Cooperative Games
Lecture 4: Game with Coalition structures, Core and Bargaining Set
Stéphane Airiau
ILC - University of Amsterdam

Today

- If agents desire the kind of stability offered by the core, they will be unable to reach an agreement.
- They have no choice but to relax their stability requirements.
- Need a solution that allows agents to reach an agreement, but maintain some stability.
- First we will consider the problem of stability of coalition structure, i.e., a partition of the set of agents is formed first, and then, members of a coalition negotiate their payoff.
- Then, we will consider the bargaining set, which relaxes the requirements of the core.

Coalition Structure

Definition (Coalition Structure)
A coalition structure (CS) is a partition of the grand coalition into coalitions.
$\mathcal{S} = \{C_1, \ldots, C_k\}$, where $\bigcup_{i \neq j} C_i \cap C_j = \emptyset$.
We note $\mathcal{A}_N$ the set of all coalition structures over the set $N$.

The set of feasible payoff vectors for $(N, \mathcal{S})$ is
$X_{N, \mathcal{S}} = \{x \in \mathbb{R}^N \mid \forall C \in \mathcal{S}, x(C) \leq \sum_{i \in C} v(i)\}$.

Definition (Core of a CS)
The core $\text{core}(N, \mathcal{S})$ of $(N, \mathcal{S})$ is defined by
$\{x \in \mathbb{R}^N \mid \forall C \in \mathcal{S}, x(C) \leq \sum_{i \in C} v(i)\}$. We have $\text{core}(N, \mathcal{S}) = \text{core}(N, \mathcal{S})$.

The next theorems are due to Aumann and Drèze.

Definition (Superadditive cover)
The superadditive cover of $(N, \mathcal{S})$ is the game $(N, \mathcal{S})$ defined by
$\hat{v}(C) = \max_{P \in \mathcal{S}} \left\{ \sum_{i \in C} v(i) \right\}$, where $\mathcal{S}$ is the superadditive cover of $(N, \mathcal{S})$.

Theorem
Let $(N, \mathcal{S})$ be a game with coalition structure. Then

a) $\text{core}(N, \mathcal{S}) \neq \emptyset$ if $\text{core}(N, \mathcal{S}) \neq \emptyset$ and $\hat{v}(N) = \sum_{i \in N} v(i)$.

b) If $\text{core}(N, \mathcal{S}) \neq \emptyset$, then $\text{core}(N, \mathcal{S}) = \text{core}(N, \mathcal{S})$.

Proof of part a)

Let $x \in \text{core}(N, \mathcal{S})$. We show that $x \in \text{core}(N, \mathcal{S})$.

Let $C \subseteq N \setminus \emptyset$ be a partition of $C$.

By definition of the core, for every $x \subseteq N \setminus \emptyset$ (i.e., $x \in \mathcal{S}$),
$x(C) = \sum_{P \in C} x(P) \geq \sum_{P \in C} v(P)$, which is valid for all partitions of $C$. Hence, $x(C) \geq \sum_{P \in C} v(P) = \hat{v}(C)$.

We have just proved $\forall C \subseteq N \setminus \emptyset$, $x(C) \geq \hat{v}(C)$, and so $x$ is group rational.

We now need to prove that $\hat{v}(N) = \sum_{C \subseteq N \setminus \emptyset} x(C)$.

$\hat{v}(N) = \sum_{C \subseteq N \setminus \emptyset} x(C)$ since $x$ is in the core of $(N, \mathcal{S})$ (efficient).

Applying the inequality above, we have
$x(N) = \sum_{C \subseteq N \setminus \emptyset} x(C)$ since $x$ is in the core of $(N, \mathcal{S})$ (efficient).

Applying the definition of the valuation function $\hat{v}$, we have
$\hat{v}(N) = \sum_{C \subseteq N \setminus \emptyset} x(C)$.

Consequently, $\hat{v}(N) = \sum_{C \subseteq N \setminus \emptyset} x(C)$ and it follows that $x$ is efficient for the game $(N, \mathcal{S})$.

Hence $x \in \text{core}(N, \mathcal{S})$.

Proof of part b)

Let’s assume $x \in \text{core}(N, \mathcal{S})$ and $\hat{v}(N) = \sum_{i \in N} v(i)$.

We need to prove that $x \in \text{core}(N, \mathcal{S})$.

For every $x \subseteq N$, $x \in \text{core}(N, \mathcal{S})$ since $x$ is in the core of $\text{core}(N, \mathcal{S})$. Then $x \in \text{core}(N, \mathcal{S})$ is a partition of $C$.

This proves $x$ is group rational.

$x(N) = \hat{v}(N) = \sum_{P \in \mathcal{S}} v(P)$ since $x$ is efficient.

It follows that $\forall C \subseteq N \setminus \emptyset$, we must have $x(C) = \hat{v}(C)$, which proves $x$ is efficient for the CS $\mathcal{S}$, and that $x$ is efficient.

Hence, $x \in \text{core}(N, \mathcal{S})$.

proof of part b): we have just proved that $x \in \text{core}(N, \mathcal{S})$ implies that $x \in \text{core}(N, \mathcal{S})$ and $x \in \text{core}(N, \mathcal{S})$ implies that $x \in \text{core}(N, \mathcal{S})$.

This proves that if $\text{core}(N, \mathcal{S}) \neq \emptyset$, then $\text{core}(N, \mathcal{S}) = \text{core}(N, \mathcal{S})$. 
Definition (Substitutes)
Let \((N,v,\mathcal{S})\) be a game with coalition structure, and let \(i,j\) be substitutes.
A property of the core related to fairness:

**Theorem**
Let \((N,v,\mathcal{S})\) be a game with coalition structure, let \(i,j\) be substitutes, and let \(x \in \text{Core}(N,v,\mathcal{S})\).
If \(i\) and \(j\) belong to different members of \(\mathcal{S}\), then \(x_i = x_j\).

**Proof**
Let \((U_i) \subseteq \mathbb{R}^n\) be substitutes, \(\mathcal{C} \subseteq \mathbb{C}\) such that \(i \in \mathcal{C} \) and \(j \notin \mathcal{C}\).
Let \(x \in \text{Core}(N,v,\mathcal{S})\).
Since \(i\) and \(j\) are substitutes, we have
\[ v(\mathcal{C}(U_i)) = v(\mathcal{C}(U_j)) \leq v(\mathcal{C}(U_i)) - v(\mathcal{C}(U_j)) = v(i) = v(j). \]
Since \(x \in \text{Core}(N,v,\mathcal{S})\), we have \(v(\mathcal{C}(U_i)) = v(\mathcal{C}(U_j)) = v(\mathcal{C}(U_i)) - v(\mathcal{C}(U_j)) = v(i) = v(j)\).
Since \(i\) and \(j\) are substitutes, we have \(x_i = x_j\).

### Example
Let \((N,v,\mathcal{S})\) be a 7-player simple majority game, i.e.
\[ v(\mathcal{C}) = \begin{cases} 1 & \text{if } |C| \geq 4 \\ 0 & \text{otherwise} \end{cases}. \]
Let us consider \(x = (1,1,1,1,1,1,1)\). It is clear that \(x(N) = 1\).
Let us prove that \(x\) is in the pre-bargaining set of the game \((N,v,\mathcal{S})\).

Objects within members of \(\{2,3,4,5,6,7\}\) will have a counter-objection by symmetry.
Let us consider the objects \((P,y)\) of 1 against another member of \(\{2,3,4,5,6,7\}\). Since the players \(2,3,\ldots,7\) play symmetric roles, we consider an objection of 1 against 7 using successively
\[ P = \{1,2,3,4,5,6\}, \quad Q = \{1,2,3,4,5\}, \quad R = \{1,2,3,4\}, \quad S = \{1,2,3\}, \quad T = \{1,2\} \quad \text{and} \quad U = \{1\}. \]
We will look for a counter-object of player 7 using \((Q,z)\).
Let \( P = \{1, 2, 3, 4\} \). The vector \( y = (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \) is an objection when 
\[
\alpha' = -\frac{2}{3}, \quad a_1 \geq 0, \quad \sum_{k=1}^{7} a_k = \alpha' \leq 1
\]
This time, we have \( \sum_{k=2}^{7} a_k = 1 + a_1 + a_4 \leq 4 + \frac{7}{3} < 4 \),
and finally \( \sum_{k=1}^{7} a_k = 1 - \alpha' = \frac{1}{3} \neq \frac{2}{3} \).

We need to find a counter-objection to \( \{P, y\} \).

We choose \( \alpha' = -\frac{4}{5} \), \( y = (\frac{1}{5}, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \).
It is clear that \( \forall i \in \{P, Q\}; y \not\in P \}, \forall y \in P \}, Q \not\in y \} \) (for agent 2 and 3).
\[
\alpha' = 5 - \frac{4}{5} = \frac{1}{5} \quad \text{and since the}
\]
\[
\sum_{i=1}^{2} a_i \leq \frac{1}{5} = \frac{3}{5} \quad \text{which is not possible.
}
\[
\text{Hence } \exists \{Q, y\} \text{ which proves } z \text{ is feasible.}
\]

**Theorem** Let \( (N, \pi, S) \) a game with coalition structure. Assume that \( I(N, \pi, S) \neq \emptyset \). Then the bargaining set \( BS(N, \pi, S) \neq \emptyset \).

**Proof** It is possible to give a direct proof of this theorem (a bit long,
(Section 4.2 in Introduction to the Theory of Cooperative Games).
We will show this result in a different way in the lecture about the nucleolus next week.


**Definition** (weighting voting games)

A game \( (N, \pi, S, q, \alpha) \) is a weighted voting game when \( \pi \) satisfies unanimity, monotonicity and the valuation function is defined as
\[
\pi(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}
\]

We note such a game by \( (q; w_1, \ldots, w_n) \).

Let \( (N, \pi) \) be the game associated with the 6-player weighted majority game \( (3, 1, 1, 1, 1, 1) \). Agent 6 is a null player since its weight is 0. Nevertheless \( x = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0) \) \( BS(N, \pi) \).

Show it at home (a solution will be posted online). We need to consider all objections \( (P, y) \) from the an agent against the null agent, and find a counter objection \( (Q, z) \).

Agent 6 is a dummy, however, it receives a payoff of \( \frac{1}{2} \), which is larger than agents who are not dummy!

**Summary**

- We introduced the notion of games with coalition structures.
- We looked at the definition of the core, i.e., stability of the coalition structure. Games with coalition structure may have an empty core (e.g., \( (N, \pi, |N|) \) and \( \varnothing \) in exercise 4 of homework 1).
- We introduced the bargaining set, and looked at some examples.
  - pros: it is guaranteed to be non-empty.
  - cons: it may be unreasonable from above.

**Coming next**

- We will consider the Nucleolus. It can also be defined in terms of objections and counter objections, but the nature of the objection is different from the bargaining set.