Odd Chromatic Number of Graph Classes *

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Abstract. A graph is called *odd* (respectively, *even*) if every vertex has odd (respectively, even) degree. Gallai proved that every graph can be partitioned into two even induced subgraphs, or into an odd and an even induced subgraph. We refer to a partition into odd subgraphs as an *odd colouring* of G. Scott [Graphs and Combinatorics, 2001] proved that a graph admits an odd colouring if and only if it has an even number of vertices. We say that a graph G is k-odd colourable if it can be partitioned into at most k odd induced subgraphs. We initiate the systematic study of odd colouring and odd chromatic number of graph classes. In particular, we consider for a number of classes whether they have bounded odd chromatic number. Our main results are that interval graphs, graphs of bounded modular-width and graphs of bounded maximum degree all have bounded odd chromatic number.

Keywords: Graph classes · Vertex partition problem · Odd colouring · Colouring variant · Upper bounds.

1 Introduction

A graph is called *odd* (respectively even) if all its degrees are odd (respectively even). Gallai proved the following theorem (see [8], Problem 5.17 for a proof).

Theorem 1. *For every graph G, there exist:*

- *a partition (V_1, V_2) of $V(G)$ such that $G[V_1]$ and $G[V_2]$ are both even;*
- *a partition (V_1', V_2') of $V(G)$ such that $G[V_1']$ is odd and $G[V_2']$ is even.*

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This theorem has two main consequences. The first one is that every graph contains an induced even subgraph with at least $|V(G)|/2$ vertices. The second is that every graph can be *even coloured* with at most two colours, i.e., partitioned into two (possibly empty) sets of vertices, each of which induces an even subgraph of G. In both cases, it is natural to wonder whether similar results hold true when considering odd subgraphs.

The first question, known as the *odd subgraph conjecture* and mentioned already by Caro [3] as part of the graph theory folklore, asks whether there exists a constant $c > 0$ such that every graph G contains an odd subgraph with at least $|V(G)|/c$ vertices. In a recent breakthrough paper, Ferber and Krivelevich proved that the conjecture is true.

Theorem 2 ([5]). *Every graph G with no isolated vertices has an odd induced subgraph of size at least $|V(G)|/10000$.*

The second question is whether every graph can be partitioned into a bounded number of odd induced subgraphs. We refer to such a partition as an *odd colouring*, and the minimum number of parts required to odd colour a given graph G, denoted by $\chi_{\text{odd}}(G)$, as its *odd chromatic number*. This can be seen as a variant of proper (vertex) colouring, where one seeks to partition the vertices of a graph into odd subgraphs instead of independent sets. An immediate observation is that in order to be odd colourable, a graph must have all its connected components be of even order, as an immediate consequence of the handshake lemma. Scott [11] proved that this necessary condition is also sufficient. Therefore, graphs can generally be assumed to have all their connected components of even order, unless otherwise specified.

Motivated by this result, it is natural to ask how many colours are necessary to partition a graph into odd induced subgraphs. As Scott showed [11], there exist graphs with arbitrarily large odd chormatic number. On the computational side, Belmonte and Sau [2] proved that the problem of deciding whether a graph is k-odd colourable is solvable in polynomial time when $k \leq 2$, and NP-complete otherwise, similarly to the case of proper colouring. They also show that the k-odd colouring problem can be solved in time $2^{O(k \cdot rw)} \cdot n^{O(1)}$, where k is the number of colours and rw is the rank-width of the input graphs. They then ask whether the problem can be solved in FPT time parameterized by rank-width alone, i.e., whether the dependency on k is necessary. A positive answer would provide a stark contrast with proper colouring, for which the best algorithms run in time $n^{2^{O(rw)^2}}$ (see, e.g., [7]), while Fomin et al. [6] proved that there is no algorithm that runs in time $n^{2^{o(rw)}}$, unless the ETH fails.[4]

On the combinatorial side, Scott showed that there exist graphs that require $\Theta(\sqrt{n})$ colours. In particular, the *subdivided clique*, i.e., the graph obtained from a complete graph on n vertices by subdividing[5] every edge once

[4] While Fomin et al. proved the lower bound for clique-width, it also holds for rank-width, since rank-width is always at most clique-width.

[5] Subdividing an edge uv consists in removing uv, adding a new vertex w, and making it adjacent to exactly u and v.

requires exactly n colours, as the vertices obtained by subdividing the edges force their two neighbours to be given distinct colours. More generally, and by the same argument, given any graph G, the graph H obtained from G by subdividing every edge once has $\chi_{\text{odd}}(H) = \chi(G)$, and H is odd colourable if and only if $|V(H)| = |V(G)| + |E(G)|$ is even. Note that a subdivided clique is odd colourable if and only if the subdivided complete graph K_n satisfies $n \in \{k : k \equiv 0 \vee k \equiv 3 \pmod 4\}$. Surprisingly, Scott also showed that only a sublinear number of colours is necessary to odd colour a graph, i.e., every graph of even order G has $\chi_{\text{odd}}(G) \leq cn(\log\log n)^{-1/2}$. As Scott observed, this bound is quite weak, and he instead conjectures that the lower bound obtained from the subdivided clique is essentially tight:

Conjecture 1 (Scott, 2001). Every graph G of even order has $\chi_{\text{odd}}(G) \leq (1 + o(1))c\sqrt{n}$.

One way of seeing Conjecture 1 is to consider that subdivided cliques appear to be essentially the graphs that require most colours to be odd coloured. More specifically, consider the family $\mathcal{B}$ of graphs G' obtained from a graph G by adding, for every pair of vertices $u, v \in V(G)$, a vertex w_{uv} and edges uw_{uv} and vw_{uv}, and G' has even order. Note that subdivided cliques of even order are exactly those graphs in $\mathcal{B}$ where graph G is edgeless, and that the graphs in $\mathcal{B}$ have $\chi_{\text{odd}}(G') = |V(G)| \in \Theta(\sqrt{|V(G')|})$. A question closely related to Conjecture 1 is whether if a class of graphs $\mathcal{G}$ does not contain arbitrarily large graphs of $\mathcal{B}$ as induced subgraphs, then $\mathcal{G}$ has odd chromatic number $\mathcal{O}(\sqrt{n})$, i.e., they satisfy Conjecture 1. This question was already answered positively for some graph classes. In fact, the bounds provided were constant. It was shown in [2] that every cograph can be odd coloured using at most three colours, and that graphs of treewidth at most k can be odd coloured using at most $k + 1$ colours. In fact, those results can easily be extended to all graphs admitting a join, and H-minor free graphs, respectively. Using a similar argument, Aashtab et al.[1] showed that planar graphs are 4-odd colourable, and this is tight due to subdivided K_4 being planar and 4-odd colourable, as explained above. They also proved that subcubic graphs are 4-odd colourable, which is again tight due to subdivided K_4, and conjecture that this result can be generalized to all graphs, i.e., $\chi_{\text{odd}}(G) \leq \Delta + 1$, where Δ denotes the maximum degree of G. Observe that none of those graph classes contain arbitrarily large graphs from $\mathcal{B}$ as induced subgraphs. On the negative side, bipartite graphs and split graphs contain arbitrarily large graphs from $\mathcal{B}$, and therefore the bound of Conjecture 1 is best possible. In fact, Scott specifically asked whether the conjecture holds for the specific case of bipartite graphs.

Our contribution. Motivated by these first isolated results and Conjecture 1, we initiate the systematic study of the odd chromatic number in graph classes, and determine which have bounded odd chromatic number. We focus on graph classes that do not contain large graphs from $\mathcal{B}$ as induced subgraphs. Our main results are that graphs of bounded maximum degree, interval graphs and graphs of bounded modular width all have bounded odd chromatic number.

In Section 3, we prove that every graph G of even order and maximum degree Δ has $\chi_{\text{odd}}(G) \leq 2\Delta - 1$, extending the result of Aashtab et al. on subcubic graphs to graphs of bounded degree. We actually prove a more general result, which provides additional corollaries for graphs of large girth. In particular, we obtain that planar graphs of girth 11 are 3-odd colourable. We also obtain that graphs of girth at least 7 are $\mathcal{O}(\sqrt{n})$-odd colourable. While this bound is not constant, it is of particular interest as subdivided cliques have girth exactly 6.

In Section 4 we prove that every graph with all connected components of even order satisfies $\chi_{\text{odd}}(G) \leq 3 \cdot mw(G)$, where $mw(G)$ denotes the modular-width of G. This significantly generalizes the cographs result from [2] and provides an important step towards proving that graphs of bounded rank-width have bounded odd chromatic number, which in turn would imply that the Odd Chromatic Number is FPT when parameterized by rank-width alone.

Finally, we prove in Section 5 that every interval graph with all components of even order is 6-odd colourable. Additionally, every proper interval graph with all components of even order is 3-odd colourable, and this bound is tight.

We would also like to point out that all our proofs are constructive and furthermore a (not necessarily) optimal odd-colouring with the number of colours matching the upper bound can be computed in polynomial time. In particular, the proof provided in [8] of Theorem 1, upon which we rely heavily is constructive, and both partitions can easily be computed in polynomial time. An overview of known results and open cases is provided in Figure 1 below.

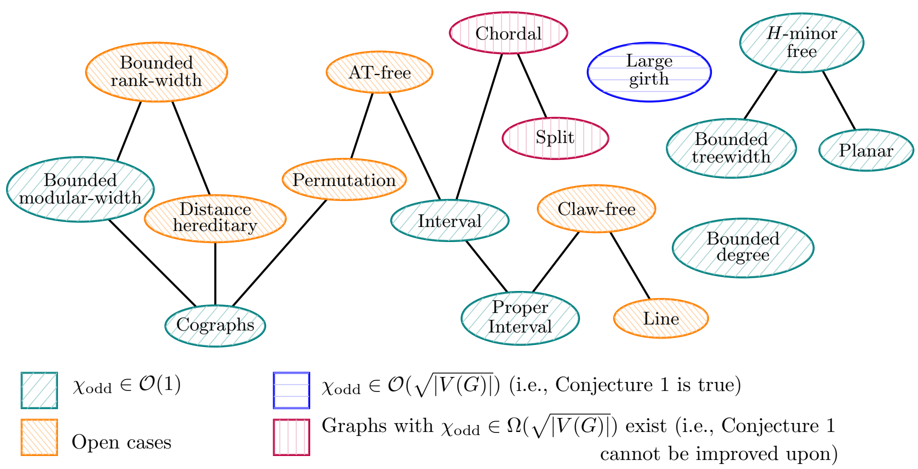

Fig. 1: Overview of known and open cases.

2 Preliminaries

For a positive integer i, we denote by $[i]$ the set of integers j such that $1 \leq j \leq i$. A partition of a set X is a tuple $\mathcal{P} = (P_1, \ldots, P_k)$ of subsets of X such that $X = \bigcup_{i \in [k]} P_i$ and $P_i \cap P_j = \emptyset$, i.e., we allow parts to be empty. Let $\mathcal{P} = (P_1, \ldots, P_k)$ be a partition of X and $Y \subseteq X$. We let $\mathcal{P}|_Y$ be the partition of Y obtained from

$(P_1 \cap Y, \ldots, P_k \cap Y)$ by removing all empty parts. A partition $(Q_1, \ldots, Q_\ell)$ of X is a *coarsening* of a partition $(P_1, \ldots, P_k)$ of X if for every P_i and every Q_j either $P_i \cap Q_j = \emptyset$ or $P_i \cap Q_j = P_i$, i.e., every Q_j is the union of P_i's.

Every graph in this paper is simple, undirected and finite. We use standard graph-theoretic notation, and refer the reader to [4] for any undefined notation. For a graph G we denote the set of vertices of G by $V(G)$ and the edge set by $E(G)$. Let G be a graph and $S \subseteq V(G)$. We denote an edge between u and v by uv. The *order* of G is $|V(G)|$. The *degree* (respectively, *open neighborhood*) of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (respectively, $N_G(v)$). We denote the subgraph induced by S by $G[S]$. $G \setminus S = G[V(G) \setminus S]$. The *maximum* degree of any vertex of G is denoted by Δ. We denote paths and cycles by tuples of vertices. The *girth* of G is the length of a shortest cycle of G. Given two vertices u and v lying in the same connected component of G, we say an edge e *separates* u and v if they lie in different connected components of $G \setminus \{e\}$.

A graph is called odd (even, respectively) if every vertex has odd (respectively, even) degree. A partition $(V_1, \ldots, V_k)$ of $V(G)$ is a *k-odd colouring*[6] of G if $G[V_i]$ induces an odd subgraphs of G for every $i \in [k]$. We say a graph is k-odd colourable if it admits a k-odd colouring. The *odd chromatic number* of G, denoted by $\chi_{\mathrm{odd}}(G)$, is the smallest integer k such that G is k-odd colourable. The empty graph (i.e., $V(G) = \emptyset$) is considered to be both even and odd. Since every connected component can be odd coloured separately, we only need to consider connected graphs.

Modular-width A set S of vertices is called a *module* if, for all $u, v \in S, N(u) \cap S = N(v) \cap S$. A partition $\mathcal{M} = (M_1, \ldots, M_k)$ of $V(G)$ is a module partition of G if every M_i is a module in G. Without loss of generality, we further ask that any module partition $\mathcal{M}$ of G, unless $G = K_1$, is non-trivial, i.e., $\mathcal{M}$ has at least two non-empty parts. Given two sets of vertices X and Y, we say that X and Y are *complete to each other* (*completely non-adjacent*, respectively) if $uv \in E(G)$ ($uv \notin E(G)$, respectively) for every $u \in X, v \in Y$. Note that for any two modules M and N in G, either M and N are non-adjacent or complete to each other. We let $G_{\mathcal{M}}$ be the module graph of $\mathcal{M}$, i.e., the graph on vertex set $\mathcal{M}$ with an edge between M_i and M_j if and only if M_i and M_j are complete to each other (non-adjacency between modules M_i, M_j in $G_{\mathcal{M}}$ corresponds to M_i and M_j being non-adjacent in G). We define the modular width of a graph G, denoted by $\mathrm{mw}(G)$, recursively as follows. $\mathrm{mw}(K_1) = 1$, the width of a module partition $(M_1, \ldots, M_k)$ of G is the maximum over k and $\mathrm{mw}(G[M_i])$ for all $i \in [k]$ and $\mathrm{mw}(G)$ is the minimum width of any module partitions of G.

3 Graphs of bounded degree and graphs of large girth

In this section, we study Scott's conjecture (Conjecture 1) as well as the conjecture made by Aashtab et al. [1] which states that $\chi_{\mathrm{odd}}(G) \leq \Delta + 1$ for any graph G. We settle Conjecture 1 for graphs of girth at least 7, and prove that

[6] This definition of odd colouring is not to be confused with the one introduced by Petrusevski and Skrekovski [10], which is a specific type of proper colouring.

$\chi_{\mathrm{odd}}(G) \leq 2\Delta - 1$ for any graph G, thus obtaining a weaker version of the conjecture of Aashtab et al. To this end, we prove the following more general theorem, which implies both of the aforementioned results.

Theorem 3. *Let $\mathcal{H}$ be a class of graphs such that:*

- *$K_2 \in \mathcal{H}$*
- *$\mathcal{H}$ is closed under vertex deletion and*
- *there is a $k \geq 2$ such that any connected graph $G \in \mathcal{H}$ satisfies at least one of the following properties:*
  - *(I) G has two pendant vertices u, v such that $N_G(u) = N_G(v)$ or*
  - *(II) G has two adjacent vertices u, v such that $d_G(u) + d_G(v) \leq k$.*

Then every graph $G \in \mathcal{H}$ with all components of even order has $\chi_{\mathrm{odd}}(G) \leq k-1$.

Proof. First notice that $\mathcal{H}$ is well defined as K_2 has the desired properties. The proof is by induction on the number of vertices. Let $|V(G)| = 2n$.

For $n = 1$, since G is connected, we have that $G = K_2$ which is odd. Therefore, $\chi_{\mathrm{odd}}(G) = 1 \leq k - 1$ (recall that $k \geq 2$). Let G be a graph of order $2n$. Notice that we only need to consider the case where G is connected as, otherwise, we can apply the inductive hypothesis to each of the components of G. Assume first that G has two pendant vertices u, v such that $N_G(u) = N_G(v) = \{w\}$. Then, since $G \setminus \{u, v\}$ is connected and belongs to $\mathcal{H}$, by induction, there is an odd colouring of $G \setminus \{u, v\}$ that uses at most $k - 1$ colours. Let $(V_1, \ldots, V_{k-1})$ be a partition of $V(G) \setminus \{u, v\}$ such that $G[V_i]$ is odd for all $i \in [k - 1]$. We may assume that $w \in V_1$. We give a partition $V'_1, \ldots, V'_{k-1}$ of $V(G)$ by setting $V'_1 = V_1 \cup \{u, v\}$ and $V'_i = V_i$ for all $i \in [k] \setminus \{1\}$. Notice that for all $i \in [k - 1]$, $G[V'_i]$ is odd. Therefore, $\chi_{\mathrm{odd}}(G) \leq k - 1$.

Thus, we assume that G has an edge $uv \in E(G)$ such that $d_G(u)+d_G(v) \leq k$. We may assume that $k \geq 3$ for otherwise the theorem follows. We consider two cases; $G \setminus \{u, v\}$ is connected and $G \setminus \{u, v\}$ is disconnected.

Assume that $G \setminus \{u, v\}$ is connected. Since $G \setminus \{u, v\}$ has $|V(G) \setminus \{u, v\}| = 2n - 2$ and belongs to $\mathcal{H}$, by induction, there is an odd colouring of it that uses at most $k - 1$ colours. Let $(V_1, \ldots, V_{k-1})$ be a partition of $V(G) \setminus \{u, v\}$, such that $G[V_i]$ is odd of all $i \in [k - 1]$. We give a partition of G into $k - 1$ odd graphs as follows. Since $|N_G(\{u, v\})| \leq k - 2$, there exists $\ell \in [k - 1]$ such that $V_\ell \cap N_G(\{u, v\}) = \emptyset$. We define a partition $(U_1, \ldots, U_{k-1})$ of $V(G)$ as follows. For all $i \in [k - 1]$, if $i \neq \ell$, we define $U_i = V_i$, otherwise we set $U_i = V_i \cup \{u, v\}$. Notice that for all $i \neq \ell$, $G[U_i]$ is odd since $U_i = V_i$. Also, since $N_{G[U_\ell]}[v] = N_{G[U_\ell]}[u] = \{u, v\}$ and $G[V_\ell]$ is odd, we conclude that $G[U_\ell]$ is odd. Thus, $\chi_{\mathrm{odd}}(G) \leq k - 1$.

Now, we consider the case where $G \setminus \{u, v\}$ is disconnected. First, we assume that there is at least one component in $G \setminus \{u, v\}$ of even order. Let U be the set of vertices of this component. By induction, $\chi_{\mathrm{odd}}(G[U]) \leq k - 1$ and $\chi_{\mathrm{odd}}(G \setminus U) \leq k - 1$. Furthermore, $|N_G(\{u, v\}) \cap U| \leq k - 3$ because $G \setminus \{u, v\}$ has at least two components. Let $(U_1, \ldots, U_{k-1})$ be a partition of U such that $G[U_i]$ is odd for all $i \in [k-1]$. Also, let $(V_1, \ldots, V_{k-1})$ be a partition of $V(G) \setminus U$

such that $G[V_i]$ is odd for all $i \in [k-1]$. We may assume that $V_i \cap \{u,v\} = \emptyset$ for all $i \in [k-3]$. Since $|N_G(\{u,v\}) \cap U| \leq k-3$, there are at least two indices $l, l' \in [k-1]$ such that $U_l \cap N_G(\{u,v\}) = U_{l'} \cap N_G(\{u,v\}) = \emptyset$. We may assume that $l = k-2$ and $l' = k-1$. We define a partition $(V'_1, \dots, V'_{k-1})$ of $V(G)$ as follows. For all $i \in [k-1]$ we define $V'_i = U_i \cup V_i$. We claim that $G[V'_i]$ is odd for all $i \in [k-1]$. To show the claim, we consider two cases; either $V'_i \cap \{u,v\} = \emptyset$ or not. If $V'_i \cap \{u,v\} = \emptyset$, since the only vertices in $V(G) \setminus U$ that can have neighbours in U are v and u we have that $G[V'_i]$ is odd. Indeed, this holds because $U_i \cap N_G(V_i) = \emptyset$ and both $G[U_i]$ and $G[V_i]$ are odd. If $V'_i \cap \{u,v\} \neq \emptyset$, then $i = k-2$ or $i = k-1$. In both cases, we know that $U_i \cap N_G(V_i) = \emptyset$ because the only vertices in $V(G) \setminus U$ that may have neighbours in U are v and u and we have assumed that u, v do not have neighbours in $U_{k-2} \cup U_{k-1}$. So, $G[V'_i]$ is odd because $U_i \cap N_G(V_i) = \emptyset$ and both $G[U_i]$ and $G[V_i]$ are odd.

Thus, we can assume that all components of $G \setminus \{u,v\}$ are of odd order. Let $\ell > 0$ be the number of components, denoted by $V_1, \dots, V_\ell$, of $G \setminus \{u,v\}$ and note that ℓ must be even. We consider two cases, either for all $i \in [\ell]$, one of $G[V_i \cup \{u\}]$ or $G[V_i \cup \{v\}]$ is disconnected, or there is at least one $i \in [\ell]$ such that both $G[V_i \cup \{u\}]$ and $G[V_i \cup \{v\}]$ are connected.

In the first case, for each V_i, $i \in [\ell]$ we call w_i the vertex in $\{u,v\}$ such that $G[V_i \cup \{w_i\}]$ is connected. Note that w_i is uniquely determined, i.e., only one of u and v can be w_i for each $i \in [\ell]$. Now, by induction, for all $i \in [\ell]$, $G[V_i \cup \{w_i\}]$ has $\chi_{\text{odd}}(G[V_i \cup \{w_i\}]) \leq k-1$. Let, for each $i \in [\ell]$, $(V^i_1, \dots, V^i_{k-1})$ denote a partition of $V_i \cup \{w_i\}$ such that $G[V^i_j]$ be odd, for all $j \in [k-1]$. Furthermore, we may assume that for each $i \in [\ell]$, if $v \in V_i \cup \{w_i\}$, then $v \in V^i_{k-2}$. Also, we can assume that for each $i \in [\ell]$, if $u \in V_i \cup \{w_i\}$, then $u \in V^i_{k-1}$. Finally, let $I = \{i \in [\ell] \mid w_i = u\}$ and $J = \{i \in [\ell] \mid w_i = v\}$.

We consider two cases. If $|I|$ is odd, then $|J|$ is odd since $\ell = |I| + |J|$ is even. Then, we claim that for the partition partition $(U_1, \dots, U_{k-1})$ of $V(G)$ where $U_i = \bigcup_{j \in [\ell]} V^j_i$ it holds that $G[U_i]$ is odd for all $i \in [k-1]$. First notice that $(U_1, \dots, U_{k-1})$ is indeed a partition of $V(G)$. Indeed, the only vertices that may belong in more than one set are u and v. However, v belongs only to some sets V^i_{k-2}, and hence it is no set U_i except U_{k-2}. Similarly, u belongs to no set U_i except U_{k-1}. Therefore, it remains to show that $G[U_i]$ is odd for all $i \in [k-1]$. We will show that for any $i \in [k-1]$ and for any $x \in U_i$, $|N_G(x) \cap U_i|$ is odd. Let $x \in U_i \setminus \{u,v\}$, for some $i \in [k-1]$. Then we know that $N_G(x) \cap U_i = N_G(x) \cap V^j_i$ for some $j \in [\ell]$. Since $G[V^j_i]$ is odd for all $i \in [k-1]$ and $j \in [\ell]$ we have that $|N_G(x) \cap U_i| = |N_G(x) \cap V^j_i|$ is odd. Therefore, we only need to consider u and v. Notice that $v \in U_{k-2} = \bigcup_{j \in [\ell]} V^j_{k-2}$ (respectively, $u \in U_{k-1} = \bigcup_{j \in [\ell]} V^j_{k-1}$). Also, v (respectively, u) is included in V^j_{k-2} (respectively, V^j_{k-1}) only if $j \in I$ (respectively, $j \in J$). Since $G[V^j_{k-2}]$ (respectively, $G[V^j_{k-1}]$) is odd for any $j \in [\ell]$ we have that $|N(v) \cap V^j_{k-2}|$ (respectively, $|N(u) \cap V^j_{k-1}|$) is odd for any $j \in I$ (respectively, $j \in J$). Finally, since $|I|$ and $|J|$ are odd, we have that $|N_G(v) \cap U_{k-2}| = \sum_{j \in I} |N(v) \cap V^j_{k-2}|$ and $|N_G(u) \cap U_{k-1}| = \sum_{j \in I} |N(u) \cap V^j_{k-1}|$ are both odd. Therefore, for any $i \in [k-1]$, $G[U_i]$ is odd and $\chi_{\text{odd}}(G) \leq k-1$.

Now, suppose that both $|I|$ and $|J|$ are even. We consider the partition $(U_1, \ldots, U_{k-1})$ of $V(G)$ where, for all $i \in [k-3]$ $U_i = \bigcup_{j \in [\ell]} V_i^j$, $U_{k-2} = \bigcup_{j \in J} V_{k-2}^j \cup \bigcup_{j \in I} V_{k-1}^j$ and $U_{k-1} = \bigcup_{j \in I} V_{k-2}^j \cup \bigcup_{j \in J} V_{k-1}^j$. We claim that for this partition it holds that $G[U_i]$ is odd for all $i \in [k-1]$. First notice that $(U_1, \ldots, U_{k-1})$ is indeed a partition of $V(G)$. Indeed, this is clear for all vertices except for v and u. However, v only belongs to sets of type V_{k-2}^i for $i \in I$, and u only belongs to sets of type V_{k-1}^i for $i \in J$. Therefore, u or v belong to no set U_i except U_{k-1}. We will show that for any $i \in [k-1]$ and $x \in U_i$, $|N_G(x) \cap U_i|$ is odd. Let $x \in U_i \setminus \{u, v\}$, for some $i \in [k-1]$. Then we know that $N_G(x) \cap U_i = N_G(x) \cap V_i^j$ for some $j \in [\ell]$. Since $G[V_i^j]$ is odd for all $i \in [k-1]$ and $j \in [\ell]$ we have that $|N_G(x) \cap U_i| = |N_G(x) \cap V_i^j|$ is odd. Therefore, we only need to consider v and u. Note that $u, v \in U_{k-1}$. Since both $|I|$ and $|J|$ are even and $U_{k-1} = \bigcup_{j \in I} V_{k-2}^j \cup \bigcup_{j \in J} V_{k-1}^j$, we have that $|N_G(v) \cap U_{k-1} \setminus \{u\}|$ and $|N_G(u) \cap U_{k-1} \setminus \{v\}|$ are both even. Finally, since $uv \in E(G)$ we have that $|N_G(v) \cap U_{k-1}|$ and $|N_G(u) \cap U_{k-1}|$ are both odd. Hence, $\chi_{\mathrm{odd}}(G) \leq k-1$.

Now we consider the case where there is at least one $i \in [\ell]$ where both $G[V_i \cup \{v\}]$ and $G[V_i \cup \{u\}]$ are connected. We define the following sets I and J. For each $i \in [\ell]$, (i) $i \in J$, if $G[V_i \cup \{v\}]$ is disconnected, and (ii) $i \in I$, if $G[V_i \cup \{u\}]$ is disconnected. Finally, for the rest of the indices, $i \in [\ell]$, which are not in $I \cup J$, it holds that both $G[V_i \cup \{v\}]$ and $G[V_i \cup \{u\}]$ are connected. Call this set of indices X and note that by assumption $|X| \geq 1$. Since $|I| + |J| + |X|$ is even, it is easy to see that there is a partition of X into two sets X_1 and X_2 such that both $I' := I \cup X_1$ and $J' := J \cup X_2$ have odd size. Let $V_I = \bigcup_{i \in I'} V_i$ and $V_J = \bigcup_{i \in J'} V_i$. Now, by induction, we have that $\chi_{\mathrm{odd}}(G[V_I \cup \{v\}]) \leq k-1$ and $\chi_{\mathrm{odd}}(G[V_J \cup \{u\}]) \leq k-1$. Assume that $(V_1^I, \ldots, V_{k-1}^I)$ is a partition of V_I and $(V_1^J, \ldots, V_{k-1}^J)$ is a partition of V_J such that for any $i \in [k-1]$, $G[V_i^I]$ and $G[V_i^J]$ are odd. Without loss of generality, we may assume that $v \in V_1^I$ and $u \in V_{k-1}^J$. Since $|X| \geq 1$, note that both $d_G(u)$ and $d_G(v)$ are at least two, which implies that $d_G(u) \leq k-2$ and $d_G(v) \leq k-2$. Therefore, there exists $i_0 \in [k-2]$ such that $N_G(v) \cap V_{i_0}^J = \emptyset$ and $j_0 \in [k-1] \setminus \{1\}$ such that $N_G(v) \cap V_{j_0}^I = \emptyset$. We reorder the sets V_i^J, $i \in [k-2]$, so that $i_0 = 1$ and we reorder the sets V_i^I, $i \in [k-1] \setminus \{1\}$ so that $j_0 = k-1$. Note that this reordering does not change the fact that $v \in V_1^I$ and $u \in V_{k-1}^J$. Consider the partition $(U_1, \ldots, U_{k-1})$ of $V(G)$, where $U_i = V_i^I \cup V_i^J$. We claim that for all $i \in [k-1]$, $G[U_i]$ is odd. Note that for any $x \in U_i$, we have $N_G(x) \cap U_i = N_G(x) \cap V_i^I$ or $N_G(x) \cap U_i = N_G(x) \cap V_i^J$. Since for any $i \in [k-1]$, $G[V_i^I]$ and $G[V_i^J]$ are odd we conclude that $G[U_i]$ is odd for any $i \in [k-1]$. $\square$

Notice that the class of graphs G of maximum degree Δ satisfies the requirements of Theorem 3. Indeed, this class is closed under vertex deletions and any connected graph in the class has least two adjacent vertices u, v such that $d_G(u) + d_G(v) \leq 2\Delta$. Therefore, the following corollary holds.

Corollary 1. *For every graph G with all components of even order,* $\chi_{\mathrm{odd}}(G) \leq 2\Delta - 1$.

Next, we prove Conjecture 1 for graphs of girth at least seven.

Corollary 2. *For every graph G with all components of even order of girth at least 7, $\chi_{\text{odd}}(G) \leq \frac{3\sqrt{|V(G)|}}{2} + 1$.* $(*)$[7].

One may wonder if graphs of sufficiently large girth have bounded odd chromatic number. In fact, this is far from being true, which we show in the next.

Proposition 1. *For every integer g and k, there is a graph G such that every component of G has even order, G is of girth at least g and $\chi_{\text{odd}}(G) \geq k$.* $(*)$

Next, we obtain the following result for sparse planar graphs.

Corollary 3. *For every planar graph G with all components of even order of girth at least 11, $\chi_{\text{odd}}(G) \leq 3$.* $(*)$

The upper bound in Corollary 3 is tight as C_{14}, the cycle of length 14, has $\chi_{\text{odd}}(C_{14}) = 3$.

4 Graphs of bounded modular-width

In this section we consider graphs of bounded modular-width and show that we can upper bound the odd chromatic number by the modular-width of a graph.

Theorem 4. *For every graph G with all components of even order, $\chi_{\text{odd}}(G) \leq 3\,\text{mw}(G)$.*

In order to prove Theorem 4 we show that every graph G is 3-colourable for which we have a module partition $\mathcal{M}$ such that the module graph $G_{\mathcal{M}}$ exhibits a particular structure, i.e., is either a star Lemma 1 or a special type of tree Lemma 2. The following is an easy consequence of Theorem 1 which will be useful to colour modules and gain control over the parity of parts in case of modules of even size.

Remark 1. For every non-empty graph G of even order, there exists a partition (V_1, V_2, V_3) of $V(G)$ with $|V_2|$, $|V_3|$ being odd such that $V[G_1]$ is odd and $G[V_2]$, $G[V_3]$ are even. This can be derived from Theorem 1 by taking an arbitrary vertex $v \in V(G)$, setting $V_3 := \{v\}$ and then using the existence of a partition (V_1, V_2) of $V(G) \setminus \{v\}$ such that $G[V_1]$ is odd and $G[V_2]$ is even.

Lemma 1. *For every connected graph G of even order with a module partition $\mathcal{M} = \{M_1, \ldots, M_k\}$ such that $G_{\mathcal{M}}$ is a star, $\chi_{\text{odd}}(G) \leq 3$.*

[7] For every result which is marked by $(*)$ the proof can be found in the full version of the paper

Proof of A. ssume that in $G_{\mathcal{M}}$ the vertices $M_2, \ldots, M_k$ have degree 1. We refer to M_1 as the centre and to $M_2, \ldots, M_k$ as leaves of $G_{\mathcal{M}}$. We further assume that $|M_2|, \ldots, |M_\ell|$ are odd and $|M_{\ell+1}|, \ldots, |M_k|$ are even for some $\ell \in [k]$. We use the following two claims.

Claim 1. If $W \subseteq V(G)$ with $G[W \cap M_i]$ is odd for every $i \in [k]$, then $G[W]$ is odd.

Proof. First observe that the degree of any vertex $v \in W \cap M_1$ in $G[W]$ is $d_{G[W \cap M_1]}(v) + \sum_{i=2}^{k} |W \cap M_i|$. Since $d_{G[W \cap M_1]}(v)$ is odd and $|W \cap M_i|$ is even for every $i \in \{2, \ldots, k\}$ (which follows from $G[W \cap M_i]$ being odd by the handshake lemma) we get that $d_{G[W]}(v)$ is odd. For every $i \in \{2, \ldots, k\}$ the degree of any vertex $v \in W \cap M_i$ in $G[W]$ is $d_{G[W \cap M_i]}(v) + |W \cap M_1|$ which is odd (again, because $|W \cap M_1|$ must be even). Hence $G[W]$ is odd. $\Diamond$

Claim 2. If $W \subseteq V(G)$ such that $G[W \cap M_i]$ is even for every $i \in [k]$, $|W \cap M_1|$ is odd and $|\{i \in \{2, \ldots, k\} : |W \cap M_i| \text{ is odd}\}|$ is odd, then $G[W]$ is odd.

Proof. Since $G_{\mathcal{M}}$ is a star and M_1 its centre we get that the degree of any vertex $v \in W \cap M_i$ for any $i \in \{2, \ldots, k\}$ is $d_{G[W \cap M_i]}(v) + |W \cap M_1|$. Since $|W \cap M_1|$ is odd and $d_{G[W \cap M_i]}(v)$ is even we get that every $v \in W \cap M_i$ for every $i \in \{2, \ldots, k\}$ has odd degree in $G[W]$. Moreover, the degree of $v \in W \cap M_1$ is $d_{G[W \cap M_1]}(v) + \sum_{i=2}^{k} |W \cap M_i|$. Since $d_{G[W \cap M_1]}(v)$ is even and $|\{i \in \{2, \ldots, k\} : |W \cap M_i| \text{ is odd}\}|$ is odd $d_{G[W]}(v)$ is odd. We conclude that $G[W]$ is odd. $\Diamond$

First consider the case that $|M_1|$ is odd. Since G is of even order this implies that there must be an odd number of leaves of $G_{\mathcal{M}}$ of odd size and hence ℓ is even. Using Theorem 1 we let (W_1^i, W_2^i) be a partition of M_i such that $G[W_1^i]$ is odd and $G[W_2^i]$ is even for every $i \in [k]$. Note that since $G[W_1^i]$ is odd $|W_1^i|$ has to be even and hence $|W_2^i|$ is odd if and only if $i \in [\ell]$. We define $V_1 := \bigcup_{i \in [k]} W_1^i$ and $V_2 := \bigcup_{i \in [k]} W_2^i$. Note that (V_1, V_2) is a partition of G. Furthermore, $G[V_1]$ is odd by Claim 1 and $G[V_2]$ is odd by Claim 2. For an illustration see Figure 2.

Now consider the case that $|M_1|$ is even. We first consider the special case that $\ell = 1$, i.e., there is no $i \in [k]$ such that $|M_i|$ is odd. In this case we let (W_1^i, W_2^i, W_3^i) be a partition of M_i for $i \in \{1, 2\}$ such that $G[W_1^i]$ is odd, $G[W_2^i]$, $G[W_3^i]$ are even and $|W_2^i|$, $|W_3^i|$ are odd which exists due to Remark 1. For $i \in \{3, \ldots, k\}$ we let (W_1^i, W_2^i) be a partition of M_i such that $G[W_1^i]$ is odd and $G[W_2^i]$ is even which exists by Theorem 1. We define $V_1 := \bigcup_{i \in [k]} W_1^i$, $V_2 := \bigcup_{i \in [k]} W_2^i$ and $V_3 := W_3^1 \cup W_3^2$. As before we observe that (V_1, V_2, V_3) is a partition of $V(G)$, $G[V_1]$ is odd by Claim 1 and $G[V_2]$, $G[V_3]$ are even by Claim 2. For an illustration see Figure 2.

Lastly, consider the case that $|M_1|$ is even and $\ell > 1$. By Remark 1 there is a partition (W_1^1, W_2^1, W_3^1) of M_1 such that $G[W_1^1]$ is odd, $G[W_2^1]$, $G[W_3^1]$ are even and $|W_2^1|$, $|W_3^1|$ are odd. For $i \in \{2, \ldots, k\}$ we let (W_1^i, W_2^i) be a partition of M_i such that $G[W_1^i]$ is odd and $G[W_2^i]$ is even which exists by Theorem 1.

We define $V_1 := \bigcup_{i\in[k]} W_1^i$, $V_2 := W_2^1 \cup \bigcup_{i=3}^{k} W_2^i$ and $V_3 := W_3^1 \cup W_2^2$. Note that (V_1, V_2, V_3) is a partition of $V(G)$. Furthermore, $G[V_1]$ is odd by Claim 1 and $G[V_3]$ is odd by Claim 2. Additionally, since $|M_1|$ is even there is an even number of $i \in \{2, \ldots, k\}$ such that $|M_i|$ is odd. Since for each $i \in \{2, \ldots, k\}$ for which $|M_i|$ is odd, $|W_1^i|$ must be odd, we get that $|\{i \in \{2, \ldots, k\} : |V_1 \cap M_i| \text{ is odd}\}|$ is odd (note that $V_1 \cap M_2 = \emptyset$ because $W_2^2 \subseteq V_3$). Hence we can use Claim 2 to conclude that $G[V_2]$ is odd. For an illustration see Figure 2. ☐

Let G be a connected graph of even order with module partition $\mathcal{M} =$

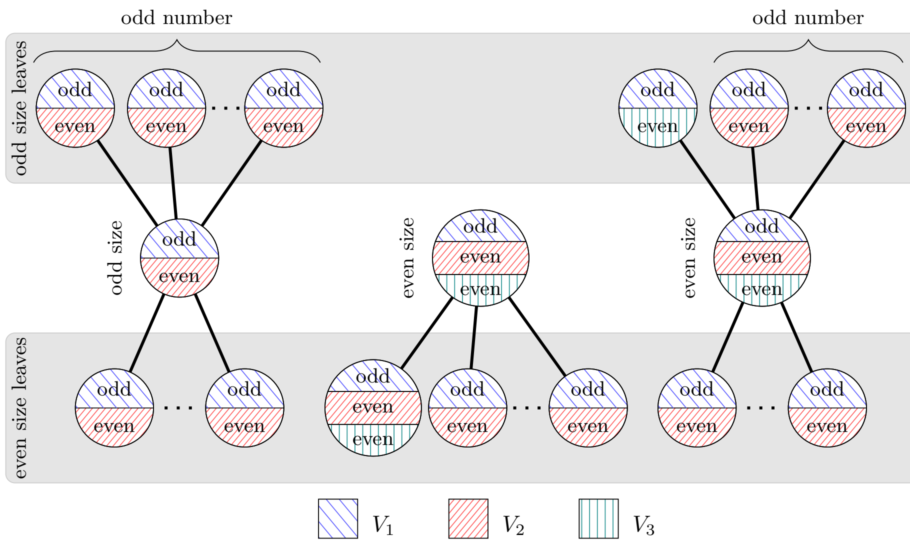

Fig. 2: Schematic illustration of the three cases in the proof of Lemma 1. Depicted is the module graph $G_{\mathcal{M}}$ along with a partition of the modules into sets V_1, V_2 and V_3 such that $G[V_i]$ is odd for $i \in [3]$.

$(M_1, \ldots, M_k)$ such that $G_{\mathcal{M}}$ is a tree. For an edge e of $G_{\mathcal{M}}$ we let X_e and Y_e be the two components of the graph obtained from $G_{\mathcal{M}}$ by removing e. We say that the tree $G_{\mathcal{M}}$ is colour propagating if the following properties hold.

(i) $|\mathcal{M}| \geq 3$.
(ii) Every non-leaf module has size one.
(iii) $|\bigcup_{M\in V(X_e)} M|$ is odd for every $e \in E(G_{\mathcal{M}})$ not incident to any leaf of $G_{\mathcal{M}}$.

Lemma 2. *For every connected graph G of even order with a module partition $\mathcal{M} = (M_1, \ldots, M_k)$ such that $G_{\mathcal{M}}$ is a colour propagating tree, $\chi_{\text{odd}}(G) \leq 2$.*

Proof. To find an odd colouring (V_1, V_2) of G, we first let (W_1^i, W_2^i) be a partition of M_i such that $G[W_1^i]$ is odd and $G[W_2^i]$ is even for every $i \in [k]$. The partitions (W_1^i, W_2^i) exist due to Theorem 1. Note that (ii) implies that for every module

M_i which is not a leaf $|W_2^i| = 1$ and $W_1^i = \emptyset$. We define $V_1 := \bigcup_{i\in[k]} W_1^i$ and $V_2 := \bigcup_{i\in[k]} W_2^i$.

To argue that (V_1, V_2) is an odd colouring of G first consider any $v \in V(G)$ such that $v \in M_i$ for some leaf M_i of $G_{\mathcal{M}}$. Condition (i) implies that $G_{\mathcal{M}}$ must have at least three vertices and hence the neighbour M_j of M_i cannot be a leaf due to $G_{\mathcal{M}}$ being a tree. Hence $|M_j| = 1$ by (ii). Hence, if $v \in W_1^i$, then $d_{G[V_1]}(v) = d_{G[W_1^i]}(v)$ since $W_1^j = \emptyset$ and therefore $d_{G[V_1]}(v)$ is odd. Further, if $v \in W_2^i$, then $d_{G[V_2]}(v) = d_{G[W_2^i]}(v) + 1$ since $|W_2^j| = 1$ and hence $d_{G[V_2]}(v)$ is odd. Hence the degree of any vertex $v \in M_i$ is odd in $G[V_1]$, $G[V_2]$ respectively.

Now consider any vertex $v \in V(G)$ such that $M_i = \{v\}$ for some non-leaf M_i of $G_{\mathcal{M}}$. Let $M_{i_1}, \ldots, M_{i_\ell}$ be the neighbours of M_i in $G_{\mathcal{M}}$. Let e_j be the edge $M_i M_{i_j} \in E(G)$ for every $j \in [\ell]$. Without loss of generality, assume that $M_i \notin V(X_{e_j})$ for every $j \in [\ell]$. By (iii) we have that $|\bigcup_{M\in V(X_{e_j})} M|$ is odd whenever M_{i_j} is not a leaf in $G_{\mathcal{M}}$. Hence, by (ii), $|X_{e_j}| \equiv |M_{i_j}| \pmod 2$ for every $j \in [\ell]$ for which M_{i_j} is not a leaf in $G_{\mathcal{M}}$. On the other hand, as a consequence of the handshake lemma we get that $|W_2^{i_j}|$ is odd if and only if $|M_{i_j}|$ is odd. Hence the following holds for the parity of the degree of v in $G[V_2]$.

$$d_{G[V_2]}(v) = |\{j \in [m] : d_{G_{\mathcal{M}}}(M_{i_j}) \geq 2\}| + \bigcup_{\substack{j\in[m] \\ d_{G_{\mathcal{M}}}(M_{i_j})=1}} |W_2^{i_j}| \equiv |V(G) \setminus M_i| \pmod 2.$$

Since G has even order, $d_{G[V_2]}(v)$ is odd and (V_1, V_2) is an odd colouring of G. $\square$

We now show that, given a graph G with module partition $\mathcal{M}$, we can decompose the graph in such a way that the module graph of any part of the decomposition is either a star or a colour propagating tree. Here we consider the module graph with respect to the module partition $\mathcal{M}$ restricted to the part of the decomposition we are considering. To obtain the decomposition we use a spanning tree $G_{\mathcal{M}}$ and inductively find a non-separating star, i.e., a star whose removal does not disconnect the graph, or a colour propagating tree. In order to handle parity during this process we might separate a module into two parts.

Lemma 3. *For every connected graph G of even order and module partition $\mathcal{M} = (M_1, \ldots, M_k)$ there is a partition $\widehat{\mathcal{M}}$ of $V(G)$ with at most $2k$ many parts such that there is a coarsening $\mathcal{P}$ of $\widehat{\mathcal{M}}$ with the following properties. $|P|$ is even for every part P of $\mathcal{P}$. Furthermore, for every part P of $\mathcal{P}$ we have that $\widehat{\mathcal{M}}|_P$ is a module partition of $G[P]$ and $G[P]_{\widehat{\mathcal{M}}|_P}$ is either a star (with at least two vertices) or a colour propagating tree.* $(*)$

Proof of Theorem 4. Without loss of generality assume that G is connected. Furthermore, let $k := \mathrm{mw}(G)$ and $\mathcal{M} = (M_1, \ldots, M_k)$ be a module partition of G. Let $\widehat{\mathcal{M}}$ be a partition of $V(G)$ with at most $2k$ parts and $\mathcal{P}$ be a coarsening of $\widehat{\mathcal{M}}$ as in Lemma 3. First observe that $\widehat{\mathcal{M}}|_P$ must contain at least two parts for every part P of $\mathcal{P}$ as $\widehat{\mathcal{M}}|_P$ is a module partition of $G[P]$. Since $\widehat{\mathcal{M}}$ has at most $2k$ parts and $\mathcal{P}$ is a coarsening of $\widehat{\mathcal{P}}$ this implies that $\mathcal{P}$ has at most k parts. Since $G[P]_{\widehat{\mathcal{M}}|_P}$ is either a star or a colour propagating tree we get that $\chi_{\text{odd}}(G[P]) \leq 3$ for every part P of $\mathcal{P}$ by Lemma 1 and Lemma 2. Using a partition (W_1^P, W_2^P, W_3^P) of $G[P]$ such that $G[W_i^P]$ is odd for every $i \in [3]$ for every part P we obtain a global partition of G into at most $3k$ parts such that each part induces an odd subgraph. □

Since deciding whether a graph is k-odd colourable can be solved in time $2^{\mathcal{O}(k\,\mathrm{rw}(G))}$ [2, Theorem 6] and $\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq \mathrm{mw}(G)$, where $\mathrm{cw}(G)$ denotes the clique-width of G and $\mathrm{rw}(G)$ rank-width, we obtain the following as a corollary.

Corollary 4. *Given a graph G and a module partition of G of width m the problem of deciding whether G can be odd coloured with at most k colours can be solved in time $2^{\mathcal{O}(m^2)}$.*

5 Interval graphs

In this section we study the odd chromatic number of interval graphs and provide an upper bound in the general case as well as a tight upper bound in the case of proper interval graphs. We use the following lemma in both proofs.

Lemma 4. *Let G be a connected interval graph and $P = (p_1, \ldots, p_k)$ a maximal induced path in G with the following property.*

(Π) $\ell_{p_1} = \min\{\ell_v : v \in V(G)\}$ and for every $i \in [k-1]$ we have that $r_{p_{i+1}} \geq r_v$ for every $v \in N_G(p_i)$.

Then every $v \in V(G)$ is adjacent to at least one vertex on P. $(*)$

To prove that the odd chromatic number of proper interval graphs is bounded by three we essentially partition the graph into maximal even sized cliques greedily in a left to right fashion.

Theorem 5. *For every proper interval graph G with all components of even order, $\chi_{\text{odd}}(G) \leq 3$ and this bound is tight.*

Proof. We assume that G is connected. Fix an interval representation of G and denote the interval representing vertex $v \in V(G)$ by $I_v = [\ell_v, r_v]$ where $\ell_v, r_v \in \mathbb{R}$. Let $P = (p_1, \ldots, p_k)$ be a maximal induced path in G as in Lemma 4. For every vertex $v \in V(G) \setminus \{p_1, \ldots, p_k\}$ let $i_v \in [k]$ be the index such that p_{i_v} is the first neighbour of v on P. Note that this is well defined by Lemma 4. For $i \in [k]$ we let Y_i be the set with the following properties.

$(\Pi 1)_i$ $\{v \in V(G) : i_v = i\} \subseteq Y_i \subseteq \{v \in V(G) : i_v = i\} \cup \{p_i, p_{i+1}\}$.
$(\Pi 2)_i$ $p_i \in Y_i$ if and only if $\left|\{p_1, \ldots, p_{i-1}\} \cup \bigcup_{j \in [i-1]} \{v \in V(G) : i_v = j\}\right|$ is even.

$(\Pi 3)_i$ $p_{i+1} \in Y_i$ if and only if $\left|\{p_1, \ldots, p_i\} \cup \bigcup_{j\in[i]}\{v \in V(G) : i_v = j\}\right|$ is odd.

First observe that $(Y_1, \ldots, Y_k)$ is a partition of $V(G)$ as $(\Pi 2)_i$ and $(\Pi 3)_i$ imply that every p_i is in exactly one set Y_i. Furthermore, $|Y_i|$ is even for every $i \in [k]$ since $(\Pi 1)_i$ and $(\Pi 3)_i$ imply that $\left|Y_i \cup \{p_1, \ldots, p_i\} \cup \bigcup_{j\in[i-1]}\{v \in V(G) : i_v = j\}\right|$ is even and $(\Pi 2)_i$ implies that $\left|(\{p_1, \ldots, p_i\} \cup \bigcup_{j\in[i-1]}\{v \in V(G) : i_v = j\}) \setminus Y_i\right|$ is even. Since $v \in V(G) \setminus \{p_1, \ldots, p_k\}$ is not adjacent to p_{i_v-1} we get that $\ell_v \in I_{p_{i_v}}$. Since G is a proper interval graph this implies that $r_{p_{i_v}} \leq r_v$ and hence v is adjacent to p_{i_v+1}. Hence $(\Pi 1)_i$ implies that $G[Y_i]$ must be a clique since $Y_i \cap \{p_1, \ldots, p_k\} \subseteq \{p_i, p_{i+1}\}$ for every $i \in [k]$. Furthermore, $N_G(Y_i)$ and Y_{i+3} are disjoint since $r_v \leq r_{p_{i+1}}$ for every $v \in Y_i$ by property (Π) and $r_{p_{i+1}} < \ell_{p_{i+3}} \leq r_w$ for every $w \in Y_{i+3}$ since P is induced. Hence we can define an odd-colouring (V_1, V_2, V_3) of G in the following way. We let $V_j := \bigcup_{i\equiv j \pmod 3} Y_i$ for $j \in [3]$. Note that since $N_G(Y_i) \cap Y_{i+3}$ we get that $d_{G[Y_i]}(v) = d_{G[V_j]}(v)$ for $i \equiv j \pmod 3$ which is odd (as Y_i is a clique of even size). Hence $G[V_j]$ is odd for every $j \in [3]$.

To see that the bound is tight consider the graph G consisting of K_4 with two pendant vertices u, w adjacent to different vertices of K_4. Clearly, G is a proper interval graph and further $\chi_{\mathrm{odd}}(G) = 3$. □

We use a similar setup (i.e., a path P covering all vertices of the graph G) as in the proof of Theorem 5 to show our general upper bound for interval graphs. The major difference is that we are not guaranteed that sets of the form $\{p_i\} \cup \{v \in V(G) : i_v = i\}$ are cliques. To nevertheless find an odd colouring with few colours of such sets we use an odd/even colouring as in Theorem 1 of $\{v \in V(G) : i_v = i\}$ and the universality of p_i. Hence this introduces a factor of two on the number of colours. Furthermore, this approach prohibits us from moving the p_i around as in the proof of Theorem 5. As a consequence we get that the intervals of vertices contained in a set Y_i span a larger area of the real line than in the proof of Theorem 5. This makes the analysis more technical.

Theorem 6. *For every interval graph G with all components of even order, $\chi_{\mathrm{odd}}(G) \leq 6$.* (∗)

Note that we currently are unaware whether the bound from Theorem6 is tight or even whether there is an interval graph G with $\chi_{\mathrm{odd}}(G) > 3$.

6 Conclusion

We initiated the systematic study of odd colouring on graph classes. Motivated by Conjecture 1, we considered graph classes that do not contain large graphs from a given family as induced subgraphs. Put together, these results provide evidence that Conjecture 1 is indeed correct. Answering it remains a major open problem, even for the specific case of bipartite graphs.

Several other interesting classes remain to consider, most notably line graphs and claw-free graphs. Note that odd colouring a line graph $L(G)$ corresponds to colouring the edges of G in such a way that each colour class induces a bipartite graph where every vertex in one part of the bipartition has odd degree, and every vertex in the other colour part has even degree. This is not to be confused with the notion of odd k-edge colouring, which is a (not necessarily proper) edge colouring with at most k colours such that each nonempty colour class induces a graph in which every vertex is of odd degree. It is known that all simple graphs can be odd 4-edge coloured, and every loopless multigraph can be odd 6-edge coloured (see e.g., [9]). While (vertex) odd colouring line graphs is not directly related to odd edge colouring, this result leads us to believe that line graphs have bounded odd chromatic number.

Finally, determining whether Theorem 4 can be extended to graphs of bounded rank-width remains open. We also believe that the bounds in Theorem 6 and Corollary 1 are not tight and can be further improved. In particular, we believe that the following conjecture, first stated in [1], is true:

Conjecture 2 (Aashtab et al., 2023). Every graph G of even order has $\chi_{\text{odd}}(G) \leq \Delta + 1$.

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