)$-free tournament T, suppose that there is a a small set Q in T with chromatic number 3. Then for any partition of Q into Q_1, Q_2, at least one part of this partition has chromatic number at least 2, and so contains a copy of C_3. Let $Y_{Q_1,Q_2} \subseteq V(D) \setminus Q$ be the set of vertices seeing all vertices of Q_1 and seen by all vertices of Q_2. Observe that Y_{Q_1,Q_2} is C_3-free, otherwise a copy of C_3 in Y_{Q_1,Q_2} together with a copy of C_3 in either Q_1 or Q_2 forms a copy of $C_3 \Rightarrow C_3$. Note that $V(T) \setminus Q$ is covered by only $2^{|Q|}$ such sets Y_{Q_1,Q_2}, and hence $\chi(T)$ is bounded.

Hence we wish to find such a small set of vertices Q with chromatic number 3. To this end, we call an arc uv of T *thick* if $N^-(u) \cap N^+(v)$ contains a copy of C_3. If T

has no thick arcs, then intuitively T should have simple structure, and thus, bounded chromatic number. Suppose that T contains a (not necessarily oriented) triangle uvw where all of the three arcs are thick. Then for each of the three thick arcs, we take its thickness-certificate (i.e., a copy of C_3) and together with u, v, w we obtain a set Q of at most 12 vertices. It is straightforward to verify that Q has chromatic number at least 3, and thus, by the argument above $\chi(T)$ is bounded. If T contains no triangle of thick arcs, then for any vertex v, the set of vertices adjacent to v by a thick arc induces a thick-arc-free tournament, which, intuitively, should have bounded chromatic number. We then easily bound the chromatic number of the sets of non-thick in-neighbors and non-thick out-neighbors of v, and hence bound the chromatic number of T.

The proof of the general case is in the same vein. Intuitively, we search for a small set Q with large chromatic number as described above. We will capture the notion of the set Q with the definition of an object called an *r-mountain*, and the notion of a triangle of thick arcs with objects called *(r, s)-cliques*. Given a digraph D, the formal definitions (which are borrowed from [2]) of *r-thick-arc*, *(r, s)-clique*, and *r-mountain* in D are defined inductively on r as follows. Every vertex of D is a 1-mountain. For every $r, s \geq 1$,

- An arc $e = uv$ of D is *r-thick* if $N^-(u) \cap N^+(v)$ contains an r-mountain. An r-mountain in $N^-(u) \cap N^+(v)$ is a *certificate* of r-thickness of e.
- An (r, s)-*clique* of D is a set $S \subseteq V(D)$ such that $|S| = s$, and for every distinct vertices $u, v \in S$, either uv or vu is an arc that is r-thick.
- Given an $(r, r+1)$-clique S and a certificate $C_{u,v}$ for every distinct $u, v \in S$, then the tournament induced on $S \cup (\bigcup_{u,v \in S} C_{u,v})$ is an $(r+1)$-*mountain* of D.

Note that if a digraph D contains an $(r, r+1)$-clique, then D contains an $(r+1)$-mountain, which is the $(r, r+1)$-clique together with certificates of all r-thick arcs of that $(r, r+1)$-clique. Hence, if D contains no $(r+1)$-mountain, then D contains no $(r, r+1)$-clique.

Lemma 3.1 ([2], Lemma 3.3)**.** *Every r-mountain has chromatic number at least r, and has at most $(r!)^2$ vertices.*

Fix two superheroes H_1 and H_2.

Lemma 3.2. *Fix $\alpha \geq 2$ and $r, s \geq 1$, suppose that there are b_0, b_1, b_2 such that*

- *Every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha - 1$ has $\chi(D) \leq b_0$.*
- *Every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha$ and containing no (r, s)-clique has $\chi(D) \leq b_1$.*
- *Every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha$ and containing no r-mountain has $\chi(D) \leq b_2$.*

Then there is b_3 such that every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha$ and containing no $(r, s+1)$-clique has $\chi(D) \leq b_3$.

Proof. A small remark is that the second hypothesis seems redundant since if D contains no $(r,2)$-clique, then D contains no r-thick arc, and so contains no r-mountain. However, the second hypothesis is necessary for the case s=1.

First, note that since H_1 and H_2 are superheroes, there is b_4 such that every H_1-free (or H_2-free) digraph D with $\alpha(D) \le \alpha$ has $\chi(D) \le b_4$. We first identify all r-thick arcs of D. Fix an arbitrary vertex v. Then $V(D) \setminus \{v\}$ can be partitioned into four sets: N^* the set of neighbors of v that are connected to v by an r-thick arc; $N^- = N^-(v) \setminus N^*$; $N^+ = N^+(v) \setminus N^*$; and $N^o(v)$.

Note that $N^o(v)$ is $(H_1 \Rightarrow H_2)$-free and $\alpha(N^o(v)) \le \alpha - 1$, and so by the first hypothesis, $\chi(N^o(v)) \le b_0$. The crucial fact is that the digraph induced by the set N^* does not contain an (r,s)-clique; indeed, an (r,s)-clique together with v would form an $(r,s+1)$-clique, a contradiction to the fact that D has no $(r,s+1)$-cliques. Hence by the second hypothesis, $\chi(N^*) \le b_1$.

Claim 3.3. *There is b_5 such that either $\chi(N^-) \le b_5$ or $\chi(N^+) \le b_5$.*

Proof. Suppose that $\chi(N^-) > b_4$, then N^- contains a copy of H_1, say $\hat{H}_1$. Note that

$$N^+ = \big(M^+(\hat{H}_1) \cap N^+\big) \cup \big(N^o(\hat{H}_1) \cap N^+\big) \cup \big(N^-(\hat{H}_1) \cap N^+\big).$$

- If $\chi(M^+(\hat{H}_1) \cap N^+) > b_4$, then $M^+(\hat{H}_1) \cap N^+$ contains a copy of H_2, say $\hat{H}_2$, and we have $\hat{H}_1 \Rightarrow \hat{H}_2$ forming a copy of $H_1 \Rightarrow H_2$, a contradiction. Hence, $\chi(M^+(\hat{H}_1) \cap N^+) \le b_4$
- For each $u \in \hat{H}_1$, we have $\alpha(N^o(u) \cap N^+) \le \alpha - 1$, so $\chi(N^o(u) \cap N^+) \le b_0$.
- For each $u \in \hat{H}_1$, if $\chi(N^-(u) \cap N^+) \ge b_2$, then $N^-(u) \cap N^+$ contains a r-mountain. This means that uv is an r-thick arc, contradicting $u \notin N^*$. Hence $\chi(N^-(u) \cap N^+) \le b_2$, for each $u \in \hat{H}_1$.

Thus we have,

$$\begin{aligned}\chi(N^+) &\le \chi(M^+(\hat{H}_1) \cap N^+) + \sum_{u \in \hat{H}_1} \Big(\chi(N^o(u) \cap N^+) + \chi(N^-(u) \cap N^+)\Big) \\ &\le b_4 + |H_1|(b_0 + b_2).\end{aligned}$$

Set $b_5 := b_4 + |H_1|(b_0 + b_2)$. We have just shown that if $\chi(N^-) > b_4$, then $\chi(N^+) \le b_5$. Hence either $\chi(N^-) \le b_4 \le b_5$ or $\chi(N^+) \le b_5$. This proves the claim. ◊

Note that $N^+(v) \subseteq N^+ \cup N^*$, and so $\chi(N^+(v)) \le \chi(N^+) + \chi(N^*)$. Similarly, $\chi(N^-(v)) \le \chi(N^-) + \chi(N^*)$. Hence for every $v \in V(D)$, either $\chi(N^+(v)) \le b_5 + b_1$ or $\chi(N^-(v)) \le b_5 + b_1$. Let R be the set of all vertices $v \in V(D)$ with $\chi(N^+(v)) \le b_5 + b_1$ and B be the set of all vertices $v \in V(D)$ with $\chi(N^-(v)) \le b_5 + b_1$. Note that $R \cup B = V(D)$.

Observe that R is a digraph with $\alpha(R) \le \alpha$, and $\chi(N^+_R(v)) \le \chi(N^+_D(v)) \le b_5 + b_1$ for every $v \in R$. Then applying Theorem 2.3 to R with $t = b_5 + b_1$, there is b_6 such that $\chi(R) \le b_6$. Similarly, by Theorem 2.5, there is b_7 such that $\chi(B) \le b_7$. Hence $\chi(D) \le \chi(R) + \chi(B) \le b_6 + b_7$. Setting $b_3 := b_6 + b_7$ completes the proof of Lemma 3.2. □

Recall that if D contains no $(r+1)$-mountain, then D contains no $(r, r+1)$-clique. We are now ready to show that digraphs that do not contain an r-mountain have bounded chromatic number.

Lemma 3.4. *Let $\alpha \geq 2$, and suppose that every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha - 1$ has $\chi(D) \leq b_0$ for some b_0. Then for every r, there exists $g_\alpha(r)$ such that every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha$ and not containing an r-mountain has $\chi(D) \leq g_\alpha(r)$.*

Proof. We proceed by induction on r. If D contains no 1-mountain, then D has no vertices, and we can set $g_\alpha(r) := 0$. Now suppose by induction that $g_\alpha(r)$ exists. We will show that $g_\alpha(r+1)$ exists. First, we claim the following.

(A) For every s, there exists function $g'_{\alpha,r}(s)$ such that if D contains no (r, s)-clique, then $\chi(D) \leq g'_{\alpha,r}(s)$.

We prove (A) by induction on s. For $s = 1$, if D contains no $(r, 1)$-clique, then D has no vertex, so $g'_{\alpha,r}(s) = 0$. Suppose, by induction, that $g'_{\alpha,r}(s)$ exists. Let D be a digraph not containing a $(r, s+1)$-clique. Applying Lemma 3.2 with $b_1 = g'_{\alpha,r}(s)$ and $b_2 = g_\alpha(r)$, we deduce that $g'_{\alpha,r}(s+1)$ exists. This proves (A).

If D contains no $(r+1)$-mountain, then D contains no $(r, r+1)$-clique, implying $\chi(D) \leq g'_{\alpha,r}(r+1)$. Set $g_\alpha(r+1) := g'_{\alpha,r}(r+1)$. This completes the proof. □

To prove Theorem 1.4, it suffices to prove the following lemma.

Lemma 3.5. *For every integer $\alpha \geq 1$, there exists $f(\alpha)$ such that every $(H_1 \Rightarrow H_2)$-free digraph D with $\alpha(D) \leq \alpha$ has $\chi(D) \leq f(\alpha)$.*

Proof. We proceed by induction on α. Since $H_1 \Rightarrow H_2$ is a hero (see [2], Theorem 3.2), Lemma 3.5 is true for $\alpha = 1$. Suppose that Lemma 3.5 is true for $\alpha - 1$ and let $c_0 = f(\alpha - 1)$. Since both H_1 and H_2 are superheroes, there exists c_1 such that if D is any H_1-free or H_2-free digraph, then $\chi(D) \leq c_1 - 1$. Let D be a $(H_1 \Rightarrow H_2)$-free digraph with vertex set V and $\alpha(D) \leq \alpha$. If D does not contain a $(c_0 + 2c_1)$-mountain, then by applying Lemma 3.4 to D with $b_0 = c_0$, there is c_2 such that $\chi(D) \leq c_2$. Thus, it remains to consider the case that D contains a $(c_0 + 2c_1)$-mountain.

Claim 3.6. *There is c_3 such that if D contains a $(c_0 + 2c_1)$-mountain, then $\chi(D) \leq c_3$.*

Proof. Let Q be a $(c_0 + 2c_1)$-mountain of D. Then by Lemma 3.1, $|Q| \leq ((c_0 + 2c_1)!)^2$ and $\chi(Q) \geq c_0 + 2c_1$. For every partition Q into three sets Q_0, Q_1, Q_2, let Y_{Q_0,Q_1,Q_2} be the set of vertices $v \in V \setminus Q$ such that $Q_0 \subseteq N^o(v), Q_1 \subseteq N^+(v)$, and $Q_2 \subseteq N^-(v)$. Note that for every vertex $v \in V \setminus Q$, there always exists a partition of Q into some sets Q_0, Q_1, Q_2 such that v is non-adjacent with every vertex in Q_0, sees every vertex in Q_1 and is seen by every vertex in Q_2. Hence, $V \setminus Q$ can be written as the union of all possible Y_{Q_0,Q_1,Q_2}. There are $3^{|Q|}$ sets Y_{Q_0,Q_1,Q_2}.

(B) $\chi(Y_{Q_0,Q_1,Q_2}) \leq c_1$ for every partition (Q_0, Q_1, Q_2) of Q.

Indeed, if $Y_{Q_0,Q_1,Q_2} = \emptyset$, then (B) clearly holds. Otherwise, $Q_0 \subseteq N^o(v)$ for any $v \in Y_{Q_0,Q_1,Q_2}$. Note that the digraph Q_0 is $(H_1 \Rightarrow H_2)$-free and $\alpha(Q_0) \leq \alpha - 1$, and so by induction hypothesis, $\chi(Q_0) \leq c_0$. This gives $\chi(Q_1 \cup Q_2) \geq \chi(Q) - \chi(Q_0) \geq 2c_1$, implying that either $\chi(Q_1) \geq c_1$ or $\chi(Q_2) \geq c_1$. If $\chi(Q_1) \geq c_1$, then Q_1 contains a copy of H_2, say $\hat{H}_2$. If $\chi(Y_{Q_0,Q_1,Q_2}) \geq c_1$, then Y_{Q_0,Q_1,Q_2} contains a copy of H_1, say $\hat{H}_1$. Then $\hat{H}_1 \Rightarrow \hat{H}_2$ forms a copy of $H_1 \Rightarrow H_2$, a contradiction. Hence, $\chi(Y_{Q_0,Q_1,Q_2}) \leq c_1$. A similar argument establishes the case $\chi(Q_2) \geq c_1$. This proves (B).

Hence

$$\begin{aligned}
\chi(D) &\leq \chi(Q) + \chi(V \setminus Q) \\
&\leq |Q| + \chi\Big(\bigcup_{(Q_0,Q_1,Q_2)} Y_{Q_0,Q_1,Q_2}\Big) \\
&\leq |Q| + \sum_{(Q_0,Q_1,Q_2)} \chi(Y_{Q_0,Q_1,Q_2}) \\
&\leq ((c_0 + 2c_1)!)^2 + 3^{((c_0+2c_1)!)^2} c_1.
\end{aligned}$$

Set $c_3 := ((c_0 + 2c_1)!)^2 + 3^{((c_0+2b_1)!)^2} c_1$. This completes proof of the claim. $\lozenge$

Hence $\chi(D) \leq \max(c_2, c_3)$. Setting $f(\alpha) := \max(c_2, c_3)$ completes the proof of Lemma 3.5, thus proving Theorem 1.4. $\square$

4 Proof of Theorem 1.5

It is proved in [14] that for each integer $k \geq 1$, every tournament with at least 2^{k-1} vertices contains a copy of T_k. Let $\mathcal{R}(a, b)$ be the Ramsey number of (a, b) (i.e., the smallest n such that any graph on n vertices either contains an independent set of order a or a clique of order b).

Proposition 4.1. *For each integer $k \geq 1$, every T_k-free digraph D with $\alpha(D) \leq \alpha$ has at most $\mathcal{R}(\alpha + 1, 2^{k-1})$ vertices.*

Proof. Suppose for a contradiction that there is a T_k-free digraph D with at least $\mathcal{R}(\alpha + 1, 2^{k-1})$ vertices. Then the underlying graph of D contains either an independent set of size $\alpha + 1$ or a clique of size 2^{k-1}. The former case is impossible since $\alpha(D) \leq \alpha$. Thus D contains a tournament of size 2^{k-1}, and hence contains a copy of T_k, a contradiction. $\square$

Recall that Theorem 1.5 states that if H is a superhero, then so are $\Delta(H, T_k, T_1)$ and $\Delta(H, T_1, T_k)$ for any integer $k \geq 1$. We will prove that if H is a superhero, then so is $\Delta(H, T_k, T_1)$ for any $k \geq 1$. This is sufficient. Indeed, if H is a superhero, then so is H_{rev}, the digraph obtained from H by reversing all its arcs. Thus, $\Delta(H_{rev}, T_k, T_1)_{rev} = \Delta(H, T_1, T_k)$ is also a superhero.

Theorem 4.2. *For every superhero H and every pair of integers $k, \alpha \geq 1$, there is a number $f(H, k, \alpha)$ such that every $\Delta(H, T_k, T_1)$-free digraph D with $\alpha(D) \leq \alpha$ has $\chi(D) \leq f(H, k, \alpha)$.*

The idea of the proof of Theorem 4.2 is as follows. Fix a large number c and call a subset B of $V(D)$ such that $\chi(B) = c$ a *bag*. We aim to find a longest chain of disjoint bags $B_1, \dots, B_t$ in $V(D)$, together with a partition $V(D) \setminus \bigcup B_i$ into sets we call *zones* $Z_0, \dots, Z_t$ such that there is no *backward arc* in D (where uv is a backward arc if the bag or zone containing u has higher index than the bag or zone containing v). Then, using maximality of t, we show that the chromatic number of every zone is bounded. Once proving this, we observe that all B_i and Z_i have bounded chromatic number and since D has no backward arc, D has bounded chromatic number. However, the requirement that there is no backward arc in D is too strong and so we will have to slightly relax it. In doing so, we will need to allow some backward arcs, but we will want to do so in a very controlled manner. This leads us to the following definitions.

For an integer $c \geq 1$ and a $\Delta(H, T_k, T_1)$-free digraph D with $\alpha(D) \leq \alpha$, a set $B \subseteq V(D)$ such that $\chi(B) = c$ is called a *c-bag*. A family of pairwise disjoint c-bags $B_1, \dots, B_t$ is a *c-bag-chain* if for every i and every $v \in B_i$, we have $\chi(N^+(v) \cap B_{i-1}) \leq c_1$ and $\chi(N^-(v) \cap B_{i+1}) \leq c_1$, where c_1 is a fixed number satisfying

- $c_1 \geq \mathcal{R}(\alpha + 1, 2^{k-1})$, and
- $c_1 \geq \chi(D)$ for every H-free digraph D with $\alpha(D) \leq \alpha$.

Since H is a superhero, such c_1 clearly exists. Note that every T_k-free digraph D with $\alpha(D) \leq \alpha$ has at most c_1 vertices by Proposition 4.1, and so has chromatic number at most c_1.

Proof of Theorem 4.2. We proceed by induction on α. Since $\Delta(H, T_k, T_1)$ is a hero (see [2], Theorem 4.1), the theorem holds for $\alpha = 1$. Suppose that Theorem 4.2 is true for $\alpha - 1$ and let $c_0 = f(H, k, \alpha - 1)$. Let D be a $\Delta(H, T_k, T_1)$-free digraph with vertex set V and $\alpha(D) \leq \alpha$. An important observation is that for every $v \in V$, the digraph $N^o(v)$ is $\Delta(H, T_k, T_1)$-free and has independence number at most $\alpha - 1$, and hence

$$\chi(N^o(v)) \leq f(H, k, \alpha - 1) = c_0. \tag{4.1}$$

We also would like to recall some useful formulas. For every $v \in X \subseteq V$,

$$\chi(N^+(X)) \leq \sum_{v \in X} \chi(N^+(v)) \text{ and } \chi(N^o(X)) \leq \sum_{v \in X} \chi(N^o(v)), \tag{4.2}$$

and if $Y \subseteq V$ and $Y \cap X = \emptyset$, then (recalling that $M^+(X)$ is the set of vertices seen by all vertices of X)

$$Y = (M^+(X) \cap Y) \cup \big((N^-(X) \cup N^o(X)) \cap Y\big). \tag{4.3}$$

Let $h = |H|$ and set $c := 2(c_0 + c_1)(h + k)$. Let us assume that $B_1, \dots, B_t$ is a c-bag-chain of D with t as large as possible. In the proof of this theorem, we drop prefix c- of c-bag and c-bag-chain for convenience. By definition of bag-chain, every bag has few backward arcs with bags preceding or succeeding it. In the following claim, we show that in fact every bag has few backward arcs with any other bag.

Claim 4.3. *For every i and $v \in B_i$, and for every $r > 0$,* *(a)* $\chi(N^+(v) \cap B_{i-r}) \leq c_1$, *and*

(b) $\chi(N^-(v) \cap B_{i+r}) \leq c_1$.

Proof. We proceed by induction on r. For $r = 1$, both (a) and (b) holds by definition of bag-chain. Suppose that both statements are true for $r - 1$. We now prove (a) for r. Suppose for a contradiction that there is v in some B_i such that $\chi(N^+(v) \cap B_{i-r}) > c_1$. Then $N^+(v) \cap B_{i-r}$ has a copy of H, say $\hat{H}$. Then by applying (4.3) we have

$$B_{i-1} = \Big(M^+(\hat{H}) \cap B_{i-1}\Big) \cup \Big(\big(N^-(\hat{H}) \cup N^o(\hat{H})\big) \cap B_{i-1}\Big),$$

and

$$B_{i-1} = \Big(N^-(v) \cap B_{i-1}\Big) \cup \Big(\big(N^+(v) \cup N^o(v)\big) \cap B_{i-1}\Big).$$

Thus (by using the fact that if $A = B \cup C = B' \cup C'$, then $A = (B \cap B') \cup C \cup C'$) we have

$$\begin{aligned} B_{i-1} =& \Big(M^+(\hat{H}) \cap N^-(v) \cap B_{i-1}\Big) \cup \Big(\big(N^-(\hat{H}) \cup N^o(\hat{H})\big) \cap B_{i-1}\Big) \\ & \cup \Big(\big(N^+(v) \cup N^o(v)\big) \cap B_{i-1}\Big). \end{aligned} \tag{4.4}$$

For each $x \in \hat{H}$, by (4.1) we have $\chi(N^o(x) \cap B_{i-1}) \leq c_0$, and by induction hypothesis of (b) applied to x and $r - 1$, we have $\chi(N^-(x) \cap B_{i-1}) \leq c_1$. We also have $\chi(N^o(v) \cap B_{i-1}) \leq c_0$ by (4.1) and $\chi(N^+(v) \cap B_{i-1}) \leq c_1$ by definition of a bag-chain. Combining with (4.4) and (4.2) we have

$$\begin{aligned} \chi(M^+(\hat{H}) \cap N^-(v) \cap B_{i-1}) \geq& \ \chi(B_{i-1}) - \sum_{x \in \hat{H} \cup \{v\}} \chi(N^o(x) \cap B_{i-1}) \\ & - \sum_{x \in \hat{H}} \chi(N^-(x) \cap B_{i-1}) - \chi(N^+(v) \cap B_{i-1}), \\ \geq& \ 2(c_0 + c_1)(h + k) - c_0(h + 1) - c_1 h - c_1, \\ >& \ c_1. \end{aligned}$$

Then there exists a copy of T_k in $M^+(\hat{H}) \cap N^-(v) \cap B_{i-1}$, say $\hat{T_k}$. Note that by construction, we have all arcs from $\hat{H}$ to $\hat{T_k}$, from $\hat{T_k}$ to v and from v to $\hat{H}$. Then $\Delta(\hat{H}, \hat{T_k}, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction.

The proof of (b) for r is similar but not symmetric. In order to obtain a copy of $\Delta(H, T_k, T_1)$, we first get a copy of T_k in B_{i+r}, and then a copy of H in B_{i+1}. This proves the claim. ♢

We next prove a stronger statement that every bag has few backward arcs with the union of all other bags preceding or succeeding it. Let $B_{i,j} = \bigcup_{s=i}^{j} B_s$ for any $1 \leq i \leq j \leq t$. (If $i < 1$ or $j > t$ or $j < i$, we set $B_{i,j} = \emptyset$.)

Claim 4.4. *For every i and $v \in B_i$,*

- $\chi(N^+(v) \cap B_{1,i-2}) \leq c_1$, *and*
- $\chi(N^-(v) \cap B_{i+2,t})) \leq c_1$.

Proof. We repeat the same argument as in the proof of Claim 4.3. Towards a contradiction, suppose that the first statement is false: there is v in some B_i such that $\chi(N^+(v) \cap B_{1,i-2}) > c_1$. Then $N^+(v) \cap B_{1,i-2}$ has a copy of H, say $\hat{H}$. For each $x \in \hat{H}$, we have $\chi(N^o(x) \cap B_{i-1}) \leq c_0$ by (4.1), and $\chi(N^-(x) \cap B_{i-1}) \leq c_1$ by Claim 4.3. We also have $\chi(N^o(v) \cap B_{i-1}) \leq c_0$ and $\chi(N^+(v) \cap B_{i-1}) \leq c_1$. Thus by the same computation as in Claim 4.3, we obtain a copy of T_k in B_{i-1} and reach the contradiction. The proof of the second statement is similar. ♢

From Claim 4.4 we have the following immediate corollary.

Claim 4.5. *For every i and $v \in B_i$,*

- $\chi(N^+(v) \cap B_{1,i-1}) \leq 2c_1$.
- $\chi(N^-(v) \cap B_{i+1,t}) \leq 2c_1$.

We now show that the union of all bags has bounded chromatic number. We note that in the following proof, we will use only two hypotheses: Claim 4.5 and that $\chi(B_i)$ is bounded for every i. We will re-use the arguments in this proof for subsequent claims.

Claim 4.6. $\chi(B_{1,t}) \leq 8c(c_1 + 1)$.

Proof. An arc uv with $u \in B_j, v \in B_i$, and $j > i$ is called a *backarc* with *span* $j - i$. For every i and every $v \in B_i$, if $|N^-(v) \cap B_{i+1,t}| < c_1$, let $F_v := N^-(v) \cap B_{i+1,t}$. If $|N^-(v) \cap B_{i+1,t}| \geq c_1$, let $F_v \subseteq N^-(v) \cap B_{i+1,t}$ consist of c_1 vertices whose backarcs to v have largest possible spans. Let us show that

(A) For every backarc uv with $u \notin F_v$, if $u \in B_j$ and $v \in B_i$ then $\chi(B_{i,j}) \leq 4c$.

If $j = i + 1$, then (A) clearly holds since $\chi(B_{i,i+1}) \leq \chi(B_i) + \chi(B_{i+1}) \leq 2c$. Thus we may suppose that $j \geq i + 2$. Since $u \in N^-(v) \cap B_{i+1,t}$ but $u \notin F_v$, it follows from the definition of F_v that $|F_v| = c_1$. Hence $|F_v \cup \{u\}| = c_1 + 1$. Then there is a copy of T_k in $F_v \cup \{u\}$, say $\hat{T}_k$. Note that $\hat{T}_k \subseteq B_{j,t}$.

We have a formula similar to (4.4):

$$
\begin{aligned}
B_{i+1,j-1} =& \Big(M^-(\hat{T}_k) \cap N^+(v) \cap B_{i+1,j-1}\Big) \cup \Big(\big(N^+(\hat{T}_k) \cup N^o(\hat{T}_k)\big) \cap B_{i+1,j-1}\Big) \cup \\
& \cup \Big(\big(N^-(v) \cup N^o(v)\big) \cap B_{i+1,j-1}\Big).
\end{aligned}
$$

For each $x \in \hat{T}_k \cup \{v\}$, we have $\chi(N^o(x) \cap B_{i+1,j-1}) \leq c_0$. Note also that from Claim 4.5, we have $\chi(N^+(x) \cap B_{i+1,j-1}) \leq 2c_1$ for every $x \in \hat{T}_k$ and $\chi(N^-(v) \cap B_{i+1,j-1}) \leq 2c_1$. Furthermore, if $\chi(M^-(\hat{T}_k) \cap N^+(v) \cap B_{i+1,j-1}) > c_1$, then $M^-(\hat{T}_k) \cap N^+(v) \cap B_{i+1,j-1}$ contains a copy of H, say $\hat{H}$. Then $\Delta(\hat{H}, \hat{T}_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. Hence

$$
\begin{aligned}
\chi(B_{i+1,j-1}) \leq & \; \chi\Big(M^-(\hat{T}_k) \cap N^+(v) \cap B_{i+1,j-1}\Big) + \sum_{x \in \hat{T}_k \cup \{v\}} \chi\Big(N^o(x) \cap B_{i+1,j-1}\Big) \\
& + \sum_{x \in \hat{T}_k} \chi\Big(N^+(x) \cap B_{i+1,j-1}\Big) + \chi\Big(N^-(v) \cap B_{i+1,j-1}\Big) \\
\leq & \; c_1 + c_0(k+1) + 2c_1 k + 2c_1 \\
\leq & \; 3(c_1 + c_0)(k+1) \leq 2c,
\end{aligned}
$$

since $c = 2(c_0 + c_1)(h + k)$. This gives $\chi(B_{i,j}) \le \chi(B_i) + \chi(B_{i+1,j-1}) + \chi(B_j) \le 4c$, which proves (A).

Now let G be the undirected graph with vertex set $B_{1,t}$ and $uv \in E(G)$ if $u \in F_v$ or $v \in F_u$. Then B_i is a stable set in G for every i. We now color the vertices of G by $c_1 + 1$ colors as follows. First, color all B_t properly by color 1. Suppose that we have already colored $B_{i+1}, \ldots, B_t$. Every vertex v in B_i is incident (in G) with at most c_1 vertices in $B_{i+1,t}$ (those belonging to F_v) and independent (in G) with all other vertices in B_i, so we can always properly color v, and so properly color B_i. When the process of coloring ends, we obtain a partition of $B_{1,t}$ into $c_1 + 1$ sets of colors, say $X_1, \ldots, X_{c_1+1}$, where each X_s is a stable set in G. We now claim that the chromatic number of the graph induced by X_s is small.

(B) $\chi(X_s) \le 8c$ for every $1 \le s \le c_1 + 1$.

To prove (B), we define a sequence of indices $i_1, i_2, \ldots$ inductively as follows. Let $i_1 = 1$, and for every $r \ge 1$, let $i_{r+1} > i_r$ be the smallest index such that $\chi(B_{i_r,i_{r+1}}) > 4c$. The sequence ends by i_ℓ with $\chi(B_{i_\ell,t}) \le 4c$ (i.e., there is no $i_{\ell+1}$ satisfying the condition). Set $A_r := B_{i_r,i_{r+1}-1}$ for every $1 \le r \le \ell - 1$ and $A_\ell := B_{i_\ell,t}$. Then $B_{1,t} = \bigcup_{r=1}^{\ell} A_r$, and by definition of the sequence, $\chi(B_{i_r,i_{r+1}-1}) \le 4c$ for every $1 \le r \le \ell - 1$. In other words, for every $1 \le r \le \ell$,

$$\chi(A_r) \le 4c. \tag{4.5}$$

Suppose that there is a backarc uv with $u \notin F_v$ and $u \in A_r, v \in A_{r'}$, where $r \ge r' + 2$. Suppose that $u \in B_j$ and $v \in B_{j'}$. Then $j \ge i_r$ since $B_j \subseteq A_r = B_{i_r,i_{r+1}-1}$, and $j' < i_{r'+1}$ since $B_{j'} \subseteq A_{r'} = B_{i_{r'},i_{r'+1}-1}$, and so $j' < i_{r-1}$ since $r' + 1 \le r - 1$. Also note that $\chi(B_{i_{r-1},i_r}) > 4c$ by definition of the sequence. Thus we have

$$\chi(B_{j',j}) \ge \chi(B_{i_{r-1},i_r}) \ge \chi(B_{i_r,i_{r+1}}) > 4c,$$

which contradicts (A) in that $\chi(B_{j',j}) \le 4c$ if $u \notin F_v$. This shows that for any $r \ge r' + 2$, there is no backarc uv with $u \in A_r, v \in A_{r'}$ and $u \notin F_v$.

Now fix an arbitrary X_s and let $X_{s,r} = X_s \cap A_r$ for every $1 \le r \le \ell$. Observe that if uv is a backarc with $u, v \in X_s$, then $u \notin F_v$ (otherwise, $u \in F_v$, so $uv \in G$ and so X_s is not stable in G, a contradiction). Hence combining with the observation in the paragraph above, we have that there is no backarc from $X_{s,r}$ to $X_{s,r'}$ for any r, r' with $r \ge r' + 2$; in other words, $X_{s,r'} \to X_{s,r}$ for any r, r' with $r \ge r' + 2$. Recall from (4.5) that $\chi(X_{s,r}) \le 4c$. Thus $\chi\big(\bigcup_{r\ge1} X_{s,2r-1}\big) \le 4c$ and $\chi\big(\bigcup_{r\ge1} X_{s,2r}\big) \le 4c$. This gives $\chi(X_s) = \chi\big(\bigcup_{r\ge1} X_{s,r}\big) \le 8c$, which proves (B).

Hence

$$\chi(B_{1,t}) \le \sum_{s=1}^{c_1} \chi(X_s) \le 8c(c_1 + 1).$$

This completes the proof of Claim 4.6. ◊

We now turn our attention to the vertices not in the bags. We partition $V \setminus B_{1,t}$ into sets $Z_0, \ldots, Z_t$ called *zones* as follows. For every $x \in V \setminus B_{1,t}$, if i is the largest index such that $\chi(N^-(x) \cap B_i) > c_1$, then $x \in Z_i$; otherwise, $x \in Z_0$. We first show that every Z_i has few backward arcs with any B_j sufficiently far from it.

Claim 4.7. *For every i and $v \in Z_i$,*

- $\chi(N^-(v) \cap B_{i+r}) \leq c_1$ *for every* $r \geq 1$, *and*
- $\chi(N^+(v) \cap B_{i-r}) \leq c_1$ *for every* $r \geq 2$.

Proof. The former inequality is obvious by the partition criterion. For the latter, the proof follows the same idea as that of Claim 4.3, but is a bit more involved. Suppose for a contradiction that $\chi(N^+(v) \cap B_{i-r}) > c_1$ for some $r \geq 2$. Then there is a copy of H in $N^+(v) \cap B_{i-r}$, say $\hat{H}$. Since $\chi(N^-(v) \cap B_i) > c_1$ by partition criterion, there is a copy of T_k in $N^-(v) \cap B_i$, say $\hat{T}_k$. Then by applying (4.3) we have

$$B_{i-1} = \Big(M^+(\hat{H}) \cap B_{i-1}\Big) \cup \Big(\big(N^-(\hat{H}) \cup N^o(\hat{H})\big) \cap B_{i-1}\Big),$$

$$B_{i-1} = \Big(M^-(\hat{T}_k) \cap B_{i-1}\Big) \cup \Big(\big(N^+(\hat{T}_k) \cup N^o(\hat{T}_k)\big) \cap B_{i-1}\Big),$$

and

$$B_{i-1} = \Big(\big(N^+(v) \cup N^-(v)\big) \cap B_{i-1}\Big) \cup \Big(N^o(v) \cap B_{i-1}\Big).$$

Let $R = M^+(\hat{H}) \cap M^-(\hat{T}_k) \cap \big(N^+(v) \cup N^-(v)\big) \cap B_{i-1}$. Then we have

$$\begin{aligned} B_{i-1} = R \cup \Big(\big(\, N^-(\hat{H}) \cup N^o(\hat{H})\big) \cap B_{i-1}\Big) \cup \\ \cup \Big(\big(N^+(\hat{T}_k) \cup N^o(\hat{T}_k)\big) \cap B_{i-1}\Big) \cup \big(N^o(v) \cap B_{i-1}\big). \end{aligned}$$

For each $x \in \hat{H} \cup \hat{T}_k \cup \{v\}$, we have $\chi(N^o(x) \cap B_{i-1}) \leq c_0$. By Claim 4.3, we have $\chi(N^-(x) \cap B_{i-1}) \leq c_1$ for each $x \in \hat{H}$ and $\chi(N^+(x) \cap B_{i-1}) \leq c_1$ for each $x \in \hat{T}_k$. Then

$$\begin{aligned} \chi(R) &\geq \chi(B_{i-1}) - \sum_{x \in \hat{H} \cup \hat{T}_k \cup \{v\}} \chi(N^o(x) \cap B_{i-1}) \\ &\quad - \sum_{x \in \hat{H}} \chi(N^-(x) \cap B_{i-1}) - \sum_{x \in \hat{T}_k} \chi(N^+(x) \cap B_{i-1}) \\ &\geq 2(c_0 + c_1)(h + k) - c_0(h + k + 1) - c_1 h - c_1 k \\ &> 2c_1. \end{aligned}$$

Let $R_1 = R \cap N^+(v)$ and $R_2 = R \cap N^-(v)$. Note that $R = R_1 \cup R_2$, and so either $\chi(R_1) > c_1$ or $\chi(R_2) > c_1$. If $\chi(R_1) > c_1$, there is a copy of H in R_1, say $\hat{H}'$. Then $\Delta(\hat{H}', \hat{T}_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. If $\chi(R_2) > c_1$, there is a copy of T_k in R_2, say $\hat{T}'_k$. Then $\Delta(\hat{H}, \hat{T}'_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction again. This proves the claim. $\Diamond$

Claim 4.7 shows that every zone has few backward arcs with every bag sufficiently far from it. The next claim shows the converse (i.e., every bag has few backward arcs with every zone sufficiently far from it).

Claim 4.8. *For every i and $v \in B_i$,*

- $\chi(N^+(v) \cap Z_{i-r}) \leq c_1$ *for every* $r \geq 2$, *and*

- $\chi(N^-(v) \cap Z_{i+r}) \le c_1$ *for every* $r \ge 3$.

Proof. We repeat the argument in the proof of Claim 4.3. Suppose for a contradiction that the first statement is false: there is v in some B_i and $r \ge 2$ such that $\chi(N^+(v) \cap Z_{i-r}) > c_1$. Then $N^+(v) \cap Z_{i-r}$ has a copy of H, say $\hat{H}$. For each $x \in \hat{H}$, we have $\chi(N^o(x) \cap B_{i-1}) \le c_0$ by (4.1), and $\chi(N^-(x) \cap B_{i-1}) \le c_1$ by the first inequality in Claim 4.7. We also have $\chi(N^o(v) \cap B_{i-1}) \le c_0$ and $\chi(N^+(v) \cap B_{i-1}) \le c_1$ by definition of a bag-chain. Thus by the same computation as in Claim 4.3, we obtain a copy of T_k in B_{i-1} and reach the contradiction. The proof of the second statement is similar. $\Diamond$

The next claim is a counterpart of Claim 4.5 for zones, and will be used to bound the chromatic number of the union of zones.

Claim 4.9. *For every* i *and* $v \in Z_i$,

- $\chi(N^+(v) \cap \bigcup_{s=0}^{i-3} Z_s) \le c_1$, *and*
- $\chi(N^-(v) \cap \bigcup_{s=i+3}^{t} Z_s) \le c_1$.

Proof. Suppose for a contradiction that $\chi(N^+(v) \cap \bigcup_{s=0}^{i-3} Z_s) > c_1$, then there is a copy of H in $N^+(v) \cap \bigcup_{s=0}^{i-3} Z_s$, say $\hat{H}$. For each $x \in \hat{H}$, we have $\chi(N^o(x) \cap B_{i-2}) \le c_0$ by (4.1), and $\chi(N^-(x) \cap B_{i-2}) \le c_1$ by the first inequality in Claim 4.7. We also have $\chi(N^o(v) \cap B_{i-2}) \le c_0$ and $\chi(N^+(v) \cap B_{i-2}) \le c_1$ by the second inequality in Claim 4.7. Thus by the same computation as in Claim 4.3, we obtain that $\chi(M^+(\hat{H}) \cap N^-(v) \cap B_{i-2}) > c_1$ implying that there is a copy of T_k in $M^+(\hat{H}) \cap N^-(v) \cap B_{i-2}$, reaching a contradiction. The proof of the second statement is similar, where we use B_{i+1} instead of B_{i-2}. $\Diamond$

In order to bound the chromatic number of the union of zones, we also need that each zone has bounded chromatic number. We will prove this by employing the assumption that the bag-chain $B_1, \ldots, B_t$ is of maximal length.

Claim 4.10. *No zone* Z_i *contains a bag-chain of length 6.*

Proof. Suppose that some Z_i contains a bag-chain of length 6, say $Y_1, \ldots, Y_6$. Note that we have

- $\chi(N^+(v) \cap B_{i-3}) \le c_1$ for every $v \in Y_1$ by Claim 4.7 and $\chi(N^-(v) \cap Y_1) \le c_1$ for every $v \in B_{i-3}$ by Claim 4.8; and
- $\chi(N^-(v) \cap B_{i+3}) \le c_1$ for every $v \in Y_6$ by Claim 4.7 and $\chi(N^+(v) \cap Y_6) \le c_1$ for every $v \in B_{i+3}$ by Claim 4.8.

Thus by definition of a bag-chain, $B_1, \ldots, B_{i-3}, Y_1, Y_2, \ldots, Y_6, B_{i+3}, \ldots, B_t$ is a bag-chain of length $t+1$, which contradicts the maximality of t. Hence no Z_i contains a bag-chain of length 6. $\Diamond$

Lemma 4.11. *There is* c' *such that if a* $\Delta(H, T_k, T_1)$*-free digraph* D' *with* $\alpha(D') \le \alpha$ *does not contain any* c*-bag-chain of length* 6*, then* $\chi(D') \le c'$.

We defer the proof of Lemma 4.11 for now.

We now show that this is sufficient to prove the theorem. To do so, we group zones by indices modulo 3 and follow a similar argument as in Claim 4.6. Fix $0 \le s \le 2$, and for every $i \equiv s \mod 3$, $1 \le i \le t$, let $Z^s_{\lfloor i/3 \rfloor} := Z_i$. By Claim 4.9, for every j and for every $v \in Z^s_j$, we have

- $\chi(N^+(v) \cap \bigcup_{r<j} Z^s_r) \le c_1$, and
- $\chi(N^-(v) \cap \bigcup_{r>j} Z^s_r) \le c_1$.

By applying Lemma 4.11 on $D' = Z_i$ and using Claim 4.10, it follows that $\chi(Z_i) \le c'$ for every i. Thus, $\chi(Z^s_j) \le c'$ for every j. We can repeat exactly the argument of Claim 4.6 (with c replaced by c' and by using Claim 4.9 instead of Claim 4.5) to deduce

$$\chi\Big(\bigcup_{r\ge 0} Z^s_r\Big) \le 8c'c_1$$

for every $s = 0, 1, 2$. (Remark: we may assume that $c' > c$, which is necessary for the argument at the end of the proof of Property (A) that $3(c_1 + c_0)(k + 1) \le 2c < 2c'$.)

Hence

$$\chi(\bigcup_{s=0}^{t} Z_s) \le \sum_{s=0}^{2} \chi\Big(\bigcup_{r\ge 0} Z^s_r\Big) \le 24c'c_1.$$

From Claims 4.6 and the above inequality, we have

$$\chi(D) \le \chi(B_{1,t}) + \chi(\cup_{i=0}^{t} Z_i) \le 8c(c_1 + 1) + 24c'c_1,$$

which proves Theorem 4.2. □

4.1 Proof of Lemma 4.11

Lemma 4.11 asserts that having no bag-chain of length 6 is enough to force a digraph (with bounded independence number and $\Delta(H, T_k, T_1)$-free) to have bounded chromatic number. It is trivial by definition that having no bag-chain of length 1 forces bounded chromatic number. In the next lemma, we show that having no bag-chain of length 2 can force bounded chromatic number, which contains most of the difficulty of the proof of Lemma 4.11. In the following lemma, c, c_0, c_1, h, k are the values as in the proof of Theorem 4.2. Recall that $c = 2(c_0 + c_1)(h + k)$.

Lemma 4.12. *For every $d \ge c$, there is $g(d)$ such that if a $\Delta(H, T_k, T_1)$-free digraph D with $\alpha(D) \le \alpha$ contains no d-bag-chain of length 2, then $\chi(D) \le g(d)$.*

Proof. Let J be the tournament $H \Rightarrow T_k$. By Theorem 1.4, J is a superhero and contains both H and T_k as subtournaments. Let D be a digraph with vertex set V satisfying the hypotheses of the lemma. A copy of J in D is called a *ball*. A ball $\hat{J}$ is colored *red* if $\chi(M^+(\hat{J})) \le d$ and *blue* if $\chi(M^-(\hat{J})) \le d$ (note that we do not color vertices of a ball but color the ball as a single object). A ball certainly can be both red and blue, in which case we color it arbitrarily with one of the colors.

Claim 4.13. *Every ball is either red or blue.*

Proof. Suppose for a contradiction that a ball $\hat{J}$ is neither red nor blue. Then there are $B_1 \subseteq M^-(\hat{J})$ and $B_2 \subseteq M^+(\hat{J})$ such that $\chi(B_1) = d$ and $\chi(B_2) = d$. Suppose that $\chi(N^+(v) \cap B_1) > c_1$ for some $v \in B_2$. Let $\hat{H}$ be a copy of H in $N^+(v) \cap B_1$. Let $\hat{T}_k$ be a copy of T_k in $\hat{J}$. Then $\Delta(\hat{H}, \hat{T}_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. Hence $\chi(N^+(v) \cap B_1) \leq c_1$ for every $v \in B_2$. Similarly, $\chi(N^-(v) \cap B_2) \leq c_1$ for every $v \in B_1$, for otherwise a copy of T_k in $N^-(v) \cap B_2$ would yield a contradiction. Hence B_1, B_2 is a d-bag-chain of length 2, a contradiction. ♢

For every vertex v of V, we color v as follows. If there are $c_1 + 1$ vertex-disjoint red balls $\hat{J}_1, \dots, \hat{J}_{c_1+1}$ such that $\hat{J}_i$ has complete arcs to $\hat{J}_j$ for every $i < j$, and $v \in \hat{J}_{c_1+1}$, then we color v *red*. If there are $c_1 + 1$ blue balls $\hat{J}_1, \dots, \hat{J}_{c_1+1}$ such that $\hat{J}_i$ has complete arcs to $\hat{J}_j$ for every $i < j$, and $v \in \hat{J}_1$, then we color v *blue*. If v satisfies both conditions, we color v arbitrarily. After the process of coloring, we obtain a partition of V into R the set of red vertices, B the set of blue vertices, and U the set of uncolored vertices.

Claim 4.14. *There is d_1 such that $\chi(U) \leq d_1$.*

Proof. Let K be the tournament of $J \Rightarrow J \Rightarrow \cdots \Rightarrow J$ ($2c_1 + 2$ times J). Since J is a superhero, then so is K by Theorem 1.4. Hence there is d_1 such that every K-free digraph D' with $\alpha(D') \leq \alpha$ has chromatic number at most d_1. Suppose that U contains a copy of K, say $\hat{K}$. Since every ball is either red or blue, we can find $c_1 + 1$ vertex-disjoint monochromatic balls $\hat{J}_1, \dots, \hat{J}_{c_1}$ in $\hat{K}$ such that $\hat{J}_i$ has complete arcs to $\hat{J}_j$ for every $i < j$. Then either vertices of $\hat{J}_1$ are blue or vertices of $\hat{J}_{c_1+1}$ are red, a contradiction with the fact that all vertices of U are uncolored. Hence U is K-free, and so $\chi(U) \leq d_1$. ♢

It remains to show that R and B have bounded chromatic number, which can be done by applying Theorems 1.8. To do so, we need to prove that $N^+(v)$ has bounded chromatic number for every $v \in R$.

Claim 4.15. *There is d_2 such that $\chi(N^+(v)) \leq d_2$ for every $v \in R$.*

Proof. Fix $v \in R$. Then there are vertex-disjoint red balls $\hat{J}_1, \dots, \hat{J}_{c_1+1}$ where $v \in \hat{J}_{c_1+1}$ and $\hat{J}_i$ has complete arcs to $\hat{J}_j$ for every $i < j$. Let $L = \bigcup_{i=1}^{c_1} \hat{J}_1$. Note that v is seen by all vertices of L. For every $u \in L$, we have $\chi(N^+(v) \cap N^o(u)) \leq c_0$. Hence

$$\chi(N^+(v) \cap N^o(L)) \leq \sum_{u \in L} \chi(N^+(v) \cap N^o(u)) \leq |L| c_0.$$

For every partition of L into L_1, L_2 (L_1 or L_2 may be empty), let $Y_{L_1,L_2} = N^+(v) \cap M^-(L_1) \cap M^+(L_2)$. Observe that for every vertex $x \in N^+(v) \setminus N^o(L)$ (note that $x \notin L$ since $v \in \hat{J}_{c_1+1}$), there is a partition of L into some L_1, L_2 such that x sees all vertices of L_1 and is seen by all vertices of L_2, and so $x \in Y_{L_1,L_2}$. Hence

$$N^+(v) \setminus N^o(L) = \bigcup_{(L_1,L_2)} Y_{L_1,L_2}.$$

We now show that $\chi(Y_{L_1,L_2}) \leq d$ for every Y_{L_1,L_2}.

- If, for some $1 \le i \le c_1$, there is $\hat{J}_i \subseteq L_2$, recall that $\chi(M^+(\hat{J}_i)) \le d$ since $\hat{J}_i$ is red. Since $Y_{L_1,L_2} \subseteq M^+(L_2) \subseteq M^+(\hat{J}_i)$, we have that $\chi(Y_{L_1,L_2}) \le d$.

- Otherwise, if there is no $\hat{J}_i \subseteq L_2$, then L_1 contains at least one vertex of each $\hat{J}_i$, $1 \le i \le c_1$, and so $|L_1| \ge c_1$. Hence L_1 has a copy of T_k, say $\hat{T}_k$. Note that all vertices of $\hat{T}_k$ see v since $v \in \hat{J}_{c_1+1}$. If $\chi(Y_{L_1,L_2}) > c_1$, then Y_{L_1,L_2} contains a copy of H, say $\hat{H}$, then $\Delta(\hat{H}, \hat{T}_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. Hence $\chi(Y_{L_1,L_2}) \le c_1 \le d$.

Note that $|L| = |J|c_1 = (h+k)c_1$. Thus, there are $2^{(h+k)c_1}$ possible ways to partition L into L_1, L_2. Hence

$$\begin{aligned}\chi(N^+(v)) &\le \chi\Big(N^+(v) \cap N^o(L)\Big) + \chi\Big(N^+(v) \setminus N^o(L)\Big)\\ &\le |L|c_0 + \sum_{(L_1,L_2)} \chi(Y_{L_1,L_2})\\ &\le (h+k)c_1c_0 + 2^{(h+k)c_1}d.\end{aligned}$$

Set $d_2 := (h+k)c_1c_0 + 2^{(h+k)c_1}d$. This completes the proof of the claim. ◊

Then $\chi(N^+(v) \cap R) \le \chi(N^+(v)) \le d_2$ for every $v \in R$. Then by applying Theorem 2.3 for digraph R with $t = d_2$, we have $\chi(R) \le d_3$ for some d_3.

We now prove that B has bounded chromatic number. The proof is slightly different from that of Claim 4.15 due to asymmetry of $\Delta(H, T_k, T_1)$.

Claim 4.16. *There is d_4 such that $\chi(N^-(v)) \le d_4$ for every $v \in B$.*

Proof. Fix $v \in B$. Then there are vertex-disjoint blues balls $\hat{J}_1, \ldots, \hat{J}_{c_1+1}$ where $v \in \hat{J}_1$ and $\hat{J}_i$ has complete arcs to $\hat{J}_j$ for every $i < j$. Let $L = \bigcup_{i=2}^{c_1+1} \hat{J}_i$. Note that v sees all vertices of L. For every $u \in L$, we have

$$\chi(N^-(v) \cap N^o(L)) \le \sum_{u \in L} \chi(N^-(v) \cap N^o(u)) \le |L|c_0.$$

For every partition of L into L_1, L_2 (L_1 or L_2 may be empty), let $Y_{L_1,L_2} = N^-(v) \cap M^-(L_1) \cap M^+(L_2)$. Observe that for every vertex $x \in N^-(v) \setminus N^o(L)$ (note that $x \notin L$ since $v \in \hat{J}_1$), there is a partition of L into some L_1, L_2 such that x sees all vertices of L_1 and is seen by all vertices of L_2, and so $x \in Y_{L_1,L_2}$. Hence

$$N^-(v) \setminus N^o(L) = \bigcup_{(L_1,L_2)} Y_{L_1,L_2}.$$

We now show that $\chi(Y_{L_1,L_2}) \le d$ for every Y_{L_1,L_2}.

- If there is $\hat{J}_i \subseteq L_2$, then $\hat{J}_i$ contains a copy of H, say $\hat{H}$. Note that v is in the first ball, so v sees all vertices of $\hat{H}$. If $\chi(Y_{L_1,L_2}) > c_1$, then Y_{L_1,L_2} contains a copy of T_k, say $\hat{T}_k$, then $\Delta(\hat{H}, \hat{T}_k, v)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. Hence $\chi(Y_{L_1,L_2}) \le c_1 \le d$.

- If $\hat{J}_{c_1+1} \subseteq L_1$, recall that $\chi(M^-(\hat{J}_{c_1+1})) \leq d$ since $\hat{J}_{c_1+1}$ is blue. Since $Y_{L_1,L_2} \subseteq M^-(L_1) \subseteq M^-(\hat{J}_{c_1+1})$, we have that $\chi(Y_{L_1,L_2}) \leq d$.
- Otherwise, we have two remarks: (1) $\hat{J}_{c_1+1}$ must have a vertex in L_2, say z. (2) Every $\hat{J}_i, 2 \leq i \leq c_1$ must have a vertex in L_1. Then $|L_1 \cup \{v\}| > c_1$, and so $L_1 \cup \{v\}$ contains a copy of T_k, say $\hat{T}'_k$, such that all vertices of $\hat{T}'_k$ are in one of the $\hat{J}_i$, $1 \leq i \leq c_1$ (note that $v \in \hat{J}_1$). Observe that z is seen by all vertices of $\hat{T}'_k$ since $z \in \hat{J}_{c_1+1}$. If $\chi(Y_{L_1,L_2}) > c_1$, then Y_{L_1,L_2} contains a copy of H, say $\hat{H}'$, then $\Delta(\hat{H}', \hat{T}'_k, z)$ forms a copy of $\Delta(H, T_k, T_1)$, a contradiction. Hence $\chi(Y_{L_1,L_2}) \leq c_1 \leq d$.

Hence using a computation similar to the claim above, we have $\chi(N^-(v)) \leq d_4$, where $d_4 := (h+k)c_1c_0 + 2^{(h+k)c_1}d$. This completes the proof of the claim. ◊

Then $\chi(N^-(v) \cap B) \leq \chi(N^-(v)) \leq d_4$ for every $v \in B$. Then by applying Theorem 2.5 to the digraph B with $t = d_4$, we have $\chi(B) \leq d_5$ for some d_5. Hence

$$\chi(D) \leq \chi(B) + \chi(R) + \chi(U) \leq d_1 + d_3 + d_5.$$

This completes the proof of Lemma 4.12. □

We are now ready to prove Lemma 4.11.

Claim 4.17. *If a $\Delta(H, T_k, T_1)$-free digraph D with $\alpha(D) \leq \alpha$ contains no c-bag-chain of length 6, then $\chi(D) \leq g(g(g(c)))$ where g is the function in Lemma 4.12.*

Proof. Suppose for a contradiction that $\chi(D) \geq g(g(g(c)))$. We will show that D contains a c-bag-chain of length 8. By applying Lemma 4.12 to D with $d := g(g(c))$, we have that D contains a $g(g(c))$-bag-chain of length two, say X_1, X_2. Hence $\chi(X_1) = \chi(X_2) = g(g(c))$ and

- $\chi(N^+(v) \cap X_1) \leq c_1$ for every $v \in X_2$, and
- $\chi(N^-(v) \cap X_2) \leq c_1$ for every $v \in X_1$.

Apply Lemma 4.12 again to X_1 (respectively, X_2) with $d := g(c)$, we obtain a $g(c)$-bag-chain of length two, say Y_1, Y_2 in X_1 (respectively, Y_3, Y_4 in X_2). Since $Y_2 \subset X_1$ and $Y_3 \subset X_2$, we have

- $\chi(N^+(v) \cap Y_2) \leq c_1$ for every $v \in Y_3$, and
- $\chi(N^-(v) \cap Y_3) \leq c_1$ for every $v \in Y_2$.

Hence by definition, $Y_1, \ldots, Y_4$ forms a $g(c)$-bag-chain of length 4. Note that $\chi(Y_s) = g(c)$ for every $1 \leq s \leq 4$. Repeating the argument we obtain a c-bag-chain of length 2 inside each Y_s, and hence obtain a c-bag-chain $B_1, \ldots, B_8$ of length 8 inside D. This contradicts the fact that D has no c-bag-chain of length 6, and so completes the proof. ◊

Claim 4.17 proves Lemma 4.11, concluding the proof of Theorem 4.2.

5 Triangle-free digraphs

In this section, we prove Theorem 1.7. We present an efficient algorithm to color a C_3-free digraph whose independence number is $\alpha(D) \leq \alpha$ with at most $35^{\alpha-1}\alpha!$ colors. For a digraph D, let $n = |V(D)|$ denote the size of its vertex set. Let $poly(n)$ denote the function n^k for some rational number $k > 0$. With respect to the time complexity of our algorithm, our main goal is to show that it is a polynomial depending only on n (i.e., $poly(n)$). We therefore do not optimize the running time nor do we provide a tight analysis. Moreover, note that each time we argue that a subroutine can be performed in $poly(n)$ time, the (implicit) value of k may be different.

For each integer $\alpha \geq 1$, define $h(\alpha)$ to be the minimum number such that every C_3-free digraph D with $\alpha(D) \leq \alpha$ has chromatic number at most $h(\alpha)$. Conjecture 1.6 asserts that $h(\alpha) \leq \alpha^\ell$ for some ℓ. Clearly $h(1) = 1$ since every C_3-free tournament is acyclic. However, $h(\alpha)$ is still unknown for all $\alpha \geq 2$. We believe that $h(2) = 2$ or 3 even though the best bound we have is around 25 (by tweaking the proof of Theorem 1.7). Theorem 1.7 gives an exponential upper bound for $h(\alpha)$ by the function $g(\alpha) := 35^{\alpha-1}\alpha!$. Since our proof of Theorem 1.7 will use induction, we will assume that a C_3-free digraph with independence number $\alpha - 1$ can be colored with at most $g(\alpha-1)$ colors and we will prove it for α. Note that $g(1) = 1$. Let us restate Theorem 1.7.

Theorem 5.1. *Let D be a C_3-free digraph where $\alpha(D) \leq \alpha$. Then $\chi(D) \leq 35^{\alpha-1}\alpha!$ and this coloring can be found in time $poly(n)$.*

The rest of this section is devoted to proving Theorem 5.1. We begin with some observations regarding the size of a dominating set in a digraph.

Proposition 5.2. *A digraph D has an acyclic dominating set, and this set can be found in time $poly(n)$.*

Proof. We proceed by induction on n. The statement clearly holds for $n = 1$. For $n > 1$, pick an arbitrary vertex v. Then $V(D) \setminus v = N^o(v) \cup N^-(v) \cup N^+(v)$. Applying induction to the subgraph, $D[N^o(v) \cup N^-(v)]$, we obtain an acyclic dominating set S'. Then $S := S' \cup \{v\}$ is a dominating set of D. Note that $S' \subseteq N^o(v) \cup N^-(v)$, so v does not see any vertex of S'. Hence S is an acyclic set since S' is an acyclic set. The running time for this procedure is $poly(n)$. □

Proposition 5.3. *Given a C_3-free digraph D, there is a set $Y \subseteq V(D)$ of size at most $\alpha(D)$ such that $V(D) = Y \cup N^o(Y) \cup N^+(Y)$, and this set can be found in time $poly(n)$.*

Proof. First apply Proposition 5.2 to obtain an acyclic dominating set S of D. Then apply Proposition 2.1 to $D[S]$ to obtain a stable dominating set Y of $D[S]$ of size at most $\alpha(D)$ in time $poly(n)$.

We now show that Y is a set with the desired properties. Suppose for a contradiction that there is $v \notin Y \cup N^o(Y) \cup N^+(Y)$. Then $Y \subseteq N^+(v)$ and $v \notin S$ since Y dominates all vertices of S. There is $u \in S$ seeing v since S is a dominating set of D. Note that $u \notin Y$; otherwise this contradicts $Y \subseteq N^+(v)$. There is $y \in Y$ seeing u since Y is a dominating set of S. Then u, v, y are distinct vertices where u sees v, v sees y, and y sees u. Hence we obtain a copy of C_3 in D, a contradiction. □

We now present some definitions and useful lemmas. First, we re-define a *bag* so that it retains the useful properties of a bag as defined in Section 4 and so that it can be tested efficiently.

Definition 5.4. *For a digraph D, we say that $B \subseteq V(D)$ is a* bag *of D, if every three distinct vertices $\{x, y, z\} \in V(D) \setminus B$ have a common neighbor in B.*

Recall that u and v are neighbors if either uv or vu is an arc. We can check in $poly(n)$ time (e.g. $O(n^4)$) whether or not a set B is a bag of D by exhaustively checking all triples in $V(D) \setminus B$. Suppose that the n vertices of a digraph can be partitioned into disjoint sets such that the subgraph induced on each set has independence number at most $\alpha - 1$. Let $t(\alpha - 1, n)$ denote the maximum (over all such possible partitions of the vertices) total time required by our algorithm (the algorithm Color-Digraph, which we will define shortly) to color all of the subgraphs, each with at most $g(\alpha - 1)$ colors. The following claim follows from this definition.

Claim 5.5. *Suppose $\sum_{i=1}^{\ell} n_i = n$, where $n_i \geq 1$ is an integer. Then $\sum_{i=1}^{\ell} t(\alpha, n_i) \leq t(\alpha, n)$.*

We now fix an arbitrary C_3-free digraph D such that $\alpha(D) \leq \alpha$, and we omit the subscript D from the relevant notation when the context is clear.

Claim 5.6. *If $S \subset V(D)$ is not a bag of D, then:*

(a) We can partition S into three disjoint sets, S_1, S_2 and S_3, such that $\alpha(D[S_i]) \leq \alpha - 1$ for $i \in \{1, 2, 3\}$. This procedure takes time $poly(n)$.

(b) We can color S with $3 \cdot g(\alpha - 1)$ colors in time $t(\alpha - 1, |S|)$.

Proof. If S is not a bag of D, then we can, in time $poly(n)$, find a triple $\{x, y, z\} \in V(D) \setminus S$ such that every $v \in S$ is not incident to at least one of x, y or z. Thus, each vertex $v \in S$ belongs to either $N^o(x) \cap S, N^o(y) \cap S$ or $N^o(z) \cap S$. Each of these sets has independence number at most $\alpha - 1$. The total time for this procedure is $poly(n)$. The second assertion follows from (a) and from the definition of the function $t(\alpha, n)$. ♢

Definition 5.7. *A bag $B \subseteq V(D)$ is* poor *if for every vertex $v \in B$, either $N^-(v) \cap B$ or $N^+(v) \cap B$ is not a bag of D.*

We can check in $poly(n)$ time (e.g. $O(n^5)$) if a bag B is poor by testing whether or not $N^-(v) \cap B$ and $N^+(v) \cap B$ are bags for every $v \in B$.

Claim 5.8. *If $B \subseteq V(D)$ is a poor bag, then we can color B with $8\alpha \cdot g(\alpha - 1)$ colors in time $poly(n) + t(\alpha - 1, |B|)$.*

Proof. Since B is poor, then for each $v \in B$, either $N^-(v) \cap B$ or $N^+(v) \cap B$ is not a bag of D. Then we can partition B into two sets, L and R, where $N^-(v) \cap B$ is not a bag for every $v \in L$, and $N^+(v) \cap B$ is not a bag for every $v \in R$.

Applying Proposition 5.3 to R, we can find a set $Y_R \subseteq R$ (respectively $Y_L \subseteq L$) such that $|Y_R| \leq \alpha$ and $R \subseteq Y_R \cup N^+(Y_R) \cup N^o(Y_R)$. And so:

$$R = \bigcup_{v \in Y_R} \Big(\big(N^+(v) \cup N^o(v) \cup \{v\} \big) \cap R \Big).$$

For $v \in Y_R$, note that $N^+(v) \cap R$ is not a bag, and so by (a) from Claim 5.6, we can partition $N^+(v) \cap R$ into three sets, each with independence number at most $\alpha - 1$. Additionally, we have the set $N^o(v) \cap R$, which also has independence number at most $\alpha - 1$. Overall, we can partition $R \setminus Y_R$ into 4α sets, each with independence number $\alpha - 1$. The same argument can be applied to Y_L to obtain 8α disjoint sets. Therefore, in time *poly*(n), we can partition $B \setminus Y_R \cup Y_L$, and in time $t(\alpha - 1, |B|)$ we can color $B \setminus \{Y_r \cup Y_L\}$ with $8\alpha \cdot g(\alpha - 1)$ colors. We can then color each $v \in Y_R$ (respectively, $v \in Y_L$) with an arbitrary color used to color the set $N^o(v) \cap R$ (respectively, $N^o(v) \cap L$). $\Diamond$

In general, we do not know how to color a bag efficiently, and a bag may be very large (e.g. $V(D)$ is a bag). Our aim is therefore to find poor bags, since these can be colored using Claim 5.8. The first step of our algorithm is to find a chain of poor bags.

Definition 5.9. *A sequence of pairwise disjoint bags $B_1, \ldots, B_t$ forms a* chain of bags *if $B_i \to B_{i+1}$ for every $i \in [1, t)$.*

Recall that $B_i \to B_{i+1}$ means there is no arc from B_{i+1} to B_i. Moreover, if each B_i is a poor bag, then this sequence is a *chain of poor bags*. Given a chain of bags $C = \{B_1, B_2, \ldots, B_t\}$ for D, we say that $v \in C$ if $v \in B_i$ for some $i \in [1, t]$. We can partition the vertices in $V(D) \setminus C$ into sets $Z_0, \ldots, Z_t$, which we call *zones*, as follows. For every $v \in V(D) \setminus C$, let i be the largest index such that v is seen by at least one vertex in B_i. Then vertex v is assigned to zone Z_i. Otherwise, we assign v to zone Z_0. This partition is unique and can be done in time *poly*(n). As in the case of the bags and zones used in Section 4, these bags and zones exhibit useful properties. The proofs we present here are similar, but much simpler.

Claim 5.10. *Let $C = \{B_1, \ldots, B_t\}$ be a chain of bags, and let $Z_0, Z_1, \ldots, Z_t$ be a partition of the vertices in $V(D) \setminus C$. For every i, the following properties hold:*

(a) $B_i \to B_{i+r}$ for every $r \geq 1$,

(b) $Z_i \to B_{i+r}$ for every $r \geq 1$,

(c) $B_i \to Z_{i+r}$ for every $r \geq 2$,

(d) $Z_i \to Z_{i+r}$ for every $r \geq 3$.

Proof. Property (a) holds for $r = 1$ by definition of a chain of bags. Suppose that (a) holds for $r - 1 > 1$, and suppose that there is an arc uv with $u \in B_{i+r}$ and $v \in B_i$. Since B_{i+1} is a bag, there is $x \in B_{i+1}$ such that x is a common neighbor of u and v. Then by induction hypothesis, vx, xu are arcs, and so vxu is a copy of C_3, a contradiction. Hence (a) holds for r.

Property (b) holds for all $r \geq 1$ by the partitioning criterion of vertices into zones.

To prove property (c), suppose that there is an arc zv with $z \in Z_{i+r}$ and $v \in B_i$ for some $r \geq 2$. Then there is $u \in B_{i+r}$ such that uz is an arc by the partitioning criterion of vertices into zones. Since B_{i+1} is a bag, there is $x \in B_{i+1}$ such that x is a common neighbor of u, v, z. By property (a), vx and xu are arcs. If xz is an arc, then vxz is a copy of C_3. Otherwise, zx is an arc, and so xuz is a copy of C_3. Either way, we reach the contradiction, and so (c) holds for every $r \geq 2$.

To prove property (d), suppose that there is an arc uv with $u \in Z_{i+r}$ and $v \in Z_i$ for some $r \geq 3$. Since B_{i+1} is a bag, there is $x \in B_{i+1}$ such that x is a common neighbor of both u and v. By property (b), vx is an arc, and by property (c), xu is an arc. Hence vxu is a copy of C_3, a contradiction. Hence (d) holds for every $r \geq 3$. $\diamondsuit$

We now show how to find a chain of poor bags, which can be colored using Claim 5.8.

FIND-CHAIN(D, B)

1. If there is $v \in B$ such that both $N^+(v) \cap B$ and $N^-(v) \cap B$ are bags of D, then:

   Return $\big($FIND-CHAIN$(D, N^-(v) \cap B)$, v, FIND-CHAIN$(D, N^+(v) \cap B)$ $\big)$.

2. Otherwise, return B.

The routine FIND-CHAIN(D, B) returns $B_1, v_1, B_2, \ldots, v_{t-1}, B_t$ (i.e., a sequence of sets B_i alternating with vertices v_i). We say that the sequence $B_1, B_2, \ldots, B_t$ is the chain of poor bags output by the procedure FIND-CHAIN(D, B). Later on, we will use the vertices v_i in the output sequence to facilitate the coloring of vertices outside the chain. Observe that if B is a poor bag or if B is not a bag, then FIND-CHAIN(D, B) returns a single set, namely B.

Claim 5.11. *If bag $B \subseteq V(D)$ is not poor, then* FIND-CHAIN(D, B) *returns a chain of poor bags $B_1, \ldots, B_t$ for some $t \geq 2$ in time poly(n).*

Proof. Let $B_1, \ldots, B_t$ be the chain of poor bags output by FIND-CHAIN(D, B). The bags in this chain are pairwise disjoint. From Step 1 of FIND-CHAIN, it follows that each B_i is a bag. Furthermore, each B_i must be poor; otherwise, Step 1 would be applied to B_i to return poor bags inside B_i. Observe that for every pair of consecutive bags B_i, B_{i+1} in the chain, $B_{i+1} \subseteq N^+(v_i)$ and $B_i \subseteq N^-(v_i)$. If there is an arc xy with $x \in B_{i+1}$ and $y \in B_i$, then $v_i xy$ is a copy of C_3, a contradiction. Hence $B_1, \ldots, B_t$ is a chain of poor bags by definition. The procedure FIND-CHAIN(D, B) runs in time *poly*(n). $\diamondsuit$

Claim 5.12. *If* FIND-CHAIN(D, B) *returns a chain of t poor bags, then we can color B with $8\alpha \cdot g(\alpha - 1) + (t - 1) \cdot g(\alpha - 1)$ colors in time poly$(n) + t(\alpha - 1, |B|)$.*

Proof. Suppose that FIND-CHAIN(D, B) returns $B_1, v_1, B_2, \ldots, v_{t-1}, B_t$ and that $C = \{B_1, \ldots, B_t\}$ is the chain of poor bags output by the procedure. Since each B_i is a poor bag, it can be colored using Claim 5.8. By Claim 5.10, $B_i \to B_j$ for every $i < j$, and so we can color the vertices in C using $8\alpha \cdot g(\alpha - 1)$ colors in time *poly*$(n) + t(\alpha - 1, |C|)$ time.

Observe that each $v \in B \setminus C$ is either (i) some v_i in the output sequence returned by FIND-CHAIN(D, B), or (ii) belongs to $N^o(v_i)$ for some v_i in this output sequence. For any $v \in V(D)$, the set $N^o(v)$ has independence number $\alpha - 1$. Therefore, the vertices in $\bigcup_{i=1}^{t-1} N^o(v_i)$ can be colored with $(t-1) \cdot g(\alpha - 1)$ colors. Note that v_i can be colored with an arbitrary color from the color palette used for $N^o(v_i)$. This coloring can be found in time *poly*$(n) + t(\alpha - 1, |B \setminus C|)$. $\diamondsuit$

Corollary 5.13. *Either* Find-Chain(D, B) *returns a chain of* t *poor bags, or* B *can be colored using* $8\alpha \cdot g(\alpha - 1) + (t - 2) \cdot g(\alpha - 1)$ *colors.*

We now have the tools to outline our main coloring algorithm.

> Color-Digraph(D)
>
> If D is acyclic, color D with one color and terminate.
>
> Otherwise:
>
> 1. Run Find-Chain$(D, V(D))$ and let $C := \{B_1, \ldots, B_t\}$ denote the chain of poor bags that is output.
> 2. Assign each vertex in $V(D) \setminus C$ to a zone Z_i for $i \in [0, t]$.
> 3. While Find-Chain(D, Z_i) returns a chain of poor bags $B'_1, B'_2, \ldots, B'_k$ for $k \geq 3$:
>   (a) Update chain: $C := \{B_1, \ldots, B_{i-2}, B'_1, B'_2, \ldots, B'_k, B_{i+1}, \ldots, B_t\}$.
>   (b) Re-assign each vertex in $V(D) \setminus C$ to a zone.
> 4. Color all vertices in the chain C with $8\alpha \cdot g(\alpha - 1)$ colors.
> 5. Color all vertices in the zones of C with $3(8\alpha + 1) \cdot g(\alpha - 1)$ colors.

Claim 5.14. Color-Digraph(D) *colors* D *with at most* $35\alpha \cdot g(\alpha - 1)$ *colors.*

Proof. If $V(D)$ is a poor bag, then we can apply Claim 5.8. Hence we may suppose that $V(D)$ is not a poor bag.

Note that the updated chain C resulting from Step 3 (a) is still a chain of poor bags, due to properties (b) and (c) from Claim 5.10. After Step 3 finishes, the chain C is *maximal* in that the procedure Find-Chain(D, Z_i) will not find a chain of three poor bags in any zone Z_i. Using Corollary 5.13, we can therefore color each zone using $(8\alpha + 1) \cdot g(\alpha - 1)$ colors. Applying property (d) from Claim 5.10, we can use at most $(24\alpha + 3) \cdot g(\alpha - 1)$ colors to color all vertices in $V(D) \setminus C$. Each bag B_i in the chain C is a poor bag, so we can color B_i with $8\alpha \cdot g(\alpha - 1)$ colors by Claim 5.8. Moreover, since $B_i \to B_j$ for every $i < j$ (by property (a) from Claim 5.10), we need $8\alpha \cdot g(\alpha - 1)$ colors to color the entire chain C. Thus, we can color D with $(32\alpha + 3) \cdot g(\alpha - 1) \leq 35\alpha \cdot g(\alpha - 1)$ colors. ◊

Claim 5.15. *The procedure* Color-Digraph(D) *uses* $g(\alpha)$ *colors.*

Proof. We proceed by induction on α. If $\alpha = 1$, then we use one color, since every C_3-free tournament is acyclic. Suppose that the algorithms colors each C_3-free digraph D' where $\alpha(D') \leq \alpha - 1$ using at most $g(\alpha - 1)$ colors. Then by Claim 5.14, the procedure Color-Digraph(D) colors D with at most

$$35\alpha \cdot g(\alpha - 1) \quad = \quad 35\alpha \cdot 35^{\alpha-2}(\alpha - 1)! \; = \; 35^{\alpha-1}\alpha!$$

colors. ◊

Claim 5.16. Color-Digraph(D) *runs in time poly*(n).

Proof. We now analyze the running time. Finding a maximal chain C takes $poly(n)$ time, as does the procedure of partitioning the vertices in $V(D) \setminus C$ into zones. Once we have found this partitioning, we can color the vertices in C in time $poly(n)+t(\alpha-1,|C|)$ using Claim 5.8, and we can color each zone Z_i in time $poly(n) + (\alpha - 1, |Z_i|)$ using Claim 5.12 and Corollary 5.13. So applying Claim 5.5, the total running time of Color-Digraph(D) is at most $poly(n) + t(\alpha - 1, n)$. This leads to the following recurrence relation:

$$\begin{aligned} t(\alpha, n) &= poly(n) + t(\alpha - 1, n) \\ &= \alpha \cdot poly(n). \end{aligned}$$

Since $\alpha \leq n$, we have that $t(\alpha, n) = poly(n)$.

We note that our algorithm is actually just partitioning $V(D)$ into disjoint subsets, where each subset has independence number at most $\alpha - 1$. In other words, it first partitions the vertices in $V(D)$ into sets where each set is a poor bag or not a bag (for example, a zone Z_i is either a poor bag or not a bag). Then, it further partitions these sets into sets with independence number at most most $\alpha-1$. (For example, by Claim 5.6, a set that is not a bag can be partitioned into three disjoint sets, each with independence number at most $\alpha-1$. Similarly, in the proof of Claim 5.8, a poor bag is partitioned into 8α disjoint sets, each with independence number at most $\alpha-1$.) Once the subgraphs on these induced subsets are colored (recursively) using $g(\alpha-1)$ colors, then certain subsets are allowed to use the same color palette, and these color palettes can be coordinated in time $poly(n)$. The initial partitioning procedure and the final coordinating procedure require $poly(n)$ time, while the recursive coloring requires $t(\alpha - 1, n)$ time. ♢

Theorem 5.1 follows from Claims 5.14, 5.15 and 5.16.

Finally, we remark that Theorem 1.7 yields a bound on the size of a maximum acyclic subgraph of a C_3-free digraph D in terms of α.

Theorem 5.17. *Let $D = (V, A)$ be a C_3-free digraph where $\alpha(D) \leq \alpha$. Then D contains an acyclic subset of arcs, $A' \subseteq A$, with cardinality $|A'| \geq |A| \cdot (\frac{1}{2} + c_\alpha)$, where c_α is a constant depending on α.*

Proof. Since D can be colored with $g(\alpha)$ colors, there is a color class containing at least $n/g(\alpha)$ vertices, and by Turan's Theorem, its induced subgraph contains at least

$$\frac{n}{2 \cdot g(\alpha)} \left(\frac{n}{\alpha \cdot g(\alpha)} - 1 \right)$$

arcs. Thus, D contains a maximum acyclic subgraph of size at least

$$\frac{|A|}{2} + \frac{n}{4 \cdot g(\alpha)} \left(\frac{n}{\alpha \cdot g(\alpha)} - 1 \right).$$

Let c be an absolute constant. Then

$$c_\alpha = \frac{1}{c \cdot \alpha \cdot g(\alpha)^2}$$

satisfies the theorem. □

References

[1] P. Aboulker, P. Charbit, R. Naserasr, private communication.

[2] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour, and S. Thomassé. Tournaments and colouring. *Journal of Combinatorial Theory, Series B*, **103** (2013) 1–20.

[3] D. Bokal, G. Fijavž, M. Juvan, P.M. Kayll, and B. Mohar. The circular chromatic number of a digraph. *Journal of Graph Theory*, **46** (2004) 227–240.

[4] K. Choromanski, M. Chudnovsky, and P. Seymour. Tournaments with near-linear transitive subsets. *Journal of Combinatorial Theory, Series B*, **109** (2014), 228–249.

[5] M. Chudnovsky. The Erdős-Hajnal Conjecture – A Survey. *Journal of Graph Theory*, **75** (2014) 178–190.

[6] M. Chudnovsky, R. Kim, C.-H. Liu, P. Seymour, and S. Thomassé, Domination in tournaments, *Journal of Combinatorial Theory, Series B*, **130** (2018) 98–113.

[7] A. Harutyunyan, T.-N. Le, S. Thomassé, H. Wu. Coloring tournaments: from local to global. *arXiv preprint:1702.01607*.

[8] A. Harutyunyan and C. McDiarmid. Large Acyclic sets in oriented graphs. *Unpublished note.*

[9] A. Harutyunyan, and B. Mohar. Strengthened Brooks Theorem for digraphs of girth three. *Electronic Journal of Combinatorics*, **18** (2011) P.195.

[10] A. Harutyunyan, and B. Mohar. Gallai's Theorem for List Coloring of Digraphs. *SIAM Journal on Discrete Mathematics*, **25**(1) (2011) 170–180.

[11] A. Harutyunyan, and B. Mohar. Two results on the digraph chromatic number. *Discrete Mathematics* **312**(10) (2012) 1823–1826.

[12] P. Keevash, Z. Li, B. Mohar, B. Reed, Digraph girth via chromatic number, *SIAM Journal on Discrete Mathematics*, **27**(2) (2013) 693–696.

[13] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, **33** (1982) 265–270.

[14] R. Stearns, The voting problem, *American Mathematical Monthly*, **66** (1959) 761–763.