Cooperative Games

Lecture 2: The core

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- We all want to work together and get $v(N)$, but we all have different views about how to share the fruits of our work. We can use the values of other coalitions as arguments in favor of a distribution.
- A condition for a coalition to form:

  all agents prefer to be in it.

  i.e., none of the participants wishes she were in a different coalition or by herself ➪ **Stability**
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts for which no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition:

➪ Is the grand coalition stable $\Leftrightarrow$ Is the core non-empty

Today

- Definition of the core
- Some geometrical representation of the core for games with up to three agents
- Convex games and the core

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(\mathcal{C})$ to any subset $\mathcal{C}$, i.e., $v : 2^N \to \mathbb{R}$.

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Imputation)

An **imputation** is a payoff distribution x that is efficient and individually rational, i.e.:

- $\sum_{i \in N} x_i = v(N)$ (efficiency)
- for all $i \in N$, $x_i \geqslant v(\{i\})$ (individual rationality)

Definition (Group rationality)

$\forall \mathcal{C} \subseteq N$, $\sum_{i \in \mathcal{C}} x(i) \geqslant v(\mathcal{C})$.

The core relates to the stability of the grand coalition:
No group of agents has any incentive to change coalition.

Definition (*core* of a game (N,v))

Let (N,v) be a TU game, and assume we form the grand coalition N. The **core** of (N,v) is the set:

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x \text{ is a group rational imputation}\}$$

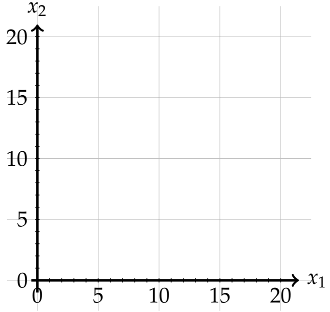
Equivalently,

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}.$$

Weighted graph games

$$N = \{1,2\}$$
$$v(\{1\}) = 5, v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

$$core(N,v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 5, x_2 \geqslant 5, x_1 + x_2 = 20\}$$



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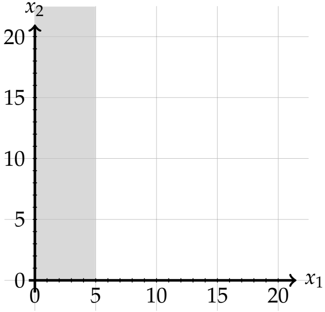

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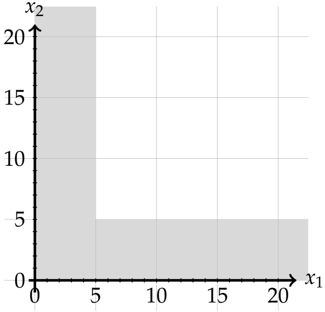

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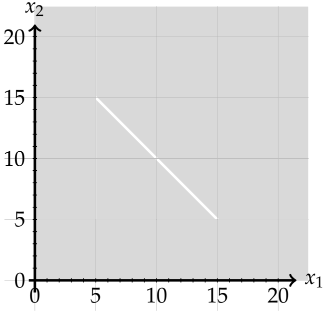

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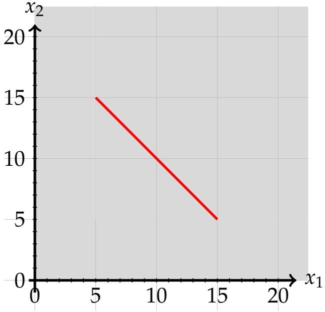

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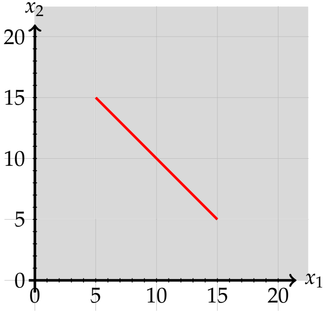

The core may not be fair: the core only considers stability.

three-player majority game

$$N = \{1,2,3\}$$
$$v(\{i\}) = 0$$
$$v(\{\mathcal{C}\}) = \alpha \text{ for } |\mathcal{C}| = 2$$
$$v(N) = 1$$

$$(x_1,x_2,x_3) \in Core(N,v) \Leftrightarrow \begin{cases} \forall i \in N,\ x_i \geqslant 0 \\ \forall (i,j) \in N^2\ i \neq j,\ x_i + x_j \geqslant \alpha \\ \sum_{i\in N} x_i = 1 \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall i \in N\ 0 \leqslant x_i \leqslant 1-\alpha & (1) \\ \sum_{i\in N} x_i = 1 & (2) \end{cases}$$

$Core(N,v)$ is nonempty iff $\alpha \leqslant \frac{2}{3}$
(by summing (1) for all $i \in N$ and using (2))

what happens when $\alpha > \frac{2}{3}$ and the core is empty?

Example with barycentric coordinate

$$\begin{array}{lll} & v(\{1\}) = 1 & v(\{1,2\}) = 4 & \\ v(\emptyset) = 0 & v(\{2\}) = 0 & v(\{1,3\}) = 3 & v(\{1,2,3\}) = 8 \\ & v(\{3\}) = 1 & v(\{2,3\}) = 5 & \end{array}$$

$$\text{set of imputations } \mathcal{I} = \left\{ \sum_{i=1}^{3} x_i = 8, x_1 \geqslant 1, x_2 \geqslant 0, x_3 \geqslant 1 \right\}$$

$\mathcal{I}$ is a triangle with vertices: $(7,0,1)$, $(1,6,1)$, $(1,0,7)$.
On the plane:
$x_1 + x_2 + x_3 = 8$

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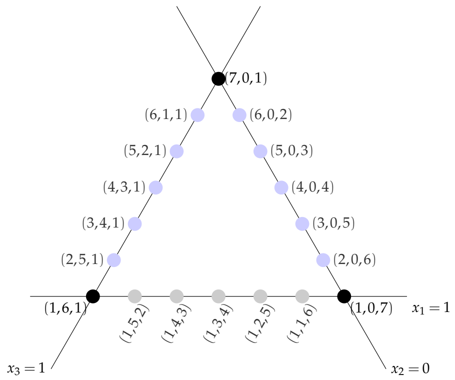

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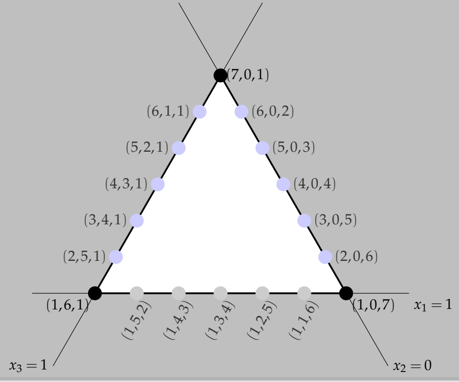

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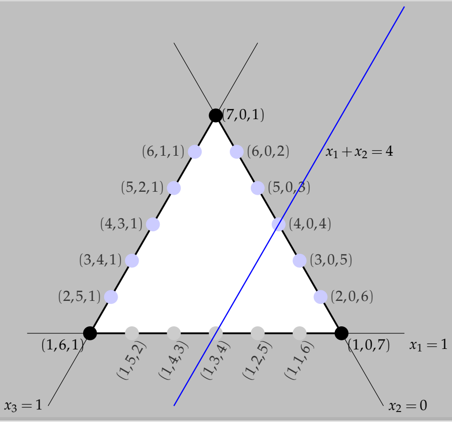

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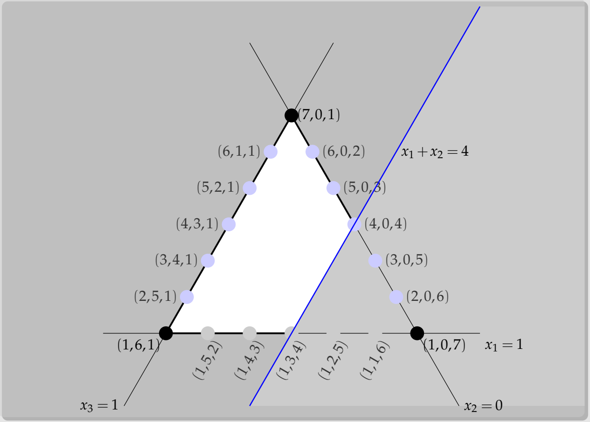

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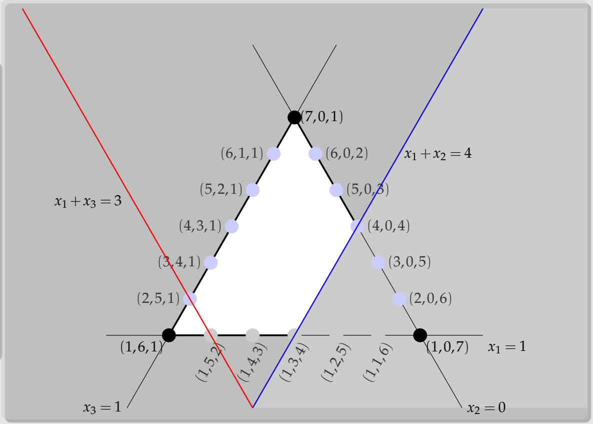

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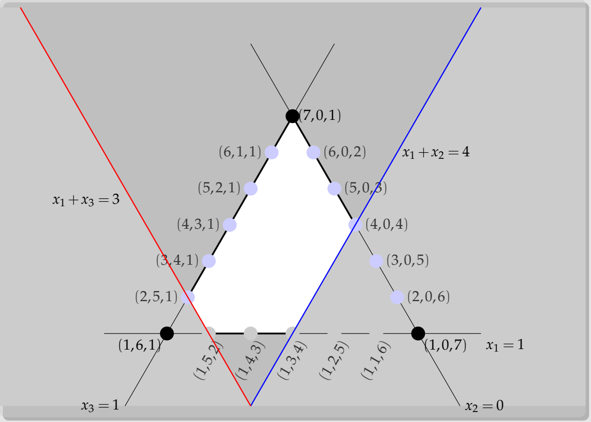

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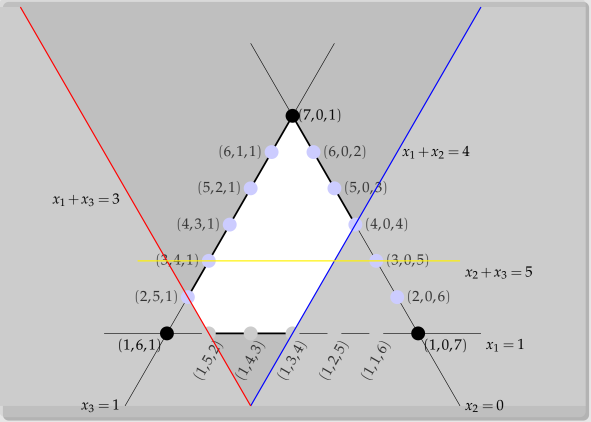

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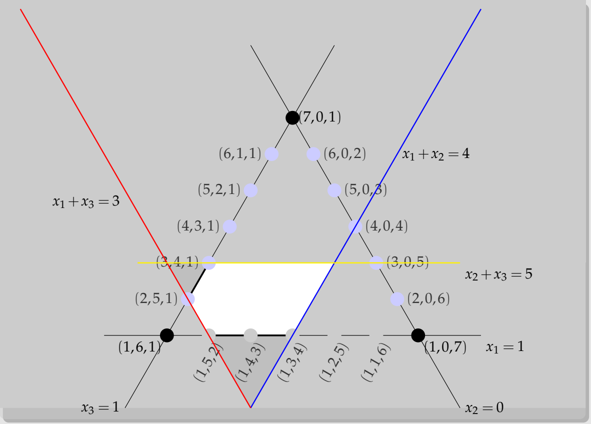

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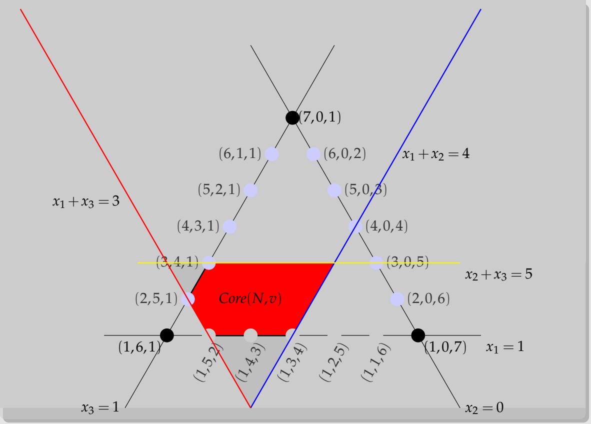

Issues with the core

- The core may not always be non-empty
- When the core is not empty, it may not be 'fair'
- It may not be easy to compute

➩ Are there classes of games that have a non-empty core?

➩ Is it possible to characterize the games with non-empty core?

Definition (Convex games)

A game (N,v) is **convex** iff
$\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$, $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.

TU-game is convex if the marginal contribution of each player increases with the size of the coalition she joins.

Bankruptcy game (E,c) $E \geqslant 0$ is the estate, there are n claimants and $c \in \mathbb{R}^n_+$ is the claim vector (c_i is the claim of the i^{th} claimant). $v(\mathcal{C}) = \max\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i\}$

Theorem

Each bankruptcy game is convex

Theorem

A convex game has a non-empty core

Minimum cost spanning tree games

- N be the set of customers
- 0 be the supplier
- $N_* = N \cup \{0\}$
- $c_{i,j}$ is the cost of connecting i and j by the edge e_{ij} for $(i,j) \in N_*^2$, $i \neq j$
- for a coalition $\mathcal{C}$, $T_{\mathcal{C}} = (\mathcal{C}, E_{\mathcal{C}})$ is the minimum cost spanning tree spanning over the set of edges $\mathcal{C} \cup \{0\}$.
- the cost function is $c(S) = \sum_{(i,j) \in E_{\mathcal{C}}} c_{ij}$
- A **minimum cost spanning tree game** is the associated cost game.

Theorem

Every minimum cost spanning tree game has a non-empty core.

The Bondareva Shapley theorem:
a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C})\chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Example: $n = 3$, $\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

$\lambda(\{1,2\})\chi_{\{1,2\}}$	$\langle \frac{1}{2} \quad \frac{1}{2} \quad 0 \rangle$
$\lambda(\{1,3\})\chi_{\{1,3\}}$	$\langle \frac{1}{2} \quad 0 \quad \frac{1}{2} \rangle$
$\lambda(\{2,3\})\chi_{\{2,3\}}$	$\langle 0 \quad \frac{1}{2} \quad \frac{1}{2} \rangle$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Characterization of games with non-empty core

Definition (Balanced game)

A game is **balanced** iff for each balanced map λ we have $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leqslant v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Summary

- We introduced the core: a stability solution concept.
- We looked at some examples and geometrical representation
- We saw that the core can be empty.
- We proved that convex games have a non-empty core.
- We proved that Minimum Cost Spanning Tree game have a non-empty core
- We started to look at a characterization of the Bondareva-Shapley theorem

Coming next

- Characterization of games with non-empty core (Bondareva Shapley theorem), informal introduction to linear programming.
- Application of Bondareva-Shapley to market games.
- Other games with non-empty core.
- Computational complexity of the core.