Sparsity and homomorphisms of graphs and digraphs

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joint work

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Theorem (Erdős 1959, Canad. J. Math.)
\[ \forall g, k \exists \text{ graph } G \text{ s.t. } \text{girth}(G) \geq g \text{ and } \chi(G) \geq k. \]
Chromatic number and sparse graphs

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\[ \forall g, k \exists \text{graph } G \text{ s.t. } \text{girth}(G) \geq g \text{ and } \chi(G) \geq k. \]

Remark: Bollobas and Sauer (1976 Canad. J. Math.) showed that \( G \) can be taken to be \textit{uniquely} \( k \)-colorable.
Coloring and homomorphisms

Definition
A homomorphism from graph $G$ to $H$ is a mapping $\phi: V(G) \rightarrow V(H)$ that preserves adjacencies.

Proposition
$G$ is $k$-colorable if and only if $G \rightarrow K_k$. 
Extending Erdős

- Erdős’ theorem implies that \( \exists \) sparse \( G \) s.t. \( G \not\rightarrow K_k \) for any \( k \)
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- Instead of \( K_k \) look at arbitrary graph \( H \).
Extending Erdős

- Erdős’ theorem implies that $\exists$ sparse $G$ s.t. $G \not\rightarrow K_k$ for any $k$
- Instead of $K_k$ look at arbitrary graph $H$.
- Clearly, $\exists$ $G$ (of arbitrary girth) s.t. $G \leftrightarrow H$. 
Extending Erdős

- Erdős’ theorem implies that \( \exists \) sparse \( G \) s.t. \( G \not\rightarrow K_k \) for any \( k \)
- Instead of \( K_k \) look at arbitrary graph \( H \).
- Clearly, \( \exists \) \( G \) (of arbitrary girth) s.t. \( G \not\rightarrow H \).
- **Question**: Does there exist graph \( G^* \) “diluted” from \( G \) s.t. \( G^* \not\rightarrow H \)?
“Diluting” $G$

Idea: $G$ and $H$ given. Suppose $G \notightarrow H$. Does there exist a sparse graph $G^*$ s.t.

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Theorem (Zhu 1996 J. Graph Theory)

$G$ and $H$ graphs, and $G \notightarrow H$. Then \( \forall g \ \exists \ G^* \text{ with:} \)

\[ \text{girth}(G^*) \geq g, \ G^* \rightarrow G \text{ and } G^* \notightarrow H. \]
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Remark: Set $G = K_r$ and $H = K_{r-1}$ to recover Erdős’ theorem.
Digraphs

Digraphs here will have no loops and no multiple arcs but digons are allowed.
Digraphs

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We write $D \rightarrow_{ac} C$
Fact: Let $G$ and $H$ be graphs, $D$ and $C$ the bidirected digraphs of $G$ and $H$, respectively. Then

$$G \rightarrow H \iff D \rightarrow_{ac} C.$$
Analog of Zhu’s theorem

Theorem (H, Kayll, Mohar, Rafferty, 2012 Canad. J. Math)

Let $D$ and $C$ be digraphs, and $D \rightarrow_{ac} C$. Then $\forall g \exists D^*$ with:

$$girth(D^*) \geq g, \ D^* \rightarrow_{ac} D \text{ and } D^* \leftrightarrow_{ac} C.$$
Unique colorability: Cores

**Definition**

Let $G$ and $H$ be graphs (digraphs). $G$ is uniquely $H$-colorable if every homomorphism (or acyclic homomorphism) from $G$ to $H$ is surjective and any two homomorphisms $\phi, \psi$ of $G$ differ by some automorphism $\pi$ of $H$ (i.e., $\phi = \pi \circ \psi$).

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Generalizing Bollobas-Sauer

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\[ \forall g \text{ and every core } H, \exists \text{ graph } H^* \text{ of girth } g \text{ that is uniquely } H\text{-colorable.} \]
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Remark: Setting \( H = K_k \) gives Bollobas-Sauer.
Theorem (H, Kayll, Mohar Rafferty 2012, Canad. J. Math.)

∀ and every core D, ∃ graph D* of girth g that is uniquely H-colorable.

Remark: Has applications on coloring of digraphs and digraph circular chromatic number.
Digraph analog

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Theorem
Let \( 1 \leq d \leq k \) be relative prime integers. Then \( \forall g, \exists \ \text{digraph } D \text{ of girth at least } g \text{ and } \chi_c(D) = \frac{k}{d}. \)
Thank You