Sparsity and homomorphisms of graphs and digraphs

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joint work

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Chromatic number and sparse graphs

Theorem (Erdős 1959, Canad. J. Math.)
\[ \forall g, k \exists \text{ graph } G \text{ s.t. } \text{girth}(G) \geq g \text{ and } \chi(G) \geq k. \]
Theorem (Erdős 1959, Canad. J. Math.)
\[ \forall g, k \exists \text{ graph } G \text{ s.t. } \text{girth}(G) \geq g \text{ and } \chi(G) \geq k. \]

Remark: Bollobas and Sauer (1976 Canad. J. Math.) showed that \( G \) can be taken to be \textit{uniquely} \( k \)-colorable.
Definition
A homomorphism from graph $G$ to $H$ is a mapping $\phi : V(G) \rightarrow V(H)$ that preserves adjacencies.

Proposition
$G$ is $k$-colorable if and only if $G \rightarrow K_k$. 
Extending Erdős

- Erdős’ theorem implies that ⋄ sparse $G$ s.t. $G \not\rightarrow K_k$ for any $k$

Instead of $K_k$, look at arbitrary graph $H$.

Clearly, ⋄ $G$ (of arbitrary girth) s.t. $G \not\rightarrow H$.

Question: Does there exist graph $G^*$ "diluted" from $G$ s.t. $G^* \not\rightarrow H$?
Extending Erdős

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- Instead of $K_k$ look at arbitrary graph $H$. 
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- Instead of \( K_k \) look at arbitrary graph \( H \).
- Clearly, \( \exists \ G \) (of arbitrary girth) s.t. \( G \not\leftrightarrow H \).
Extending Erdős

- Erdős’ theorem implies that \( \exists \) sparse \( G \) s.t. \( G \not\rightarrow K_k \) for any \( k \).
- Instead of \( K_k \) look at arbitrary graph \( H \).
- Clearly, \( \exists \ G \) (of arbitrary girth) s.t. \( G \not\leftrightarrow H \).
- **Question:** Does there exist graph \( G^{*} \) “diluted” from \( G \) s.t. \( G^{*} \not\leftrightarrow H \)?
“Diluting” $G$

Idea: $G$ and $H$ given. Suppose $G \not\rightarrow H$. Does there exist a sparse graph $G^*$ s.t.

\[ G^* \rightarrow G \]
\[ G^* \not\rightarrow H \]
“Diluting” $G$

Idea: $G$ and $H$ given. Suppose $G \nrightarrow H$. Does there exist a sparse graph $G^*$ s.t.

$$G^* \rightarrow G$$

$$G^* \nrightarrow H$$

Theorem (Zhu 1996 J. Graph Theory)

$G$ and $H$ graphs, and $G \nrightarrow H$. Then $\forall g \exists G^*$ with:

$$\text{girth}(G^*) \geq g$$

$G^* \rightarrow G$ and $G^* \nrightarrow H$. 

Remark: Set $G = K_2$ and $H = K_2 - 1$ to recover Erdős’ theorem.
“Diluting” $G$

Idea: $G$ and $H$ given. Suppose $G \not
rightarrow H$. Does there exist a sparse graph $G^*$ s.t.

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Theorem (Zhu 1996 J. Graph Theory)

$G$ and $H$ graphs, and $G \not
rightarrow H$. Then $\forall g \exists G^*$ with:

$$\text{girth}(G^*) \geq g, \ G^* \rightarrow G \text{ and } G^* \not
rightarrow H.$$ 

Remark: Set $G = K_r$ and $H = K_{r-1}$ to recover Erdős’ theorem.
Digraphs

Digraphs here will have no loops and no multiple arcs but digons are allowed.
Digraphs

$D$ and $C$ digraphs. $\phi : V(D) \rightarrow V(C)$ is an acyclic homomorphism if

1. $\forall v \in V(C), \phi^{-1}(v)$ is acyclic;
2. for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$ or $\phi(u) \phi(v)$ is an arc in $C$.
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We write $D \rightarrow_{ac} C$
Fact: Let $G$ and $H$ be graphs, $D$ and $C$ the bidirected digraphs of $G$ and $H$, respectively. Then

$$G \rightarrow H \iff D \rightarrow_{ac} C.$$
Analog of Zhu’s theorem

Theorem (H, Kayll, Mohar, Rafferty, 2012 Canad. J. Math)

$D$ and $C$ digraphs, and $D \not\rightarrow_{ac} C$. Then $\forall g \exists D^*$ with:

girth$(D^*) \geq g$, $D^* \rightarrow_{ac} D$ and $D^* \not\rightarrow_{ac} C$. 
Unique colorability: Cores

**Definition**

Let $G$ and $H$ be graphs (digraphs). $G$ is uniquely $H$-colorable if every homomorphism (or acyclic homomorphism) from $G$ to $H$ is surjective and any two homomorphisms $\phi, \psi$ of $G$ differ by some automorphism $\pi$ of $H$ (i.e., $\phi = \pi \circ \psi$).

Graph (digraph) $H$ is a core if it is uniquely $H$-colorable.
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Generalizing Bollobas-Sauer

Theorem (Bollobas-Sauer 1976, Canad. J. Math.)
\[ \forall g, k \exists G \text{ of girth } g \text{ that is uniquely } k\text{-colorable.} \]
Generalizing Bollobas-Sauer

Theorem (Bollobas-Sauer 1976, Canad. J. Math.)
\[ \forall g, k \exists G \text{ of girth } g \text{ that is uniquely } k\text{-colorable}. \]

Theorem (Zhu 1996, J. Graph Theory)
\[ \forall g \text{ and every core } H, \exists \text{ graph } H^* \text{ of girth } g \text{ that is uniquely } H\text{-colorable}. \]
Generalizing Bollobas-Sauer

Theorem (Bollobas-Sauer 1976, Canad. J. Math.)
∀g, k ∃ G of girth g that is uniquely k-colorable.

Theorem (Zhu 1996, J. Graph Theory)
∀g and every core H, ∃ graph H* of girth g that is uniquely H-colorable.

Remark: Setting $H = K_k$ gives Bollobas-Sauer.
Theorem (H, Kayll, Mohar Rafferty 2012, Canad. J. Math.)

\[ \forall g \text{ and every core } D, \exists \text{graph } D^* \text{ of girth } g \text{ that is uniquely } H\text{-colorable.} \]

Remark: Has applications on coloring of digraphs and digraph circular chromatic number.
Digraph analog

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∀g and every core D, ∃ graph D* of girth g that is uniquely H-colorable.
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Theorem
\[ \forall g, k \exists \text{ digraph } D \text{ of girth } g \text{ that is uniquely } k\text{-colorable}. \]
The applications

Theorem
∀g, k ∃ digraph D of girth g that is uniquely k-colorable.

Theorem
Let 1 ≤ d ≤ k be relative prime integers. Then ∀g, ∃ digraph D of girth at least g and $\chi_c(D) = \frac{k}{d}$. 
Thank You