Cooperative Games

Lecture 4: The Bargaining Set

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Today

- If agents desire the kind of stability offered by the core, they will be unable to reach an agreement.
- ⇝ they have no choice but to **relax** their stability requirements.
- Need a solution that allows agents to reach an agreement, but maintain some stability.
- Then, we will consider the bargaining set, which relaxes the requirements of the core.

A second solution concept:

The bargaining set.

R.J. Aumann and M. Maschler. **The bargaining set for cooperative games**, in *Advances in game theory (Annals of mathematics study)*, 1964.

M. Davis and M. Maschler. **Existence of stable payoff configurations for cooperative games**, *Bulletin of of the American mathematical society*, 1963.

Let $(N,v,\mathcal{S})$ be a game with coalition structure and x an imputation.
The bargaining set models stability in the following sense:

Any **argument** from an agent i against a payoff distribution x is of the following form:

> *I get too little in the imputation x, and agent j gets too much! I can form a coalition that excludes j in which some members benefit and all members are at least as well off as in x.*

The argument is **ineffective** for the bargaining set if agent j can answer the following:

> *I can form a coalition that excludes agent i in which all agents are at least as well off as in x, and as well off as in the payoff proposed by i for those who were offered to join i in the argument.*

Definition (Objection)

Let $(N,v,\mathcal{S})$ be a game with coalition structure, $x \in X_{(N,v,\mathcal{S})}$ (the set of all feasible payoff vectors for $(N,v,\mathcal{S})$), $\mathcal{C} \in \mathcal{S}$ be a coalition, and i and j two distinct members of $\mathcal{C}$ ($(i,j) \in \mathcal{C}^2$, $i \neq j$).
An **objection of** i **against** j is a pair (P,y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where p is the size of P
- $y(P) \leqslant v(P)$ (y is a feasible payoff distribution for the agents in P)
- $\forall k \in P$, $y_k \geqslant x_k$ and $y_i > x_i$ (agent i strictly benefits from y, and the other members of P do not do worse in y than in x.)

An objection (P,y) of i against j is a **potential threat** by coalition P, which contains i but not j, to deviate from x.
The goal is not to change $\mathcal{S}$, but to obtain a side payment from j to i, i.e., to modify x within $X_{(N,v,\mathcal{S})}$.

Definition (Counter-objection)

An **counter-objection to** (P,y) is a pair (Q,z) where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.
- $z \in \mathbb{R}^q$ where q is the size of Q
- $z(Q) \leqslant v(Q)$ (z is a feasible payoff distribution for the agents in Q)
- $\forall k \in Q$, $z_k \geqslant x_k$ (the members of Q get at least the value in x)
- $\forall k \in Q \cap P$ $z_k \geqslant y_k$ (the members of Q which are also members of P get at least the value promised in the objection)

In a counter-objection, agent j must show that it can protect its payoff x_j in spite of the existing objection of i.

Definition (Stability)

Let $(N,v,\mathcal{S})$ a game with coalition structure. A vector $x \in X_{(N,v,\mathcal{S})}$ is **stable** iff for each objection at x there is a counter-objection.

Definition (Pre-bargaining set)

The **pre-bargaining set (*preBS*)** is the set of all stable members of $X_{(N,v,\mathcal{S})}$.

Lemma

Let $(N,v,\mathcal{S})$ a game with coalition structure, we have

$$Core(N,v,\mathcal{S}) \subseteq preBS(N,v,\mathcal{S}).$$

This is true since, if $x \in Core(N,v,\mathcal{S})$, no agent i has any objection against any other agent j.

Example

Let (N,v) be a 7-player simple majority game, i.e.

$$v(\mathcal{C}) = \begin{cases} 1 \text{ if } |\mathcal{C}| \geqslant 4 \\ 0 \text{ otherwise} \end{cases}.$$

Let us consider $x = \langle -\frac{1}{5}, \frac{1}{5}, \ldots, \frac{1}{5} \rangle$. It is clear that $x(N) = 1$.

Let us prove that **x is in the pre-bargaining set** of the game $(N,v,\{N\})$.

Objections within members of $\{2,3,4,5,6,7\}$ will have a counter-objection by symmetry. ✔

Let us consider the objections (P,y) of 1 against another member of $\{2,3,4,5,6,7\}$. Since the players $\{2,\ldots,7\}$ play symmetric roles, we consider an objection (P,y) of 1 against 7 using successively as P $\{1,2,3,4,5,6\}$, $\{1,2,3,4,5\}$, $\{1,2,3,4\}$, $\{1,2,3\}$, $\{1,2\}$ and $\{1\}$q. We will look for a counter-objection of player 7 using (Q,z).

- $P=\{1,2,3,4,5,6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that (P,y) is an objection. $y = \langle \alpha, \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \dots, \frac{1}{5}+\alpha_6 \rangle$.
  The conditions for (P,y) to be an objection are the following:
  - each agent is as well off as in x: $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$
  - y is feasible for coalition P: $\sum_{i=2}^{6}\left(\alpha_i+\frac{1}{5}\right)+\alpha \leqslant 1$.

  w.l.o.g $0 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \alpha_4 \leqslant \alpha_5 \leqslant \alpha_6$.

  Then $\sum_{i=2}^{6}\left(\frac{1}{5}+\alpha_i\right)+\alpha = \frac{5}{5}+\sum_{i=2}^{6}\alpha_i+\alpha = 1+\sum_{i=2}^{6}\alpha_i+\alpha \leqslant 1.$

  Then $\sum_{i=2}^{6}\alpha_i \leqslant -\alpha < \frac{1}{5}$.

➩ We need to find a counter-objection for (P,y).
claim: we can choose $Q=\{2,3,4,7\}$ and
$z = \langle \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \frac{1}{5}+\alpha_4, \frac{1}{5}+\alpha_5 \rangle$
$z(Q) = \frac{1}{5}+\alpha_2+\frac{1}{5}+\alpha_3+\frac{1}{5}+\alpha_4+\frac{1}{5}+\alpha_5 = \frac{4}{5}+\sum_{i=2}^{5}\alpha_i \leqslant 1$ since $\sum_{i=2}^{5}\alpha_i \leqslant \sum_{i=2}^{6}\alpha_i < \frac{1}{5}$ so z is feasible.
It is clear that $\forall i \in Q$, $z_i \geqslant x_i$ ✔and that $\forall i \in Q \cap P$, $z_i \geqslant y_i$ ✔
Hence, (Q,z) is a counter-objection. ✔

- $P=\{1,2,3,4,5\}$. The vector $y = \langle \alpha, \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \frac{1}{5}+\alpha_4, \frac{1}{5}+\alpha_5 \rangle$ is an objection when

  $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\sum_{i=2}^{5}(\frac{1}{5}+\alpha_i)+\alpha \leqslant 1$

  This time, we have $\sum_{i=2}^{5}(\frac{1}{5}+\alpha_i)+\alpha = \frac{4}{5}+\sum_{i=2}^{5}\alpha_i+\alpha \leqslant 1$

  then $\sum_{i=2}^{5}\alpha_i \leqslant 1-\frac{4}{5}-\alpha = \frac{1}{5}-\alpha$ and finally $\sum_{i=2}^{5}\alpha_i \leqslant \frac{1}{5}-\alpha < \frac{2}{5}$.

➩ We need to find a counter-objection to (P,y)
claim: we can choose $Q=\{2,3,6,7\}$, $z = \langle \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$
It is clear that $\forall i \in Q$, $z_i \geqslant x_i$ ✔and $\forall i \in P \cap Q$ $z_i \geqslant y_i$ (for agent 2 and 3).
$z(Q) = \frac{1}{5}+\alpha_2+\frac{1}{5}+\alpha_3+\frac{1}{5}+\frac{1}{5} = \frac{4}{5}+\alpha_2+\alpha_3$. We have $\alpha_2+\alpha_3 < \frac{1}{5}$, otherwise, we would have $\alpha_2+\alpha_3 \geqslant \frac{1}{5}$ and since the α_i are ordered, we would then have $\sum_{i=2}^{5}\alpha_i \geqslant \frac{2}{5}$, which is not possible. Hence $z(Q) \leqslant 1$ which proves z is feasible ✔

Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P=\{1,2,3,4\}$, $y = \langle \alpha, \frac{1}{5}+\alpha_1, \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\sum_{i=2}^{4}\alpha_i+\alpha \leqslant \frac{2}{5} \Rightarrow \sum_{i=2}^{4}\alpha_i \leqslant \frac{2}{5}-\alpha < \frac{3}{5}$.

➩ $Q=\{2,5,6,7\}$, $z = \langle \frac{1}{5}+\alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ since $\alpha_2 \leqslant \frac{1}{5}$

- $|P| \leqslant 3$ $P=\{1,2,3\}$, $v(P)=0$, $y=\langle \alpha, \alpha_1, \alpha_2 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\alpha_1+\alpha_2 \leqslant -\alpha < \frac{1}{5}$

➩ $Q=\{4,5,6,7\}$, $z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ will be a counter argument (1 cannot provide more than $\frac{1}{5}$ to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

➩ $x \in preBS(N,v,\mathcal{S})$.✔

Bargaining set

In the example, agent 1 gets $-\frac{1}{5}$ when $v(\mathcal{C}) \geqslant 0$ for all coalition $\mathcal{C} \subseteq N$! This shows that the pre-bargaining set may **not** be individually rational.

Let $I(N,v,\mathcal{S}) = \left\{x \in X_{(N,v,\mathcal{S})} \mid x_i \geqslant v(\{i\}) \forall i \in N\right\}$ be the **set of individually rational payoff vector** in $X_{(N,v,\mathcal{S})}$.

Lemma

If a game is weakly superadditive, $I(N,v,\mathcal{S}) \neq \emptyset$.

Definition (Bargaining set)

Let $(N,v,\mathcal{S})$ a game in coalition structure.
The **bargaining set** (BS) is defined by
$$BS(N,v,\mathcal{S}) = I(N,v,\mathcal{S}) \cap preBS(N,v,\mathcal{S}).$$

Lemma

We have $Core(N,v,\mathcal{S}) \subseteq BS(N,v,\mathcal{S})$.

Theorem

Let $(N,v,\mathcal{S})$ a game with coalition structure. Assume that $I(N,v,\mathcal{S}) \neq \emptyset$. Then the bargaining set $BS(N,v,\mathcal{S}) \neq \emptyset$.

Proof

It is possible to give a direct proof of this theorem (a bit long, (Section 4.2 in **Introduction to the Theory of Cooperative Games**)).
We will show this result in a different way in the lecture about the nucleolus next week. □

B. Peleg and P. Sudhölter **Introduction to the Theory of Cooperative Games**, Springer, 2007.

Definition (weighted voting games)

A game $(N, w_{i \in N}, q, v)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as
$$v(S) = \begin{cases} 1 \text{ when } \sum_{i \in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$
We note such a game by $(q : w_1, \dots, w_n)$

Let (N,v) be the game associated with the 6-player weighted majority game (3:1,1,1,1,1,0).
Agent 6 is a null/dummy player since its weight is 0.
Nevertheless $\langle \frac{1}{7}, \dots, \frac{1}{7}, \frac{2}{7} \rangle \in BS(N,v)$.

Proof

This will be part of homework 2 □

Agent 6 is a dummy, however, it receives a payoff of $\frac{2}{7}$, which is larger than agents who are not dummy!

Remember: $mc_i^{max} = \max_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$
x is **reasonable from above** if $\forall i \in N$ $x^i < mc_i^{max}$

➪ mc_i^{max} is the strongest **threat** that an agent can use against a coalition.

The bargaining set is not **Reasonable from above**: the dummy agent gets more than $\max_{\mathcal{C} \subseteq N \setminus \{6\}} (v(\mathcal{C} \cup \{6\}) - v(\mathcal{C})) = 0$.✘

Lemma

The core is reasonable for above and from below

Proof

Since the core satisfies IR, it must be reasonable from below. Let (N,v) be a game, $x \in C(N,v)$ and $i \in N$. Then $x(N) = v(N)$ and $x(N \setminus \{i\}) \geqslant v(N \setminus \{i\})$.Then
$$x_i = v(N) - x(N \setminus \{i\}) \leqslant v(N) - v(N \setminus \{i\}) \leqslant mc_i^{max}.$$
□

Summary

- We introduced the bargaining set, and looked at some examples.

pros: it is guaranteed to be non-empty,
when the core is non-empty, it is contained in the bargaining set.
cons: it may not be reasonable from above.

Coming next

- We will consider the Nucleolus. It can also be defined in terms of objections and counter objections, but the nature of the objection is different from the bargaining set.