Cooperative Games
Lecture 6: The Kernel

Stéphane Airiau

ILLC - University of Amsterdam
One last stability concept from the bargaining set family: 

**The kernel.**

Excess

**Definition** (Excess)

For a TU game \((N,v)\), the excess of coalition \(C\) for a payoff distribution \(x\) is defined as \(e(C,x) = v(C) - x(C)\).

We saw that a positive excess can be interpreted as an amount of complaint for a coalition. We can also interpret the excess as a potential to generate more utility.
Let \((N,v)\) be a TU game, \(S \in \mathcal{S}_N\) a coalition structure and \(x\) a payoff distribution. Objections and counter-objections are exchanged between members of the same coalition in \(S\). Objections and counter-objections take the form of coalitions, i.e., they do not propose another payoff distribution.

Let \(C \in S, k \in C, l \in C\).

**Objection:** A coalition \(P \subseteq N\) is an objection of \(k\) against \(l\) to \(x\) iff \(k \in P,\ l \notin P\) and \(x_l > v(\{l\})\).

"\(P\) is a coalition that contains \(k\), excludes \(l\) and which sacrifices too much (or gains too little)."

**Counter-objection:** A coalition \(Q \subseteq N\) is a counter-objection to the objection \(P\) of \(k\) against \(l\) at \(x\) iff \(l \in Q,\ k \notin Q\) and \(e(Q,x) \geq e(P,x)\).

"\(k\)'s demand is not justified: \(Q\) is a coalition that contains \(l\) and excludes \(k\) and that sacrifices even more (or gains even less)."
A first definition

Remember that the set of feasible payoff vectors for \((N,v,S)\) is \(X_{(N,v,S)} = \{x \in \mathbb{R}^n \mid \text{for every } C \in S : x(C) \leq v(C)\}\).

**Definition** (Kernel)

Let \((N,v,S)\) be a TU game in coalition structure. The **kernel** is the set of imputations \(x \in X_{(N,v,S)}\) s.t. for any coalition \(C \in S\), for each objection \(P\) of an agent \(k \in C\) over any other member \(l \in C\) to \(x\), there is a counter-objection of \(l\) to \(P\).
Another definition

**Definition (Maximum surplus)**

For a TU game \((N,v)\), the **maximum surplus** \(s_{k,l}(x)\) of agent \(k\) over agent \(l\) with respect to a payoff distribution \(x\) is the **maximum excess** from a coalition that includes \(k\) but does exclude \(l\), i.e.,

\[
s_{k,l}(x) = \max_{C \subseteq N \mid k \in C, l \notin C} e(C, x).
\]

**Definition (Kernel)**

Let \((N,v,S)\) be a TU game with coalition structure. The **kernel** is the set of imputations \(x \in X_{(N,v,S)}\) such that for every coalition \(C \in S\), if \((k,l) \in C^2, k \neq l\), then we have either \(s_{kl}(x) \geq s_{lk}(x)\) or \(x_k = v(\{k\})\).

\(s_{kl}(x) < s_{lk}(x)\) calls for a transfer of utility from \(k\) to \(l\) unless it is prevented by individual rationality, i.e., by the fact that \(x_k = v(\{k\})\).
Properties

Theorem
Let \((N,v,S)\) a game with coalition structure, and let \(\text{Imp} \neq \emptyset\). Then we have:

(i) \(\text{Nu}(N,v,S) \subseteq K(N,v,S)\)
(ii) \(K(N,v,S) \subseteq BS(N,v,S)\)

Theorem
Let \((N,v,S)\) a game with coalition structure, and let \(\text{Imp} \neq \emptyset\). The kernel \(K(N,v,S)\) and the bargaining set \(BS(N,v,S)\) of the game are non-empty.

Proof
Since the Nucleolus is non-empty when \(\text{Imp} \neq \emptyset\), the proof is immediate using the theorem above. \(\square\)
Proof of (i)

Let \( x \notin K(N,v,S) \), we want to show that \( x \notin Nu(N,v,S) \).

\( x \notin K(N,v,S) \), hence, there exists \( C \in CS \) and \( (k,l) \in C^2 \) such that \( s_{lk}(x) > s_{kl}(x) \) and \( x_k > v(\{k\}) \).

Let \( y \) be a payoff distribution corresponding to a transfer of utility

\[
\epsilon > 0 \text{ from } k \text{ to } l: \quad y_i = \begin{cases} 
  x_i & \text{if } i \neq k \text{ and } i \neq l \\
  x_k - \epsilon & \text{if } i = k \\
  x_l + \epsilon & \text{if } i = l 
\end{cases}
\]

Since \( x_k > v(\{k\}) \) and \( s_{lk}(x) > s_{kl}(x) \), we can choose \( \epsilon > 0 \) small enough s.t.

- \( x_k - \epsilon > v(\{k\}) \)
- \( s_{lk}(y) > s_{kl}(y) \)

We need to show that \( e(y)^\triangleright \leq_{lex} e(x)^\triangleright \).

Note that for any coalition \( S \subseteq N \) s.t. \( e(S,x) \neq e(S,y) \) we have either

- \( k \in S \) and \( l \notin S \) (\( e(S,x) > e(S,y) \) since \( e(S,y) = e(S,x) + \epsilon > e(S,x) \))
- \( k \notin S \) and \( l \in S \) (\( e(S,x) < e(S,y) \) since \( e(S,y) = e(S,x) - \epsilon < e(S,x) \))
Proof of (i)

Let \(\{B_1(x), \ldots, B_M(x)\}\) a partition of the set of all coalitions s.t.

- \((S, T) \in B_i(x)\) iff \(e(S, x) = e(T, x)\). We denote by \(e_i(x)\) the common value of the excess in \(B_i(x)\), i.e. \(e_i(x) = e(S, x)\) for all \(S \in B_i(x)\).

- \(e_1(x) > e_2(x) > \cdots > e_M(x)\)

In other words, \(e(x)\uparrow = \langle e_1(x), \ldots, e_1(x), \ldots, e_M(x), \ldots, e_M(x)\rangle\).

\(|B_1(x)|\times\langle e_1(x)\rangle\quad|B_M(x)|\times\langle e_M(x)\rangle\)

Let \(i^*\) be the minimal value of \(i \in \{1, \ldots, M\}\) such that there is \(C \in B_{i^*}(x)\) with \(e(C, x) \neq e(C, y)\).

For all \(i < i^*\), we have \(B_i(x) = B_i(y)\) and \(e_i(x) = e_i(y)\).
Proof of (i)

Since $s_{lk}(x) > s_{kl}(x)$ $B_{i^*}$ contains
- at least one coalition $S$ that contains $l$ but not $k$, for such coalition, we must have $e(S,x) > e(S,y)$
- no coalition that contains $k$ but not $l$. 

**If** $B_{i^*}$ contains either
- coalitions that contain both $k$ and $l$
- or coalitions that do not contain both $k$ and $l$

**Then**, for any such coalitions $S$, we have $e(S,x) = e(S,y)$, and it follows that $B_{i^*}(y) \subset B_{i^*}(x)$. 

**Otherwise**, we have $e_{i^*}(y) < e_{i^*}(x)$. 

In both cases, we have $e(y)$ is lexicographically less than $e(x)$, and hence $y$ is not in the nucleolus of the game $(N,v,S)$. 
Proof of (ii)

Let \((N,v,S)\) a TU game with coalition structure. Let \(x \in K(N,v,S)\). We want to prove that \(x \in BS(N,v,S)\). To do so, we need to show that for any objection \((P,y)\) from any player \(i\) against any player \(j\) at \(x\), there is a counter objection \((Q,z)\) to \((P,y)\). For the bargaining set, An objection of \(i\) against \(j\) is a pair \((P,y)\) where

- \(P \subseteq N\) is a coalition such that \(i \in P\) and \(j \notin P\).
- \(y \in \mathbb{R}^p\) where \(p\) is the size of \(P\)
- \(y(P) \leq v(P)\) (\(y\) is a feasible payoff for members of \(P\))
- \(\forall k \in P, y_k \geq x_k\) and \(y_i > x_i\)

An counter-objection to \((P,y)\) is a pair \((Q,z)\) where

- \(Q \subseteq N\) is a coalition such that \(j \in Q\) and \(i \notin Q\).
- \(z \in \mathbb{R}^q\) where \(q\) is the size of \(Q\)
- \(z(Q) \leq v(Q)\) (\(z\) is a feasible payoff for members of \(Q\))
- \(\forall k \in Q, z_k \geq x_k\)
- \(\forall k \in Q \cap P, z_k \geq y_k\)
Proof of (ii)

Let \((P, y)\) be an objection of player \(i\) against player \(j\) to \(x. i \in P, j \notin P, y(P) \leq v(P)\) and \(y(P) > x(P).\) We choose \(y(P) = v(P).\)

- \(x_j = v(\{j\}):\) Then \(\{j\}, v(\{j\})\) is a counter objection to \((P, y)\). ✔

- \(x_j > v(\{j\}):\) Since \(x \in K(N, v, S)\) we have
  \[s_{ji}(x) \geq s_{ij}(x) \geq v(P) - x(P) \geq y(P) - x(P)\]
  since \(i \in P, j \notin P.\)

Let \(Q \subseteq N\) such that \(j \in Q, i \notin Q\) and \(s_{ji}(x) = v(Q) - x(Q).\)

We have \(v(Q) - x(Q) \geq y(P) - x(P).\) Then, we have

\[
\begin{align*}
v(Q) & \geq y(P) + x(Q) - x(P) \\
& \geq y(P \cap Q) + y(P \setminus Q) + x(Q \setminus P) - x(P \setminus Q) \\
& > y(P \cap Q) + x(Q \setminus P) \quad \text{since } i \in P \setminus Q, y(P \setminus Q) > x(P \setminus Q)
\end{align*}
\]

Let us define \(z\) as follows
\[
\begin{align*}
  &\begin{cases} 
    x_k \text{ if } k \in Q \setminus P \\
    y_k \text{ if } k \in Q \cap P 
  \end{cases} \\
(Q, z) \text{ is a counter-objection to } (P, y). \quad ✔
\end{align*}
\]

Finally \(x \in BS(N, v, S).\)
Computing a kernel-stable payoff distribution

- There is a transfer scheme converging to an element in the kernel.
- It may require an infinite number of small steps.
- We can consider the $\epsilon$-kernel where the inequality are defined up to an arbitrary small constant $\epsilon$.

Computing a kernel-stable payoff distribution

**Algorithm 1:** Transfer scheme converging to a \( \epsilon \)-Kernel-stable payoff distribution for the CS \( S \)

\[
\text{compute-}\epsilon\text{-Kernel-Stable}(N, v, S, \epsilon) \\
\text{repeat} \\
\quad \text{for each coalition } C \in S \text{ do} \\
\quad \quad \text{for each member } (i,j) \in C, i \neq j \text{ do} \\
\quad\quad\quad \text{// compute the maximum surplus} \\
\quad\quad\quad \text{// for two members of a coalition in } S \\
\quad\quad\quad s_{ij} \leftarrow \max_{R \subseteq N | (i \in R, j \notin R)} v(R) - x(R) \\
\quad\quad \delta \leftarrow \max_{(i,j) \in C^2, C \in S} s_{ij} - s_{ji}; \\
\quad\quad (i^*, j^*) \leftarrow \arg\max_{(i,j) \in N^2} (s_{ij} - s_{ji}); \\
\quad\quad \text{if } (x_{j^*} - v(\{j\}) < \frac{\delta}{2}) \text{ then} \\
\quad\quad\quad d \leftarrow x_{j^*} - v(\{j^*\}); \\
\quad\quad \text{else} \\
\quad\quad\quad d \leftarrow \frac{\delta}{2}; \\
\quad\quad\quad x_{i^*} \leftarrow x_{i^*} + d; \\
\quad\quad\quad x_{j^*} \leftarrow x_{j^*} - d; \\
\text{until } \frac{\delta}{v(S)} \leq \epsilon;
\]
The complexity for one side-payment is $O(n \cdot 2^n)$.

Upper bound for the number of iterations for converging to an element of the $\epsilon$-kernel: $n \cdot \log_2\left(\frac{\delta_0}{\epsilon \cdot v(S)}\right)$, where $\delta_0$ is the maximum surplus difference in the initial payoff distribution.

To derive a polynomial algorithm, the number of coalitions must be bounded. For example, only consider coalitions which size is bounded in $[K_1, K_2]$. The complexity of the truncated algorithm is $O(n^2 \cdot n_{\text{coalitions}})$ where $n_{\text{coalitions}}$ is the number of coalitions with size in $[K_1, K_2]$, which is a polynomial of order $K_2$.

We saw another way to use the excess to make objections and counter-objections.

We defined the kernel.

We proved that both the kernel and the bargaining set are non-empty if the set of imputations is non-empty.

**pros:**
- If the set of imputations is non-empty, the nucleolus, kernel, bargaining set are non-empty.
- There is an algorithm to compute a payoff in the kernel.

**cons:** The algorithm is not polynomial.
The **Shapley value**.
It is not a stability concept, but it tries to guarantee fairness. We will see it can be defined axiomatically or using the concept of marginal contributions.