Colorings and acyclic sets in planar graphs and digraphs

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Some Definitions

Acyclic digraph: digraph without directed cycles.
Digon: the directed cycle of length two.
An old conjecture

Conjecture (Albertson, Berman 1979)

Every $n$-vertex planar graph contains an induced forest of order at least $n/2$.
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Every \( n \)-vertex planar graph contains an induced forest of order at least \( n/2 \).

Best possible: \( K_4 \)
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Fact: every \( n \)-vertex planar graph contains an induced forest of order at least \( 2n/5 \).
Borodin’s result

Theorem (Borodin 1979)

The vertices of every planar graph can be 5-colored so that any two color classes induce a forest.
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The vertices of every planar graph can be 5-colored so that any two color classes induce a forest.

\( \exists \) a pair of color classes of total size at least \( \frac{2n}{5} \).
Albertson-Berman via coloring
Albertson-Berman via coloring

Approach: can vertices of every planar graph be colored with two colors such that each color class induces a forest?
Albertson-Berman via coloring

**Approach:** can vertices of every planar graph be colored with two colors such that each color class induces a forest?

**Vertex-arboricity, \(a(G)\), of graph \(G\):** smallest \(k\) s.t. \(V(G)\) can be \(k\)-colored with each color class inducing a forest.
∃ planar $G$ with $a(G) = 3$ (Chartrand, Kronk, Wall (1968), Raspaud, Wang (2008))
Vertex-arboricity of planar graphs

\[ \exists \text{ planar } G \text{ with } a(G) = 3 \text{ (Chartrand, Kronk, Wall (1968), Raspaud, Wang (2008))} \]

**Fact:** \( a(G) \leq 3 \) if \( G \) is planar.
Exponentially many 3-arboricities
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- Theorem (Mohar, H., 2012)

Every planar graph $G$ has at least $2^{|V(G)|/9}$ different 3-arboricities.
Exponentially many 3-arboricities

- Theorem (Mohar, H., 2012)
  
  Every planar graph $G$ has at least $2^{|V(G)|}/9$ different 3-arboricities.

- Generalizes to 3-list colorings.
Conjecture (H., 2011)

*Every* $n$-vertex oriented planar graph *has a set of at least* $3n/5$ *vertices which induces an* acyclic digraph.
Conjecture (H., 2011)

Every $n$-vertex oriented planar graph has a set of at least $\frac{3n}{5}$ vertices which induces an acyclic digraph.

Best possible.
Digraphs: a conjecture

Conjecture (H., 2011)

Every \( n \)-vertex oriented planar graph has a set of at least \( \frac{3n}{5} \) vertices which induces an acyclic digraph.

Best possible.

Albertson-Berman conjecture would imply \( \frac{n}{2} \) (instead of \( \frac{3n}{5} \)).
The dichromatic number $\chi(D)$ of digraph $D$ is the smallest $k$ s.t. $V(D)$ can be partitioned into $k$ sets $V_1, \ldots, V_k$ each of which induces an acyclic subdigraph.
The **dichromatic number** $\chi(D)$ of digraph $D$ is the smallest $k$ s.t. $V(D)$ can be partitioned into $k$ sets $V_1, ..., V_k$ each of which induces an acyclic subdigraph.

\[
\chi(G) = 3 \\
\chi(D) = 3
\]
Colorings in digraphs

The **dichromatic number** $\chi(D)$ of digraph $D$ is the smallest $k$ s.t. $V(D)$ can be partitioned into $k$ sets $V_1, ..., V_k$ each of which induces an acyclic subdigraph.

$\chi(G) = 3$  
$\chi(D) = 3$

Another old conjecture

Conjecture (Neumann-Lara, 1985)

*Every oriented planar graph* $D$ has $\chi(D) \leq 2$.  

Seems very hard to attack.
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Theorem (Mohar, H., 2013)

Every planar digraph $D$ of digirth at least five has $\chi(D) \leq 2$. 
The proof

Idea: Discharging...but messy. Configurations are graphs, not digraphs.
A reducible configuration
A reducible configuration
A reducible configuration
A reducible configuration
A reducible configuration
Open questions

Conjecture (McDiarmid, Mohar 2002)
Every oriented graph $D$ with maximum degree $\Delta$ has $\chi(D) \leq C \cdot \Delta \log \Delta$. 

planar and of digirth four $\Rightarrow \chi(D) \leq 2$?
Open questions

- $D$ planar and of digirth four $\Rightarrow \chi(D) \leq 2$?
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**Conjecture (McDiarmid, Mohar 2002)**

*Every oriented graph $D$ with maximum degree $\Delta$ has $\chi(D) \leq C \cdot \frac{\Delta}{\log \Delta}$.***
Thank You